PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 21, Fasc. 1 (2001), pp. 231–251

ON q-DEFORMED QUANTUM STOCHASTIC CALCULUS

BY

PIOTR ŚNIADY* (WROCŁAW)

Abstract. In this paper we investigate a quantum stochastic calculus built of creation, annihilation and number of particles operators which fulfill some deformed commutation relations.

Namely, we introduce a deformation of a number of particles operator which has simple commutation relations with well-known q-deformed creation and annihilation operators. Since all operators considered in this theory are bounded, we do not deal with some difficulties of a non-deformed theory of Hudson and Parthasarathy [8]. We define stochastic integrals and estimate them in the operator norm. We prove Itô's formula as well.

1. INTRODUCTION

The aim of this paper is to construct a quantum stochastic calculus in which all operators are bounded and which would unify classical examples we mention below.

1.1. Classical examples of quantum stochastic calculi. The fundamental observation which inspired the development of Hudson–Parthasarathy stochastic calculus [8] was that a family of commuting self-adjoint operators and a state τ induce (by the spectral theorem) measures which can be interpreted as joint distributions of a certain stochastic process.

The most important examples are $B(t) = A(t) + A^*(t)$, which corresponds to the Brownian motion, and $P_l(t) = \sqrt{l}\, B(t) + \Lambda(t) + lt\mathbf{1}$, which corresponds to the Poisson process with intensity l. Quantities $A(t)$, $A^*(t)$, $\Lambda(t)$ $(t \geqslant 0)$ called *annihilation, creation* and *gauge processes*, respectively, have values being unbounded operators acting on some Hilbert space called a *bosonic Fock space.*

For all s, $t \geqslant 0$ they fulfill the following commutation relations:

$$[A(t), A(s)] = [A^*(t), A^*(s)] = 0, \tag{1}$$

* Institute of Mathematics, Wrocław University.

$$[A(t), A(s)] = 0, \tag{2}$$

$$[A(t), A^*(s)] = \min(t, s)\mathbf{1}, \tag{3}$$

$$[\Lambda(t), A(s)] = A[\min(t, s)], \tag{4}$$

$$[\Lambda(t), A^*(s)] = A^*[\min(t, s)]. \tag{5}$$

In the Fock space there exists a unital cyclic vector Ω such that $A(t)\Omega = 0$. The state τ is defined as follows:

$$\tau(S) = \langle \Omega, S\Omega \rangle.$$

Stochastic integrals with respect to the Brownian motion or Poisson process can therefore be written as integrals with respect to creation, annihilation and gauge processes. A stochastic calculus in which such integrals are considered was constructed by Hudson and Parthasarathy [8]. However, the fact that operators considered in this theory are unbounded causes serious technical problems. For example, equations (1)–(5) can be treated only informally and have to be clarified in a more complicated way. Moreover, a product of two stochastic integrals (considered in Itô's formula) is not well defined and has to be evaluated in the weak sense.

The second important example is a fermionic stochastic calculus (see [1] and [2]) in which in equations (1)–(5) commutators were replaced by anticommutators.

The third group of examples is connected with free probability in which the notion of classical independence of random variables was replaced by a noncommutative notion of freeness. Biane and Speicher [3] considered integrals with respect to the free Brownian motion which are a generalization of Itô's integral. On the other hand, the approach of Kümmerer and Speicher [9] is rather related to the calculus of Hudson and Parthasarathy: a free Brownian motion is represented as a family of noncommuting self-adjoint operators $B(t) = A(t)+A^*(t)$ $(t \geqslant 0)$, where $A(t)$, $A^*(t)$ fulfill only a relation

$$A(t)A^*(s) = \min(t, s)\mathbf{1} \tag{6}$$

for all $t, s \geqslant 0$ and a state τ is defined as $\tau(S) = \langle \Omega, S\Omega \rangle$ for a unital cyclic vector Ω such that $A(t)\Omega = 0$. Stochastic integrals are evaluated with respect to $A(t)$ and $A^*(t)$ separately.

1.2. Overview of this paper. In order to avoid problems of Hudson and Parthasarathy's theory we postulate that all operators considered in our stochastic calculus should be bounded. Therefore, we shall replace commutation relations of Hudson–Parthasarathy's calculus by some deformed analogues.

We start with the q-deformed commutation relation which was postulated by Frisch and Bourret [7]:

$$a(\phi)a^*(\psi) = qa^*(\psi)a(\phi)+\langle \phi, \psi \rangle \tag{7}$$

for all $\phi, \psi \in \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space, $a(\phi)$, called an *annihilation operator*, and its adjoint $a^*(\phi)$, called a *creation operator*, are operators acting on some Hilbert space $\Gamma_{\mathscr{H}}$.

If in equation (7) we take $q = 1$, we obtain a bosonic commutation relation (3), for $q = -1$ we obtain a fermionic anticommutation relation, and for $q = 0$ we obtain a free relation (6). Therefore q-deformed commutation relation unifies these three basic cases.

In Section 2 we shall repeat Bożejko and Speicher's [6] construction of the q-deformed Fock space $\Gamma_{\mathscr{H}}$ and bounded operators $a(\phi)$, $a^*(\phi)$ which fulfill (7). Furthermore, we construct a bounded operator λ_μ which acts on $\Gamma_{\mathscr{H}}$ and is a deformation of the Hudson–Parthasarathy gauge operator and an auxiliary operator $\gamma_\mu\colon \Gamma_{\mathscr{H}} \to \Gamma_{\mathscr{H}}$ which is a deformation of the identity. We show commutation relations fulfilled by these operators. It turns out that these commutation relations allow us to write any product of these operators in a special order which is a generalization of Wick or normal ordering.

In Section 3 we define stochastic integrals with respect to four basic processes: annihilation $A(t)$, creation $A^*(t)$, gauge $\Lambda_\mu(t)$ and time $T(t)$. Since in the noncommutative probability the integrand does not commute with the increments of integrator, we have to decide if the integrand should be multiplied from the left or from the right by the integrator. In fact, we shall investigate even a more general case, namely after Biane and Speicher [3] we consider so-called bioperators and biprocesses, so that the increments of integrator are multiplied both from the left and the right by the integrand.

Just like in the classical theory we first define stochastic integrals of simple adapted biprocesses and then by some limit procedure we extend stochastic integrals to a more general class of biprocesses.

In Section 4 we show that (under certain assumptions) an integral of a stochastic process in again an integrable stochastic process and that such an iterated integral is continuous.

Section 5 is devoted to the central point of this paper, Itô's formula, which can be viewed as an integration by parts.

2. DEFORMED CREATION, ANNIHILATION AND A NUMBER OF PARTICLES OPERATORS

2.1. Fock space. Let $\mathscr{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Elements of $\mathscr{H}$ will be denoted by small Greek letters: $\phi, \psi, \ldots$

We shall denote the standard scalar product on $\mathscr{H}^{\otimes n}$ by $\langle \cdot, \cdot \rangle_{\mathrm{free}}$ and call it a *free scalar product*. $\mathscr{H}^{\otimes n}$ furnished with this scalar product will be denoted by $\mathscr{H}^{\otimes n}_{\mathrm{free}}$. By $\Gamma_{\mathrm{free}}(\mathscr{H})$ or simply Γ_{free} we shall denote the direct sum of $\mathscr{H}^{\otimes n}_{\mathrm{free}}$, $n \in N = \{0, 1, 2, \ldots\}$. The space $\mathscr{H}^{\otimes 0}$ which appears in this sum is understood as a one-dimensional space $C\Omega$ for some unital vector Ω.

