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BOX DIMENSION OF INTERPOLATIONS OF SELF-SIMILAR PROCESSES WITH STATIONARY INCREMENTS

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Abstract. We prove that under a general condition interpolation dimensions of H-sssi process converge in probability to $2-H$. The result can be applied to a wide class of H-sssi processes which includes fractional Brownian motions, (α, β)-fractional stable processes or strictly stable H-sssi processes. Moreover, we prove that for an H-sssi process with continuous sample paths the same general condition implies uniform convergence in probability of sample paths of fractal interpolations to sample paths of the interpolated process.

Key words: Fractal interpolation, interpolation dimension, box dimension, self-similar process, stationary increments, stable process.

1. INTRODUCTION

In [1] we proved that, for an H-fractional Brownian motion, sample paths of fractal interpolations converge uniformly in probability to sample paths of the interpolated process and the respective box dimensions of graphs of sample paths converge to $2-H$ with probability 1. In this paper we continue this line of research and extend the results to a wider class of self-similar processes with stationary increments. This class includes all H-sssi processes which are known to have the Hausdorff dimension of sample paths equal with probability 1 to $2-H$ (see Final remarks).

In the sequel we shall need the following definitions.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = \{X(t)\colon t \in [0, 1]\}$ be a process defined on it. A process X is *self-similar with index* $H > 0$ if, for any $a > 0$, the finite-dimensional distributions of $\{X(at)\colon t \in [0, 1]\}$ are the same as those of $\{a_H X(t)\colon t \in [0, 1]\}$. The process is *$H$-sssi* if it is self-similar with index H and has stationary increments (cf. [4], p. 309). We shall consider only measurable, separable H-sssi processes with $P(X \equiv 0) = 0$. For every such process X and

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every $n \geq 3$ a transformation $w_i^{(n)}: \Omega \times R^2 \to R^2$, $i = 1, 2, \ldots, n$, is defined as follows. For every (x, y) in R^2

$$w_i^{(n)}(\omega, (x, y)) = \begin{pmatrix} 1/n & 0 \\ c_i^{(n)}(\omega) & d_i^{(n)}(\omega) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (i-1)/n \\ f_i^{(n)}(\omega) \end{pmatrix},$$

where

(1) $$d_i^{(n)} = \frac{|X(i/n) - X((i-1)/n)|}{\max\{X(k/n):\ 0 \leq k \leq n\} - \min\{X(k/n):\ 0 \leq k \leq n\}},$$

$$c_i^{(n)} = X(i/n) - X((i-1)/n) - d_i^{(n)}(X(1) - X(0)),$$

and

$$f_i^{(n)} = X((i-1)/n) - d_i^{(n)} X(0).$$

Since $X(0) = 0$ and $X(1) \neq 0$ with probability 1 ([6], Lemma 1.2 and Theorem 2.4), it follows that scaling factors $d_i^{(n)}$ are positive with probability 1. Thus, for every $\omega \in \Omega$ modulo null events, the attractor of the family $\{w_i^{(n)}(\omega, \cdot):\ i = 1, \ldots, n\}$ is the graph of a continuous function $(F_n(\cdot))(\omega):\ [0, 1] \to R$ which interpolates the set of data $\{(i/n, (X(i/n))(\omega)):\ i = 0, 1, \ldots, n\}$ (cf. [1], 1.1). We proved in [1], Lemma 1.3, that F_n is a well-defined stochastic process. We call it the *fractal interpolation* of $(X(i/n))_{i \in \{0,1,\ldots,n\}}$ with scaling factors $d_i^{(n)}$. The *box dimension* $D_n(\omega)$ of the graph of $(F_n(\cdot))(\omega)$ is given by the formula

$$D_n(\omega) = 1 + \frac{\ln \sum_{i=1}^{n} d_i^{(n)}(\omega)}{\ln n}.$$

The random variable D_n is called the *interpolation dimension* of $(X(i/n))_{i \in \{0,1,\ldots,n\}}$.

2. VARIOUS KINDS OF CONVERGENCE OF INTERPOLATION DIMENSIONS

In order to examine convergence of interpolation dimensions we shall first define four limit operations on sequences of positive random variables.

1. Let $\mathscr{Y}_1$ be the class of sequences (Y_n) such that $((\ln Y_n)/\ln n)$ converges with probability 1 to a constant limit. The operation $l_1: \mathscr{Y}_1 \to R$ assigns to every sequence $(Y_n) \in \mathscr{Y}_1$ the number being the value of the limit of $((\ln Y_n)/\ln n)$ with probability 1.