If $E\colon \mathscr{D}(E) \to \Gamma_{\mathrm{free}}$ for $\mathscr{D}(E) \subseteq \Gamma_{\mathrm{free}}$ is a (possibly unbounded) strictly positive operator, we can introduce a new scalar product $\langle \cdot, \cdot \rangle_E = \langle \cdot, E \cdot \rangle_{\mathrm{free}}$ and a Hilbert space $\Gamma_E(\mathscr{H})$ or simply Γ_E, which is a completion of $\mathscr{D}(E)$ with respect to $\langle \cdot, \cdot \rangle_E$. The norm in Γ_E will be denoted by $\|\cdot\|_E$.

We choose now a parameter of deformation $q \in (-1, 1)$ which will be fixed in this paper.

For $n \in N$ we introduce after Bożejko and Speicher [6] a *q-deformed symmetrization operator* $P^{(n)}\colon \mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes n}$, which is a generalization of a symmetrization (for $q = 1$) and antisymmetrization (for $q = -1$) operators:

$$P^{(n)}(\psi_1 \otimes \ldots \otimes \psi_n) = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \psi_{\sigma(1)} \otimes \ldots \otimes \psi_{\sigma(n)},$$

where $\mathrm{inv}(\sigma) = \# \{(i, j)\colon i, j \in \{1, \ldots, n\},\ i < j,\ \sigma(i) > \sigma(j)\}$ is a number of inversions in permutation σ.

THEOREM 2.1. *$P^{(n)}$ is a strictly positive operator.*

The proof can be found in [6]. ∎

By $P\colon \mathscr{D}(P) \to \Gamma_{\mathrm{free}}$ ($\mathscr{D}(P) \subset \Gamma_{\mathrm{free}}$) we shall denote a closure of the direct sum of $P^{(n)}$, and by Γ we shall denote Γ_P. Since it does not lead to confusions, by $\langle \cdot, \cdot \rangle$ we shall denote both the scalar product in $\mathscr{H}$ and the q-deformed scalar product $\langle \cdot, \cdot \rangle_P$ in the Fock space Γ, and by $\|\cdot\|$ both the norm in Γ and in $\mathscr{H}$. Elements of the Fock space will be denoted by capital Greek letters: Φ, Ψ, …

From now on $\mathscr{H}^{\otimes n}$ will denote the tensor power of $\mathscr{H}$ furnished with q-deformed scalar product $\langle \cdot, \cdot \rangle$.

Let $\Pi_j\colon \Gamma \to \mathscr{H}^{\otimes j}$ denote the orthogonal projection on $\mathscr{H}^{\otimes j}$.

The state τ which plays the role of a noncommutative expectation value is defined as $\tau(X) = \langle \Omega, X\Omega \rangle$ for $X\colon \Gamma \to \Gamma$.

2.2. Operators of creation and annihilation. For $\phi \in \mathscr{H}$ we define action of operators $a(\phi)$, $a^*(\phi)\colon \Gamma_{\mathscr{H}} \to \Gamma_{\mathscr{H}}$ on simple tensors as follows:

$$a^*(\phi)(\psi_1 \otimes \ldots \otimes \psi_n) = \phi \otimes \psi_1 \otimes \ldots \otimes \psi_n, \tag{8}$$

$$a(\phi)(\psi_1 \otimes \ldots \otimes \psi_n) = \sum_{i=1}^{n} q^{i-1} \langle \phi, \psi_i \rangle \psi_1 \otimes \ldots \otimes \psi_{i-1} \otimes \psi_{i+1} \otimes \ldots \otimes \psi_n. \tag{9}$$

2.3. Number of particles operators. Now we need to introduce a deformed analogue of a number of particles operator known also as a gauge operator or a differential second quantization operator. For this deformation we require the operator to be a bounded operator and to have simple commutation relations with $a(\phi)$ and $a^*(\phi)$.

For a bounded operator $T\colon \mathscr{H} \to \mathscr{H}$ we are looking for $\lambda(T)\colon \Gamma_{\mathscr{H}} \to \Gamma_{\mathscr{H}}$ the action of which on simple tensors is defined as:

$$\lambda(T)(\psi_1 \otimes \ldots \otimes \psi_n) = \sum_{i=1}^{n} f(n) \psi_1 \otimes \ldots \otimes \psi_{i-1} \otimes T(\psi_i) \otimes \psi_{i+1} \otimes \ldots \otimes \psi_n.$$

Except the factor $f(n)$ this definition coincides with a non-deformed gauge operator. This factor was added in order to make $\lambda(T)$ a bounded operator. As will be proved in Lemma 2.3, this holds if and only if $\sup_{n\in N}|f(n)|\,n < \infty$.

The choice of $f(n) = \mu^n$ for a complex number μ ($|\mu| < 1$) seems to be the easiest solution. Therefore we define

$$(10)\qquad \lambda_\mu(T)(\psi_1\otimes\ldots\otimes\psi_n) = \sum_{i=1}^{n} \mu^n\,\psi_1\otimes\ldots\otimes\psi_{i-1}\otimes T(\psi_i)\otimes\psi_{i+1}\otimes\ldots\otimes\psi_n.$$

As we shall see in Section 2.4 in order to interchange the deformed number of particles operator with creation or annihilation operators we need to introduce for $|\mu| \leqslant 1$ an operator $\gamma_\mu\colon \Gamma_{\mathscr{H}} \to \Gamma_{\mathscr{H}}$ as follows:

$$(11)\qquad \gamma_\mu(\psi_1\otimes\ldots\otimes\psi_n) = \mu^n\,\psi_1\otimes\ldots\otimes\psi_n.$$

This operator is a *deformed identity operator* and for $\mu = 1$ is equal to identity.

THEOREM 2.2. *For $\phi\in\mathscr{H}$, $|\mu| < 1$ and a bounded $T\colon \mathscr{H}\to\mathscr{H}$, operators $a(\phi)$, $a^*(\phi)$, $\lambda_\mu(T)$ and γ_μ are bounded and*

$$\|\gamma_\mu\| = 1,\qquad \|a(\phi)\| = \|a^*(\phi)\| \leqslant \frac{\|\phi\|}{\sqrt{1-|q|}},$$

$$\|\lambda_\mu(T)\| \leqslant \|T\|\sup_{n\in N} n\,|\mu|^n.$$

Operators $a^*(\phi)$ and $a(\phi)$ as defined in equations (8) *and* (9) *are adjoint as the notation suggests. Furthermore we have*

$$[\lambda_\mu(T)]^* = \lambda_{\bar{\mu}}(T^*),\qquad \gamma_\mu^* = \gamma_{\bar{\mu}}.$$

Proof. It is obvious that for $|\mu| \leqslant 1$ the operator γ_μ is a contraction.

The second inequality will be proved in a more general context in Section 3.3.

Since $\mathscr{H}^{\otimes n}$ are mutually orthogonal invariant spaces of $\lambda_\mu(T)$, from Lemma 2.3 it follows that

$$\|\lambda_\mu(T)\|_{\Gamma\to\Gamma} = \sup_{n\in N}\|\lambda_\mu(T)\|_{\mathscr{H}^{\otimes n}\to\mathscr{H}^{\otimes n}} = \sup_{n\in N}\|\lambda_\mu(T)\|_{\mathscr{H}^{\otimes n}_{\text{free}}\to\mathscr{H}^{\otimes n}_{\text{free}}} \leqslant \|T\|\sup_{n\in N} n\,|\mu|^n.$$

The proof of the fact that $a(\phi)$ and $a^*(\phi)$ are adjoint can be found in [6]. Proofs of the remaining two equations are straightforward. ∎

LEMMA 2.3. *Suppose that $\mathscr{V}_i$ ($i = 1, 2$) are vector spaces. $\mathscr{V}_i$ furnished with scalar product $\langle\cdot,\cdot\rangle_i$ is a Hilbert space denoted by $\mathscr{K}_i$.*

If $P_i\colon \mathscr{K}_i\to\mathscr{K}_i$ are strictly positive bounded operators, we can furnish $\mathscr{V}_i$ with another scalar product $\langle\cdot, P_i\cdot\rangle_i$, and the resulting Hilbert spaces will be denoted by $\mathscr{K}_i'$.

Then operator norms of $S\colon \mathscr{K}_1\to\mathscr{K}_2$ and $S\colon \mathscr{K}_1'\to\mathscr{K}_2'$ are equal for every operator $S\colon \mathscr{V}_1\to\mathscr{V}_2$ such that $SP_1 = P_2S$.

Proof. For any polynomial $f(x)$ we have $Sf(P_1) = f(P_2)S$. Therefore, by approximating the square root by polynomials, we obtain $S\sqrt{P_1} = \sqrt{P_2}S$. Note that for $v \in \mathscr{V}_1$ we have

$$\|Sv\|_{\mathscr{K}_2'} = \|\sqrt{P_2}Sv\|_{\mathscr{K}_2} = \|S\sqrt{P_1}v\|_{\mathscr{K}_2} \leqslant \|S\|_{\mathscr{K}_1 \to \mathscr{K}_2} \|\sqrt{P_1}v\|_{\mathscr{K}_1} = \|S\|_{\mathscr{K}_1 \to \mathscr{K}_2} \|v\|_{\mathscr{K}_1'}.$$

Consequently,

$$\|S\|_{\mathscr{K}_1' \to \mathscr{K}_2'} \leqslant \|S\|_{\mathscr{K}_1 \to \mathscr{K}_2}.$$

If in the preceding calculations we replace $\mathscr{K}_i$ by $\mathscr{K}_i'$ (and vice versa) and replace P_i by P_i^{-1}, we obtain the converse inequality. ∎

2.4. Commutation relations.

THEOREM 2.4. *For $\phi, \psi \in \mathscr{H}$, a bounded operator $T: \mathscr{H} \to \mathscr{H}$ and for $|\mu|, |v| < 1$, the following equalities hold*:

$$a(\phi)a^*(\psi) = qa^*(\psi)a(\phi) + \langle \phi, \psi \rangle, \tag{12}$$

$$a(\phi)\gamma_\mu = \mu\gamma_\mu a(\phi), \tag{13}$$

$$\gamma_\mu a^*(\phi) = \mu a^*(\phi)\gamma_\mu, \tag{14}$$

$$a(\phi)\lambda_\mu(T) = \mu\lambda_\mu(T)a(\phi) + \mu\gamma_\mu a(T^*\phi), \tag{15}$$

$$\lambda_\mu(T)a^*(\phi) = \mu a^*(\phi)\lambda_\mu(T) + \mu a^*(T\phi)\gamma_\mu, \tag{16}$$

$$\lambda_\mu(T)\gamma_v = \gamma_\mu\lambda_v(T) = \lambda_{\mu v}(T), \tag{17}$$

$$\gamma_\mu\gamma_v = \gamma_{\mu v}. \tag{18}$$

If bounded operators $T_1, T_2: \mathscr{H} \to \mathscr{H}$ commute, then

$$\lambda_\mu(T_1)\lambda_v(T_2) = \lambda_v(T_2)\lambda_\mu(T_1). \tag{19}$$

The proof is straightforward and we omit it. ∎

Since γ_1 is equal to identity, we see that in the limit $q, \mu \to 1$ relations (12), (15), (16) and (17) correspond to non-deformed relations (3), (4), (5) and (2). Note that contrary to the non-deformed case among these commutation relations there is none which would allow us to interchange the order of adjacent two creation or two annihilation operators.

2.5. Algebra $\mathscr{A}$. Suppose $\mathscr{H} = \mathscr{K} \oplus \mathscr{K}^\perp$ and $\mathscr{K}^\perp$ is an infinite-dimensional separable Hilbert space. We denote by $\mathscr{A}_{\mathrm{fin}\mathscr{K}}(\mathscr{H})$ an algebra of bounded operators acting on $\mathscr{H}$, generated by operators $a(\phi)$, $a^*(\phi)$, γ_μ, $\lambda_\mu(T \oplus \mathbf{0})$ for all $\phi \in \mathscr{K}$, $|\mu| < 1$ and bounded operators $T: \mathscr{K} \to \mathscr{K}$. We shall denote by $\mathscr{A}_{\mathscr{K}}(\mathscr{H})$ the completion of $\mathscr{A}_{\mathrm{fin}\mathscr{K}}(\mathscr{H})$ in the operator norm.

2.6. Normal ordering. In algebras generated by (bosonic, fermionic or q-deformed) creation and annihilation operators one introduces normal or

Wick's ordering where in each expression one writes creation operators on the right-hand side and annihilation operators on the left-hand side. Now we introduce an analogue of such an ordering in algebras $\mathscr{A}_{\mathrm{fin}}$.