2. Let $\mathscr{Y}_2$ be the class of sequences (Y_n) such that $((\ln Y_n)/\ln n)$ converges in probability to a constant limit. The operation $l_2: \mathscr{Y}_2 \to R$ assigns to every sequence $(Y_n) \in \mathscr{Y}_2$ the number being the value of the limit in probability of $((\ln Y_n)/\ln n)$.

3. Let $\mathscr{Y}_3$ be the class of sequences (Y_n) such that $(E((\ln Y_n)/\ln n))$ converges. The operation $l_3: \mathscr{Y}_3 \to R$ assigns to every sequence $(Y_n) \in \mathscr{Y}_3$ the limit of $(E((\ln Y_n)/\ln n))$.

4. Let $\mathscr{Y}_4$ be the class of sequences (Y_n) such that $\big((\ln E(Y_n))/\ln n\big)$ is convergent. The operation l_4: $\mathscr{Y}_4 \to R$ assigns to every sequence $(Y_n) \in \mathscr{Y}_4$ the limit of $\big((\ln Y_n)/\ln n\big)$.

We shall summarize relationships between l_1–l_4.

2.1. *$\mathscr{Y}_1 \subset \mathscr{Y}_2$ and if $(Y_n) \in \mathscr{Y}_1$, then $l_1(Y_n) = l_2(Y_n)$.*

2.2. *If $(Y_n) \in \mathscr{Y}_2 \cap \mathscr{Y}_3$, then $l_2(Y_n) = l_3(Y_n)$.*

2.3. *If $(Y_n) \in \mathscr{Y}_3 \cap \mathscr{Y}_4$, then $l_4(Y_n) \geqslant l_3(Y_n)$.*

2.4. *If $(Y_n) \in \mathscr{Y}_3 \cap \mathscr{Y}_4$ and $l_4(Y_n) = l_3(Y_n) = \alpha$, then $(Y_n) \in \mathscr{Y}_2$ and $l_2(Y_n) = \alpha$.*

Assertions 2.1 and 2.2 are obvious; 2.3 is a consequence of the Jensen inequality. To prove 2.4 we need the following lemma.

2.5. LEMMA. *Let (X_n) be a sequence of random variables defined on the same probability space $(\Omega, \mathscr{F}, P)$. If*

$$\lim_{n\to\infty} E\big(\ln(X_n)\big) = \lim_{n\to\infty} \ln\big(E(X_n)\big) = \alpha,$$

then

$$\ln(X_n) \xrightarrow{P} \alpha.$$

Proof. Take an $\varepsilon > 0$. By the assumption,

$$\big|E\big(\ln(X_n)\big) - \ln\big(E(X_n)\big)\big| < \varepsilon \quad \text{for } n \text{ large enough.}$$

Thus, by the Jensen inequality,

$$0 \leqslant \ln\big(E(X_n)\big) - E\big(\ln(X_n)\big) < \varepsilon.$$

Let $y = a_n x + b_n$ be an equation of the line tangent to the graph of the logarithmic function at the point $\big(E(X_n), \ln\big(E(X_n)\big)\big)$. Then

$$a_n X_n(\omega) + b_n \geqslant \ln\big(X_n(\omega)\big) \quad \text{for every value } X_n(\omega). \tag{2}$$

Let δ_n^1 and δ_n^2 be the smallest numbers satisfying

$$a_n X_n(\omega) + b_n \geqslant \ln\big(X_n(\omega)\big) + \varepsilon \tag{3}$$

for $X_n(\omega) \leqslant E(X_n) - \delta_n^1$ or $X_n(\omega) \geqslant E(X_n) + \delta_n^2$. Take $\delta_n = \max\{\delta_n^1, \delta_n^2\}$ and let $A_n = \{\omega\colon |X_n(\omega) - E(X_n)| > \delta_n\}$. Conditions (2) and (3) imply

$$\begin{aligned}
\ln\big(E(X_n)\big) &= a_n E(X_n) + b_n \\
&= \int_{A_n} a_n X_n(\omega) + b_n \, dP(\omega) + \int_{\Omega\setminus A_n} a_n X_n(\omega) + b_n \, dP(\omega) \\
&\geqslant \int_{A_n} \ln\big(X_n(\omega)\big) + \varepsilon dP(\omega) + \int_{\Omega\setminus A_n} \ln\big(X_n(\omega)\big) dP(\omega) = E\big(\ln(X_n)\big) + \varepsilon P(A_n).
\end{aligned}$$