THEOREM 2.5. *Every element S of an algebra $\mathscr{A}_{\mathrm{fin}}$ can be written as a finite sum of products of the following form: on the left-hand side — creation operators, some $\lambda_\nu(T)$ operators, a γ_μ operator (for some μ, $|\mu| \leqslant 1$), and on the right-hand side — annihilation operators:*

$$S = \sum_i a^*(\phi_{i1})\ldots a^*(\phi_{in_i})\lambda_{\nu_{i1}}(T_{i1})\ldots\lambda_{\nu_{ik_i}}(T_{ik_i})\gamma_{\mu_i}a(\psi_{i1})\ldots a(\psi_{im_i}). \tag{20}$$

Proof. Note that an expression is in the above-mentioned form if and only if it does not contain any subexpression being the left-hand side of one of equations (12)–(18). If it does not hold, by replacing the left-hand side of the appropriate equation by the right-hand side, we obtain an expression (or a sum of expressions) which is either shorter or has the same length but a smaller number of disorderings. We can easily see that this procedure has to stop after a finite number of iterations. ∎

By $\mathscr{A}^{(k,l)}$ we shall denote the completion of the space of operators which can be written in the normal ordering (20) with exactly k creation operators $a^*(\cdot)$ and l annihilation operators $a(\cdot)$. Let

$$\mathscr{A}^{(k,\cdot)} = \overline{\bigoplus_{l\in N}\mathscr{A}^{(k,l)}} \quad\text{and}\quad \mathscr{A}^{(\cdot,l)} = \overline{\bigoplus_{k\in N}\mathscr{A}^{(k,l)}}.$$

For an integer number $n\in Z$ we define $\mathscr{A}^{[n]}$ to be a completion of the space of operators which (not necessarily in the normal ordering) contain exactly n more creators than annihilators,

$$\mathscr{A}^{[n]} = \overline{\bigoplus_{\substack{k,l\in N\\ k-l=n}}\mathscr{A}^{(k,l)}}.$$

For $S\in\mathscr{A}$, $n\in Z$, let $S^{[n]}\in\mathscr{A}^{[n]}$ be a part of S which contains exactly n more creation than annihilation operators. More precisely,

$$S^{[n]} = \sum_{\substack{i\in Z\\ i\geqslant 0,\, i+n\geqslant 0}} \Pi_{i+n}S\Pi_i.$$

Note that $\|S^{[n]}\| \leqslant \|S\|$ because

$$\|S^{[n]}\Psi\|^2 = \sum_i \|\Pi_{i+n}S\Pi_i\Psi\|^2 \leqslant \sum_i \|S\Pi_i\Psi\|^2 \leqslant \|S\|^2\sum_i \|\Pi_i\Psi\|^2 = \|S\|\,\|\Psi\|^2.$$

2.7. Extension of operators.

LEMMA 2.6. *If $\mathscr{H}_1 = \mathscr{K}\oplus\mathscr{L}_1$ and $\mathscr{H}_2 = \mathscr{K}\oplus\mathscr{L}_2$, where $\mathscr{L}_1$ and $\mathscr{L}_2$ are separable infinite-dimensional Hilbert spaces, then there exists exactly* *one continuous $*$-isomorphism V: $\mathcal{A}_{\mathcal{K}}(\mathcal{H}_1) \to \mathcal{A}_{\mathcal{K}}(\mathcal{H}_2)$ of Banach algebras, which maps operators $a(\phi)$, $a^*(\phi)$, $\lambda_\mu(T)$, $\gamma_\mu \in \mathcal{A}_{\mathcal{K}}(\mathcal{H}_1)$, respectively, on $a(\phi)$, $a^*(\phi)$, $\lambda_\mu(T)$, $\gamma_\mu \in \mathcal{A}_{\mathcal{K}}(\mathcal{H}_2)$. Moreover, this $*$-isomorphism is an isometry.*

Particularly, if $\mathcal{H}_1 \subset \mathcal{H}_2$, then this $*$-isomorphism assigns to an operator $S \in \mathcal{A}_{\mathcal{K}}(\mathcal{H}_1)$ its extension $\tilde{S} \in \mathcal{A}_{\mathcal{K}}(\mathcal{H}_2)$ (since no confusion is possible, we shall often denote both operators by the same letter).

Proof. Let U: $\mathcal{H}_1 \to \mathcal{H}_2$ be an isometry such that U limited to $\mathcal{K}$ is equal to identity. Such an isometry exists because $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same dimension.

We define now a second quantization of U, i.e. an isometry $\Gamma(U)$: $\Gamma_{\mathcal{H}_1} \to \Gamma_{\mathcal{H}_2}$ by the formula

$$\Gamma(U)(\psi_1 \otimes \ldots \otimes \psi_n) = U(\psi_1) \otimes \ldots \otimes U(\psi_n).$$

The required $*$-isomorphism is

$$\mathcal{A}_{\mathcal{K}}(\mathcal{H}_1) \ni S \mapsto \Gamma(U) S \Gamma(U)^* \in \mathcal{A}_{\mathcal{K}}(\mathcal{H}_2).$$

The uniqueness of such an isomorphism follows from the fact that it is uniquely defined on a dense subspace $\mathcal{A}_{\mathrm{fin}}$. ■

The lemma remains true if in the formulation we skip the assumption of separability, the proof of this fact is however more complicated.

3. STOCHASTIC INTEGRALS

Since we are interested in stochastic calculus, from now on we have $\mathcal{H} = \mathcal{L}^2(\boldsymbol{R}_+)$. We also introduce the notation: $\mathcal{A} = \mathcal{A}_{\mathcal{H}}$, $\mathcal{H}_t = \mathcal{L}^2(0, t)$ and $\mathcal{A}_t = \mathcal{A}_{\mathcal{H}_t}$.

We shall investigate stochastic integrals with respect to four basic stochastic processes with values in the algebra $\mathcal{A}$: annihilation $A(t) = a(\chi_{(0,t)})$, creation $A^*(t) = a^*(\chi_{(0,t)})$, gauge $\Lambda_\mu(t) = \lambda_\mu(\Pi_{(0,t)})$ and time $T(t) = t\mathbb{1}$, where $\chi_I \in \mathcal{H}$ denotes a characteristic function of a set $I \subset \boldsymbol{R}_+$ and Π_I: $\mathcal{L}^2(\boldsymbol{R}_+) \to \mathcal{L}^2(I)$ denotes the orthogonal projection.

3.1. Bioperators and biprocesses. If S: $\boldsymbol{R}_+ \to \mathcal{A}$ is a measurable function, we shall call it a *process*. If for almost all $t \in \boldsymbol{R}_+$ we have $S(t) \in \mathcal{A}_t$, we say it is *adapted.*

Elements of $\mathcal{A} \otimes \mathcal{A}$ will be called *bioperators*. A bioperator can be multiplied by an operator from the left or the right and the result is a bioperator: for F, G, $S \in \mathcal{A}$ we define

$$(F \otimes G) S = F \otimes (GS), \qquad S(F \otimes G) = (SF) \otimes G.$$

Furthermore, we define a "*musical*" *product* of a bioperator by an operator such that the result is an operator:

$$(F \otimes G) \sharp S = FSG.$$

We shall introduce a convolution: $(F \otimes G)^* = G^* \otimes F^*$.

If $R\colon \mathbf{R}_+ \to \mathscr{A}\otimes\mathscr{A}$ is a measurable function, we shall call it a *biprocess*. If for almost all $t\in\mathbf{R}_+$ we have $R(t)\in\mathscr{A}_t\otimes\mathscr{A}_t$, we say that R is *adapted*. If for almost all $t\in\mathbf{R}_+$ we have $R(t)\in\mathscr{A}_t\otimes\mathscr{A}$ or $R(t)\in\mathscr{A}\otimes\mathscr{A}_t$, we say that R is *left-adapted* or *right-adapted*, respectively.

A *simple biprocess* is a biprocess of the form $R(t) = \sum_{i=1}^{n} B_i\chi_{I_i}(t)$, where $B_i\in\mathscr{A}\otimes\mathscr{A}$ and I_i are intervals.

3.2. Stochastic integral of a simple biprocess. A *stochastic integral of a simple biprocess* is defined as a Riemann sum

$$\int\Big(\sum_{i=1}^{n} B_i\chi_{I_i}(t)\Big)\,\sharp dS = \sum_{i=1}^{n} B_i\sharp[S(s_i)-S(t_i)], \qquad \text{where } I_i = (s_i, t_i).$$

3.3. Stochastic integrals with respect to the creation and annihilation process.

3.3.1. *Tensor product* $\otimes_k$. For $\phi, \psi_1, \ldots, \psi_n\in\mathscr{H}$ we define a tensor product $\otimes_k$ as

$$\phi\otimes_k(\psi_1\otimes\ldots\otimes\psi_n) = \begin{cases}\psi_1\otimes\ldots\otimes\psi_k\otimes\phi\otimes\psi_{k+1}\otimes\ldots\otimes\psi_n, & k\leqslant n,\\ 0, & k>n,\end{cases}$$

and an operator $\mathbf{1}\otimes_k P^{(n)}\colon \mathscr{H}^{\otimes(n+1)}\to\mathscr{H}^{\otimes(n+1)}$ which for $n\geqslant k$ is defined as

$$\begin{aligned}&[\mathbf{1}\otimes_k P^{(n)}](\psi_1\otimes\ldots\otimes\psi_k\otimes\phi\otimes\psi_{k+1}\otimes\ldots\otimes\psi_n)\\ &\qquad = \sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)}\psi_{\sigma(1)}\otimes\ldots\otimes\psi_{\sigma(k)}\otimes\phi\otimes\psi_{\sigma(k+1)}\otimes\ldots\otimes\psi_{\sigma(n)}\end{aligned}$$

and is a modification of a q-deformed symmetrization operator, which does not move the factor on the $k+1$ position. For $n<k$ we take $\mathbf{1}\otimes_k P^{(n)} = 0$.

LEMMA 3.1. *There exists a positive constant $\omega(q)$ such that for each n*

$$P^{(n+1)} \leqslant \frac{1}{1-|q|}\mathbf{1}\otimes P^{(n)}, \qquad \mathbf{1}\otimes P^{(n)} \leqslant \frac{1}{\omega(q)}P^{(n+1)}.$$

There exist positive constants $c_{k,q}$ and $d_{k,q}$ such that for each $n\geqslant k$

$$P^{(n+1)} \leqslant c_{k,q}\mathbf{1}\otimes_k P^{(n)}, \qquad \mathbf{1}\otimes_k P^{(n)} \leqslant d_{k,q}P^{(n+1)}.$$

Proof. The proof of the first two inequalities can be found in [4]. Now we show the third one:

$$\begin{aligned}P^{(n+1)} &\leqslant \frac{1}{1-|q|}\mathbf{1}\otimes P^{(n)} \leqslant \ldots \leqslant \frac{1}{(1-|q|)^{k+1}}\mathbf{1}^{\otimes(k+1)}\otimes P^{(n-k)}\\ &= \frac{1}{(1-|q|)^{k+1}}\mathbf{1}\otimes_k[\mathbf{1}^{\otimes k}\otimes P^{(n-k)}]\\ &\leqslant \frac{\omega(q)}{(1-|q|)^{k+1}}\mathbf{1}\otimes_k[\mathbf{1}^{\otimes(k-1)}\otimes P^{(n-k+1)}] \leqslant \ldots \leqslant \frac{\omega(q)^{k-1}}{(1-|q|)^{k+1}}\mathbf{1}\otimes_k P^{(n)}.\end{aligned}$$

The last inequality can be proved similarly. ∎

COROLLARY 3.2. *For any* $\Psi \in \Gamma$ *we have*

$$\|\Psi\|_{1\otimes_k P} \leqslant \sqrt{d_{k,q}}\,\|\Psi\|.$$

If furthermore $\Psi \in \bigoplus_{n>k} \mathscr{H}^{\otimes n}$, *then*

$$\|\Psi\| \leqslant \sqrt{c_{k,q}}\,\|\Psi\|_{1\otimes_k P}.$$

For every $\phi \in \mathscr{H}$ *we have*

$$\|a(\phi)\| = \|a^*(\phi)\| \leqslant \frac{\|\phi\|}{\sqrt{1-|q|}}.$$

Proof. The last inequality holds since for each $\Psi \in \Gamma$ we have

$$\|a^*(\phi)\Psi\| \leqslant \frac{1}{\sqrt{1-|q|}}\|a^*(\phi)\Psi\|_{1\otimes P} = \frac{1}{\sqrt{1-|q|}}\|\phi\|\,\|\Psi\|. \ \blacksquare$$

3.3.2. *Properties of* Q_k. Let $S \in \mathscr{A}_{\mathscr{K}}(\mathscr{H})$ and let ϕ be a unital vector perpendicular to $\mathscr{H}$. By Lemma 2.6 there exists an operator S: $\mathscr{H}\oplus\phi \to \mathscr{H}\oplus\phi$ which is an extension of S: $\mathscr{H} \to \mathscr{H}$.

It is easy to see that for each $\Phi \in \Gamma_{\mathscr{H}}$ there exists an element of $\Gamma_{\mathscr{H}}$ denoted by $Q_k(S)\Phi$ such that

$$\phi\otimes_k[Q_k(S)\Phi] = (\Pi_\phi\otimes_k 1)\,Sa^*(\phi)\,\Phi, \tag{21}$$

where Π_ϕ denotes the orthogonal projection on the subspace spanned by ϕ, and $\Pi_\phi\otimes_k 1$ is an operator which on tensor products of not more than k vectors acts as $\mathbf{0}$ and on longer tensor powers acts on the $(k+1)$-st factor by Π_ϕ.

Of course, $Q_k(S)$: $\Gamma \to \Gamma$ is a linear operator. We shall prove that $Q_k(S)$ is an element of the algebra $\mathscr{A}_{\mathscr{K}}(\mathscr{H})$ and that this operator in the normal ordering has exactly k creation operators, i.e. $Q_k(S) \in \mathscr{A}_{\mathscr{K}}^{(k,\cdot)}(\mathscr{H})$.

Indeed, if S is of the form

$$S = a^*(\phi_1)\dots a^*(\phi_i)\,\lambda_{v_1}(T_1)\dots\lambda_{v_l}(T_l)\,\gamma_\mu\, a(\psi_1)\dots a(\psi_j),$$

then a simple computation shows that

$$Q_k(S) = \begin{cases} q^j v_1\dots v_l\,\mu S & \text{if } k = i,\\ 0 & \text{if } k \neq i,\end{cases}$$

and therefore $Q_k(S) \in \mathscr{A}_{\mathrm{fin}\,\mathscr{K}}^{(k,\cdot)}(\mathscr{H})$.