Thus $\lim_{n\to\infty} P(A_n) = 0$. By a simple estimation of increments of the logarithmic function we infer that δ_n can be chosen to be less than $\sqrt{2\varepsilon/(1-2\varepsilon)}\, E(X_n)$. By the assumption, $(E(X_n))$ tends to e^α; thus there is a number c such that $E(X_n) < c$ for all n. Hence, for large n,

$$P\left(\left\{\omega\colon\ |X_n(\omega) - E(X_n)| \leqslant \sqrt{\frac{2\varepsilon}{1-2\varepsilon}c}\right\}\right) > 1-\varepsilon.$$

Letting $\varepsilon \to 0$ makes

$$\sqrt{\frac{2\varepsilon}{1-2\varepsilon}c} \to 0.$$

Therefore $X_n \xrightarrow{P} e^\alpha$ and, consequently, $\ln(X_n) \xrightarrow{P} \alpha$, as required. ∎

Lemma 2.5 can be easily generalized over injective concave functions with some simple requirements on second derivatives.

Proof of 2.4. Let us assume that

$$\lim_{n\to\infty} \frac{E(\ln Y_n)}{\ln n} = \lim_{n\to\infty} \frac{\ln E(Y_n)}{\ln n} = \alpha.$$

By the Jensen inequality, applied first to the logarithmic function and second to a power function with the exponent $1/\ln n$, we obtain

$$\frac{E(\ln Y_n)}{\ln n} = E\left(\ln(Y_n^{1/\ln n})\right) \leqslant \ln E(Y_n^{1/\ln n}) \leqslant \ln\left((E(Y_n))^{1/\ln n}\right) = \frac{\ln E(Y_n)}{\ln n}.$$

Thus, also

$$\lim_{n\to\infty} \ln E(Y_n^{1/\ln n}) = \alpha.$$

By Lemma 2.5 applied to $X_n = Y_n^{1/\ln n}$,

$$Y_n^{1/\ln n} \xrightarrow{P} e^\alpha.$$

Hence $(\ln Y_n)/\ln n \xrightarrow{P} \alpha$ and $l_2(Y_n) = \alpha$, as claimed. ∎

We shall use the above results to examine the convergence of interpolation dimensions D_n with scaling factors $d_i^{(n)}$ defined in (1) for $i = 1, 2, \ldots, n$ and $n > 2$. Denote $\sum_{i=1}^n d_i^{(n)}$ by Y_n. Let $L_j(D_n) = 1 + l_j(Y_n)$ for $j = 1, \ldots, 4$ (if the limit $l_j(Y_n)$ exists). Note that the mean values which occur in the definition of L_3 and L_4 exist, since $d_i^{(n)}$ and D_n are bounded and measurable. Statements 2.1–2.4 can now be reformulated for the operations L_1–L_4.

2.6. Corollary. (i) $L_1(D_n) = \alpha \Rightarrow L_2(D_n) = \alpha \Rightarrow L_3(D_n) = \alpha$.

(ii) $L_4(D_n) \geqslant L_3(D_n)$.

(iii) $L_4(D_n) = L_3(D_n) = \alpha \Rightarrow L_2(D_n) = \alpha$.

Corollary 2.6 (iii) is of special interest to us because it allows us to prove convergence in probability of interpolation dimensions by estimating only corresponding expectations, which is easier and even can be done experimentally. In the next section we shall apply that result to interpolation dimensions of self-similar processes with stationary increments.