The general statement follows from the fact that Q_k: $\mathscr{A}_{\mathscr{K}}(\mathscr{H}) \to \mathscr{A}_{\mathscr{K}}(\mathscr{H})$ is a continuous map:

$$\begin{aligned}\|Q_k(S)\Psi\| &= \|\phi\otimes_k[Q_k(S)\Psi]\|_{1\otimes_k P} = \|[\Pi_\phi\otimes_k 1]Sa^*(\phi)\Psi\|_{1\otimes_k P}\\ &\leqslant \|Sa^*(\phi)\Psi\|_{1\otimes_k P} \leqslant \sqrt{d_{k,q}}\,\|Sa^*(\phi)\Psi\| \leqslant \sqrt{\frac{d_{k,q}}{1-|q|}}\,\|S\|\,\|\Psi\|.\end{aligned} \tag{22}$$

In the following we shall often use the notation Q_k: $\mathscr{A}_{\mathscr{K}}(\mathscr{H})\otimes\mathscr{A}_{\mathscr{K}}(\mathscr{H}) \to \mathscr{A}_{\mathscr{K}}(\mathscr{H})$ defined on simple tensors by $Q_k(P\otimes R) = Q_k(P)R$.

3.3.3. *Norm of an integral with respect to the creation and annihilation processes.*

THEOREM 3.3. *If $R\colon \mathbf{R}_+ \to \mathscr{A}\otimes\mathscr{A}$ is a simple left-adapted biprocess, then*

$$\left\|\int R(t)\sharp dA^*(t)\,\Psi\right\| \leqslant \sum_k \left(c_{k,q}\int \|Q_k[R(t)]\,\Psi\|^2\,dt\right)^{1/2}.$$

Proof. Let $R(t) = \sum_i B_i \chi_{I_i}(t)$, where I_i are disjoint intervals. Since R is left-adapted for any $\Psi \in \Gamma_{\mathscr{H}}$, we have

$$\sum_i B_i \sharp a^*(\chi_{I_i})\,\Psi = \sum_k \sum_i \chi_{I_i} \otimes_k [Q_k(B_i)\,\Psi].$$

Therefore

$$\left\|\sum_i B_i \sharp a^*(\chi_{I_i})\,\Psi\right\| \leqslant \sum_k \left\|\sum_i \chi_{I_i}\otimes_k [Q_k(B_i)\,\Psi]\right\|,$$

$$\left\|\sum_i \chi_{I_i}\otimes_k [Q_k(B_i)\,\Psi]\right\|^2 \leqslant c_{k,q}\left\|\sum_i \chi_{I_i}\otimes_k [Q_k(B_i)\,\Psi]\right\|^2_{1\otimes_k P}$$

$$= c_{k,q}\sum_i \|Q_k(B_i)\,\Psi\|^2\,\|\chi_{I_i}\|^2 = c_{k,q}\int \|Q_k[R(t)]\,\Psi\|^2\,dt,$$

which completes the proof. ∎

Now we shall define appropriate seminorms on the space of biprocesses:

$$\|R\|_{A^*} = \sum_k \left(c_{k,q}\int_0^\infty \left\|Q_k[R(t)]\right\|^2 dt\right)^{1/2}, \qquad \|R\|_A = \|R^*\|_{A^*}.$$

THEOREM 3.4. *Simple adapted (respectively, left-adapted or right-adapted) biprocesses are dense in the space of adapted (respectively, left-adapted or right--adapted) biprocesses in seminorms $\|\cdot\|_A$ and $\|\cdot\|_{A^*}$.*

The proof of an analogous fact can be found in the paper by Biane and Speicher [3]. ∎

Therefore we can define an integral with respect to the creation (or annihilation) process of a left-adapted (or, respectively, right-adapted) biprocess $R(t)$ with finite seminorm $\|\cdot\|_{A^*}$ (or $\|\cdot\|_A$) as a limit of integrals of a sequence of simple biprocesses.

We have the following

THEOREM 3.5. *If $R\colon \mathbf{R}_+ \to \mathscr{A}\otimes\mathscr{A}$ is a left-adapted biprocess and $\Psi \in \Gamma$, then*

$$\left\|\int R(t)\sharp dA^*(t)\right\| \leqslant \|R\|_{A^*},$$

$$\left\|\int R(t)\sharp dA^*(t)\,\Psi\right\| \leqslant \sum_k \left(c_{k,q}\int \|Q_k[R(t)]\,\Psi\|^2\,dt\right)^{1/2}.$$

If R: $\mathbf{R}_+ \to \mathscr{A} \otimes \mathscr{A}$ *is a right-adapted biprocess, then*

$$\left\| \int R(t) \# dA(t) \right\| \leqslant \|R\|_A.$$

3.4. Stochastic integral with respect to the gauge process. For an operator $S\colon \Gamma \to \Gamma$ and subspaces $\mathscr{V}$, $\mathscr{W}$ of Γ, we shall denote by $\|S\|_{\mathscr{V} \to \mathscr{W}}$ the operator norm of S defined as

$$\|S\|_{\mathscr{V} \to \mathscr{W}} = \sup_{\Phi \in \mathscr{W}, \|\Phi\| = 1} \ \sup_{\Psi \in \mathscr{V}, \|\Psi\| = 1} |\langle \Phi, S\Psi \rangle|.$$

For $|\mu| < 1$ we introduce a gauge seminorm of a bioperator $B \in \mathscr{A}_t \otimes \mathscr{A}_t$:

$$\|B\|_{\lambda_\mu} = \sup_{n,m \in \mathbf{N}} \sqrt{nm} \, \|B \# \lambda_\mu (\Pi_{(t,\infty)})\|_{\mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes m}};$$

and if a biprocess R is adapted, we introduce its gauge seminorm as

$$\|R\|_{\Lambda_\mu} = \sup_{t \in \mathbf{R}_+} \sup_{n,m \in \mathbf{N}} \sqrt{nm} \, \|R(t) \# \lambda_\mu (\Pi_{(t,\infty)})\|_{\mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes m}}.$$

THEOREM 3.6. *If* $R(t)$ *is a simple adapted biprocess, then the following estimation holds*:

$$\left\| \int R(t) \# d\Lambda_\mu (t) \right\| \leqslant \|R\|_{\Lambda_\mu}.$$

As we shall see in the sequel the assumption that R is simple can be omitted.

Proof. Let us consider a Hilbert space $\mathscr{H} \oplus \tilde{\mathscr{H}}$ such that there exists an operator $U\colon \mathscr{H} \oplus \tilde{\mathscr{H}} \to \mathscr{H} \oplus \tilde{\mathscr{H}}$, which restricted to $\mathscr{H}$ is a unitary operator $U\colon \mathscr{H} \to \tilde{\mathscr{H}}$ and restricted to $\tilde{\mathscr{H}}$ is equal to $\mathbf{0}$. We introduce a process $\tilde{\Lambda}_\mu(t) = \lambda_\mu (\Pi_{(0,t)^\sim})$, where $\Pi_{I^\sim}$ denotes the orthogonal projection on the subspace $U[\mathscr{L}^2(I)]$ for a set $I \subset \mathbf{R}_+$.

For any operator $X\colon \mathscr{H} \oplus \tilde{\mathscr{H}} \to \mathscr{H} \oplus \tilde{\mathscr{H}}$ we define $\lambda(X) = \lambda_1(X)$. Of course, this operator is not bounded, and therefore any manipulations with it have to be done carefully. Lemma 2.3 ensures that $\lambda(U)$ on $\mathscr{H}^{\otimes n}$ is bounded and its norm equals $\sqrt{n}$.