3. INTERPOLATION DIMENSION OF H-sssi PROCESSES

In [1] we proved that if $\{X(t)\}$ is a fractal interpolation of fractional Brownian motion with index H, then the interpolation dimensions D_n converge to $2-H$ with probability 1. Applying methods developed above we can easily extend this result to H-sssi processes. For a process $\{X(t)\}_{t\in[0,1]}$ let

$$X_{\max}^{(n)} = \max_{i\in\{0,1,\ldots,n\}} \{X(i/n)\} \quad \text{and} \quad X_{\min}^{(n)} = \min_{i\in\{0,1,\ldots,n\}} \{X(i/n)\}.$$

3.1. THEOREM. *Let $\{X(t)\}_{t\in[0,1]}$ be an H-sssi process with $E(X_{\max}^{(n)}-X_{\min}^{(n)})$ being finite. Then $L_3(D_n(X(i/n))) \geqslant 2-H$.*

Proof. Since $X(0)=0$ and $X(1)\neq 0$ with probability 1, there is a positive constant c such that

$$E(X_{\max}^{(n)}-X_{\min}^{(n)}) = c.$$

Therefore, by the Jensen inequality, we have

$$\begin{aligned}
E\left(\ln\frac{\sum_{i=1}^{n}|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\right) &\geqslant E\ln n\frac{\sum_{i=1}^{n}|X(i/n)-X((i-1)/n)|}{n}-\ln c \\
&\geqslant E\left(\ln n\sqrt[n]{\prod_{i=1}^{n}|X(i/n)-X((i-1)/n)|}\right)-\ln c \\
&= \ln n+\frac{1}{n}\sum_{i=1}^{n} E\big(\ln|X(i/n)-X((i-1)/n)|\big)-\ln c.
\end{aligned}$$

Now it suffices to estimate the value $E\big(\ln|X(i/n)-X((i-1)/n)|\big)$. By self-similarity,

$$E\big(\ln|X(i/n)-X((i-1)/n)|\big) = E\big(\ln(|X(1)-X(0)|\,n^{-H})\big) = -H\ln n+C,$$

where $C = E\big(\ln|X(1)-X(0)|\big)$. Hence, by the Jensen inequality,

$$\begin{aligned}
L_3(D_n) &= 1+\lim_{n\to\infty}\frac{E\left(\sum_{i=1}^{n}\frac{|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\right)}{\ln n} \\
&\geqslant 1+\lim_{n\to\infty}\frac{\ln n-H\ln n+C-\ln c}{\ln n} = 2-H. \ \blacksquare
\end{aligned}$$

3.2. Theorem. *Let $\{X(t)\}_{t\in[0,1]}$ be an H-sssi process. If $E(|H_1|) < \infty$, then $L_4\big(D_n(X(i/n))\big) \leqslant 2-H$.*

Proof. It is enough to prove the following inequality:

$$\lim_{n\to\infty} \ln\left(E\left(\frac{|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\right)\right)/\ln n \leqslant -\alpha.$$

Take an $\varepsilon > 0$. Let

$$A_n = \{|X(1)-X(0)| > 1/n^{\varepsilon}\}.$$

Since $X(0) = 0$ and $X(1) \neq 0$ with probability 1, we have $\lim_{n\to\infty} P(A_n) = 1$. Therefore, using self-similarity we get

$$E\left(\frac{|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\right)$$

$$\leqslant \int_{A_n} \frac{|X(i/n)-X((i-1)/n)|}{|X(1)-X(0)|}\,dP + \int_{\Omega\setminus A_n} \frac{|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\,dP$$

$$\leqslant n^{\varepsilon}\int_{A_n} |X(1)-X(0)|\,dP + P(\Omega\setminus A_n) \leqslant n^{\varepsilon} n^{-H} \int_{A_n} |X(1)-X(0)|\,dP + P(\Omega\setminus A_n).$$

Thus

$$\frac{\ln E\left(\frac{|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\right)}{\ln n} \leqslant \frac{\ln\left(n^{-H}\int_{A_n}|X(1)-X(0)|\,dP+1-P(A_n)\right)}{\ln n}.$$

Letting $n\to\infty$ we get the claim. ∎

Summarizing the above results we obtain the following corollary:

3.3. Corollary. *Let $\{X(t)\}_{t\in[0,1]}$ be an H-sssi process with $E(X_{\max}^{(n)}-X_{\min}^{(n)})$ being finite. Then the sequence D_n of interpolation dimensions converges to $2-H$ in probability.*

Proof. By Corollary 2.6 (ii) and Theorems 3.1 and 3.2 we obtain

$$2-H \geqslant L_4(D_n) \geqslant L_3(D_n) \geqslant 2-H.$$

Thus $L_4(D_n) = L_3(D_n) = 2-H$ and, in view of Corollary 2.6 (iii), also $L_2(D_n) = 2-H$.