Let $\Gamma_{\mathscr{H}} \ni \Psi = \sum \Psi_n$, $\Gamma_{\mathscr{H}} \ni \Phi = \sum \Phi_n$, where $\Psi_n, \Phi_n \in \mathscr{H}^{\otimes n}$. For any measurable set M we have

$$\begin{aligned} \Big|\Big\langle \Phi_n, \int_M R(t) \# d\Lambda_\mu(t) \Psi_m \Big\rangle\Big| &= \Big|\Big\langle \lambda(U) \Phi_n, \Big(\int R(t) \# d\tilde{\Lambda}_\mu(t)\Big) \lambda(U\Pi_M) \Psi_m \Big\rangle\Big| \\ &\leqslant \sqrt{n} \, \|\Phi_n\| \, \|\lambda(U\Pi_M) \Psi_m\| \, \Big\| \int R(t) \# d\tilde{\Lambda}_\mu(t) \Big\|_{\lambda(U)[\mathscr{H}^{\otimes m}] \to \lambda(U)[\mathscr{H}^{\otimes n}]}. \end{aligned}$$

Let $R(t) = \sum_i B_i \chi_{I_i}(t)$, where I_i are disjoint intervals. For different values of i, operators $B_i \# \lambda_\mu(\Pi_{I_i^\sim})\colon \lambda(U) \mathscr{H}^{\otimes m} \to \Gamma(\mathscr{H} \oplus \tilde{\mathscr{H}})$ have mutually orthogonal images and cokernels. It follows that

$$\|\sum_i B_i \sharp\lambda_\mu(\Pi_{I_i^\sim})\|_{\lambda(U)\mathscr{H}^{\otimes m}\to\lambda(U)\mathscr{H}^{\otimes n}} = \max_i \|B_i \sharp\lambda_\mu(\Pi_{(t_i,\infty)^\sim})\|_{\lambda(U)\mathscr{H}^{\otimes m}\to\lambda(U)\mathscr{H}^{\otimes n}}$$

$$\leqslant \max_i \|B_i \sharp\lambda_\mu(\Pi_{(t_i,\infty)^\sim})\|_{(\mathscr{H}\oplus\tilde{\mathscr{H}})^{\otimes m}\to(\mathscr{H}\oplus\tilde{\mathscr{H}})^{\otimes n}}$$

$$= \max_i \|B_i \sharp\lambda_\mu(\Pi_{(t_i,\infty)})\|_{(\mathscr{H}\oplus\tilde{\mathscr{H}})^{\otimes m}\to(\mathscr{H}\oplus\tilde{\mathscr{H}})^{\otimes n}} = \max_i \|B_i \sharp\lambda_\mu(\Pi_{(t_i,\infty)})\|_{\mathscr{H}^{\otimes m}\to\mathscr{H}^{\otimes n}},$$

where in the last equality we used Lemma 2.3 and in the last but one equality we used the fact that the second quantization $\Gamma(U+U^*)\colon \Gamma(\mathscr{H}\oplus\tilde{\mathscr{H}})\to \Gamma(\mathscr{H}\oplus\tilde{\mathscr{H}})$ of unitary operator $U+U^*$ defined as

$$\Gamma(U+U^*)(\phi_1\otimes\ldots\otimes\phi_n) = (U+U^*)(\phi_1)\otimes\ldots\otimes(U+U^*)(\phi_n)$$

is again unitary and

$$B_i \sharp\lambda_\mu(\Pi_{\mathscr{L}^2(t_i,\infty)}) = \Gamma(U+U^*)\,[B_i \sharp\lambda_\mu(\Pi_{(t_i,\infty)^\sim})]\,\Gamma(U+U^*).$$

Hence

$$(23)\qquad |\langle\Phi, \int_M R(t)\sharp dA_\mu(t)\Psi\rangle|$$

$$\leqslant \sum_{n,m} \|\Phi_n\|\,\|\lambda(U\Pi_M)\Psi_m\|\sqrt{n}\sup_i \|B_i \sharp\lambda_\mu(\Pi_{I_i})\|_{\mathscr{H}^{\otimes m}\to\mathscr{H}^{\otimes n}}$$

$$\leqslant \sum_{n,m} \|\Psi_n\|\,\|\Phi_m\|\sqrt{nm}\sup_i \|B_i \sharp\lambda_\mu(\Pi_{I_i})\|_{\mathscr{H}^{\otimes m}\to\mathscr{H}^{\otimes n}} \leqslant \|\Psi\|\,\|\Phi\|\,\|R\|_{A_\mu}. \quad\blacksquare$$

We would like to extend the definition of a stochastic integral with respect to the gauge process to all biprocesses with finite seminorm $\|\cdot\|_{A_\mu}$ by taking the limit. However, since this seminorm is of $\mathscr{L}^\infty$ type, the space of simple biprocesses is not dense in this space. However, we may have the pointwise convergence.

THEOREM 3.7. *For each adapted biprocess $R(t)$, $\|R\|_{A_\mu} < \infty$, there exists a sequence of simple adapted biprocesses R_i, $\|R_i\|_{A_\mu} \leqslant \|R\|_{A_\mu}$, such that $R_i(t) \to R(t)$ (convergence in the seminorm $\|\cdot\|_{A_\mu}$) for almost all t.*

The proof of this theorem follows the well-known proofs in the classical theory of stochastic integration and we shall omit it.

THEOREM 3.8. *If (R_i) is a sequence of simple adapted biprocesses such that $\sup_i \|R_i\|_{A_\mu} < \infty$ and $R_i(t)$ converges to some $R(t)$ in the seminorm $\|\cdot\|_{A_\mu}$, then the sequence $\int R_i \sharp dA_\mu(t)$ converges in the strong operator topology.*

Proof. It is enough to prove that for each $\varepsilon > 0$ and all vectors $\Psi \in \Gamma$

$$\lim_{N\to\infty} \sup_{i,j>N} \|[R_i - R_j]\sharp dA_\mu(t)\Psi\| \leqslant \varepsilon.$$

Let $M_{ij} = \{t\colon \|R_i(t)-R_j(t)\|_{\lambda_\mu} > \varepsilon/2\}$. We have

$$\int [R_i(t)-R_j(t)]\,\sharp dA_\mu(t)$$
$$(24)\qquad = \int_{M_{ij}} [R_i(t)-R_j(t)]\,\sharp dA_\mu(t) + \int_{R_+\setminus M_{ij}} [R_i(t)-R_j(t)]\,\sharp dA_\mu(t)$$

and the operator norm of the second summand does not exceed $\varepsilon/2$.

We shall use the notation introduced in the proof of Theorem 3.6. Since $\bigcap_N \bigcup_{i,j>N} M_{ij}$ has measure 0, we obtain

$$\lim_{N\to\infty} \sup_{i,j>N} \|\lambda(U\Pi_{M_{ij}})\Psi_m\| = 0$$

for any fixed vector $\Psi \in \Gamma$.