4. CONVERGENCE OF PATHS OF FRACTAL INTERPOLATIONS

In [1] we examined uniform convergence of sample paths of fractal interpolations to sample paths of the interpolated process. We proved the following theorem ([1], Theorem 3.3):

4.1. THEOREM. *Let $\{X(t)\}_{t\in[0,1]}$ be a stochastic process that has continuous sample paths. Let F_n be the fractal interpolation of $(X(i/n))_{i\in\{0,1,\ldots,n\}}$ with scaling factors*

$$d_i^{(n)} = \frac{|X(i/n)-X((i-1)/n)|}{\max_i\{X(i/n)\}-\min_i\{X((i-1)/n)\}}.$$

If there is a constant c such that

$$P\{\omega\colon \max_i d_i^{(n)}(\omega) \leqslant c < 1/2\} \to 1 \quad \text{as } n\to\infty,$$

then

$$\sup_{t\in[0,1]} (|F_n(t)-X(t)|) \to 0 \text{ in probability.}$$

We shall use that theorem to H-sssi processes.

4.2. THEOREM. *Let $\{X(t)\}_{t\in[0,1]}$ be an H-sssi process with continuous sample paths and $E(|X(1)|)<\infty$. Let F_n be the fractal interpolation of $(X(i/n)\colon i=0,1,\ldots,n)$ with scaling factors*

$$d_i^{(n)} = \frac{|X(i/n)-X((i-1)/n)|}{\max_i\{X(i/n)\}-\min_i\{X((i-1)/n)\}}.$$

Then

$$\sup_{t\in[0,1]} (|F_n(t)-X(t)|) \to 0 \text{ in probability.}$$

Proof. We shall prove that $\lim_{n\to\infty}\max_i d_i^{(n)} = 0$ in probability. Let $0<\varepsilon<H$. Like in the proof of Theorem 3.2, let

$$A_n = \{|X(1)-X(0)| > 1/n^{\varepsilon}\}.$$

Then

$$E\left(\frac{\max_i|X(i/n)-X((i-1)/n)|}{X_{\max}^{(n)}-X_{\min}^{(n)}}\right)$$
$$\leqslant \int_{A_n} \frac{\max_i|X(i/n)-X((i-1)/n)|}{|X(1)-X(0)|}\,dP + P(\Omega\backslash A_n)$$
$$\leqslant n^{\varepsilon} n^{-H} \int_{A_n} |X(1)-X(0)|\,dP + P(\Omega\backslash A_n) \leqslant n^{\varepsilon} n^{-H} c + P(\Omega\backslash A_n),$$

where $c = E(|X(1)-X(0)|)$. Therefore $(\max_i d_i^{(n)})$ converges to 0 in L^1, which implies convergence in probability ([4], Chapter II, § 10, Theorem 2). ■

Final remarks. Xiao and Lin [7] gave general conditions under which the Hausdorff dimension of the graph of a sample path of an H-sssi process is equal with probability one to $2-H$. They also showed that these conditions are satisfied by

fractional Brownian motions,

strictly stable processes with index α, $1 < \alpha < 2$,

(α, β)-fractional stable processes with $1 < \alpha < 2$, $-1/\alpha < \beta < 1-1/\alpha$,

and all H-sssi processes X satisfying the moment condition

$$\exists \gamma > 1/H \quad E(|X(1)|^{\gamma}) < \infty$$

(for the last condition see [5], for more details concerning H-sssi processes see [3], Chapter 7). One of the conditions used by Xiao and Lin is, for H-sssi processes, equivalent to

$$(*) \qquad E\big(\sup_{h \in [0,1]} |X(t+h) - X(t)|\big) < \infty.$$

In proofs of Theorems 3.1, 3.2 and 4.2 we use only the assumption $E(|X(1)|) < \infty$ or $E(X_{\max}^{(n)} - X_{\min}^{(n)}) < \infty$, which is a discrete version of $(*)$. Thus we can apply the results to all processes mentioned above. For these processes interpolation dimension can be used as an estimator of the Hausdorff dimension of graphs of their sample paths.

The class of H-sssi processes which satisfy $(*)$ and have continuous sample paths versions includes (cf. [2]):

fractional Brownian motions,

(α, β)-fractional stable processes with $1 < \alpha < 2$, $0 < \beta < 1-1/\alpha$,

sub-Gaussian $S\alpha S$-processes with $1 < \alpha < 2$.

For this class, fractal interpolations approximate sample paths of the interpolated process.

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