If we rewrite inequality (23) replacing M by M_{ij} and $R(t)$ by $R_i(t)-R_j(t)$, we see by the majorized convergence theorem that the first summand in (24) tends strongly to $\mathbf{0}$. ∎

The preceding theorems allow us to extend the definition of an integral with respect to the gauge process to all adapted processes with finite norm $\|\cdot\|_{A_\mu}$ and to remove from the formulation of Theorem 3.6 the assumption that the integrand is simple.

3.5. Integrals with respect to time. For a biprocess R we introduce its seminorm

$$\|R\|_T = \int_0^\infty \|R\,\sharp\mathbf{1}\|\,dt.$$

Of course, we have

$$\left\|\int R\sharp dT(t)\right\| \leqslant \|R\|_T.$$

4. ITERATED INTEGRALS

LEMMA 4.1. *If $R\colon \boldsymbol{R}_+ \to \mathscr{A}\otimes\mathscr{A}$ is a biprocess such that there exists an integer number j such that $R\colon \boldsymbol{R}_+ \to \bigoplus_{|i|\leqslant j} \mathscr{A}^{[i]}$, then for any process $S\colon \boldsymbol{R}_+ \to \mathscr{A}$ we have*

$$\|RS\|_{A_\mu},\ \|SR\|_{A_\mu} \leqslant \sqrt{j+1}\,(2j+1) \sup_{t\in\boldsymbol{R}_+} \|S(t)\|\,\|R\|_{A_\mu}.$$

Proof. We have

$$\|SR\|_{A_\mu} \leqslant \sum_{|i|\leqslant j} \sup_{t\in\boldsymbol{R}_+} \sup_{n,m} \sqrt{nm}\,\|S(t)^{[i]} R(t)\,\sharp\lambda_\mu(\Pi_{\mathscr{K}^\perp})\|_{\mathscr{H}^{\otimes n}\to\mathscr{H}^{\otimes m}}$$
$$\leqslant \sum_{|i|\leqslant j} \sup_{t\in\boldsymbol{R}_+} \sup_{n,m} \sqrt{\frac{m}{m-i}}\sqrt{n(m-i)}\,\|S(t)^{[i]}\|\,\|R(t)\,\sharp\lambda_\mu(\Pi_{\mathscr{K}^\perp})\|_{\mathscr{H}^{\otimes n}\to\mathscr{H}^{\otimes(m-i)}}$$
$$\leqslant \sqrt{j+1}\,(2j+1)\,\|S\|\,\|R\|_{A_\mu}$$

because $m-i \geqslant 1$ and $m/(m-i) \leqslant j+1$. ∎

LEMMA 4.2. *For any biprocess* $R(t)$, *a process* $S(t)$ *and* $\Psi \in \Gamma$ *we have*

$$\|RS\|_{A^*} \leqslant \|R\|_{A^*} \sup_{t \in \boldsymbol{R}_+} \|S(t)\|,$$

$$\left\|\int R(t) S(t) \sharp dA^*(t) \Psi\right\| \leqslant \|R\|_{A^*} \sup_{t \in \boldsymbol{R}_+} \|S(t) \Psi\|.$$

Proof. It is enough to notice that $Q_k[R(t)S(t)] = Q_k[R(t)]S(t)$ and recall the definition of the norm $\|\cdot\|_{A^*}$ and Theorem 3.5. ∎

LEMMA 4.3. *There exists a constant* $C_{n,q}$ *such that if* $R(t)$ *is a biprocess such that* $R(t) \in \bigoplus_{i \leqslant n} \mathscr{A}^{(i,\cdot)} \otimes \mathscr{A}$ *and* $S(t)$ *is a process such that* $S(t) \in \bigoplus_{i \leqslant n} \mathscr{A}^{(i,\cdot)}$, *then*

$$\|SR\|_{A^*} \leqslant C_{n,q} \|R\|_{A^*} \sup_t \|S(t)\|.$$

If R *is a biprocess such that* $R(t) \in \bigoplus_{i \leqslant n} \mathscr{A}^{(i,\cdot)}$ *and a sequence of processes* $S_i(t)$ *converges strongly to* $\mathbf{0}$ *and fulfills* $S_i(t) \in \bigoplus_{i \leqslant n} \mathscr{A}^{(i,\cdot)}$ *and* $\sup_i \sup_t \|S_i(t)\| < \infty$, *then integrals* $\int S_i(t) R(t) \sharp dA^*(t)$ *converge strongly to* $\mathbf{0}$ *as well.*

Proof. For a unital vector ψ orthogonal to $\mathscr{H}$ we have (see inequality (22))

$$\|Q_k[S(t)R(t)]\Psi\| \leqslant \sqrt{d_{k,q}}\, \|S(t)R(t) \sharp a^*(\phi)\Psi\|,$$

$$\|R(t) \sharp a^*(\phi)\Psi\| = \left\|\sum_{l \leqslant n} \phi \otimes_l Q_l[R(t)]\Psi\right\| \leqslant \sum_{l \leqslant n} \sqrt{c_{l,q}}\, \|Q_l[R(t)]\|\, \|\Psi\|,$$

and therefore for some constants C_1, C_2, C_3 which depend only on q and n we have

$$\|Q_k[S(t)R(t)]\| \leqslant C_1 \|S(t)\| \sum_{l \leqslant n} \|Q_l[R(t)]\|,$$

$$\|Q_k[S(t)R(t)]\|^2 \leqslant C_2 \|S(t)\|^2 \sum_{l \leqslant n} \|Q_l[R(t)]\|^2,$$

$$\|SR\|_{A^*} = \sum_{k \leqslant 2n} \left(c_{k,q} \int \|Q_k[S(t)R(t)]\|^2 dt\right)^{1/2} \leqslant C_3 \sup_{t \in \boldsymbol{R}_+} \|S(t)\| \left(\sum_{l \leqslant n} \int \|Q_l[R(t)]\|^2 dt\right)^{1/2}.$$

The second part of the lemma follows from the majorized convergence theorem. ∎

THEOREM 4.4. *Let the following assumptions be satisfied*:

1. $\tau_n\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *is a sequence of measurable functions*, $0 \leqslant \tau_n(t) \leqslant t$ *and functions* $\tau_n(t)$ *tend to* t *uniformly*;
2. S_1, S_2 *are processes*, $S_1 \in \{A^*, A, \Lambda_\mu, T\}$, $S_2 \in \{A^*, A, \Lambda_\mu, T\}$;
3. $R_1, R_2\colon \boldsymbol{R}_+ \to \mathscr{A}$ *are adapted biprocesses and their appropriate norms are finite*: $\|R_1\|_{S_1}, \|R_2\|_{S_2} < \infty$;

4. *if* $S_1 = \Lambda_\mu$, *then there exists* j *such that for each* $t \in \boldsymbol{R}_+$ *we have*

$$\int_0^t R_2 \,\sharp dS_2 \in \bigoplus_{|i| \leqslant j} \mathscr{A}^{[i]};$$

5. *if* $S_1 = A$, *then there exists* j *such that for each* $t \in \boldsymbol{R}_+$ *we have*

$$R_1(t) \in \bigoplus_{i \leqslant j} \mathscr{A} \otimes \mathscr{A}^{(\cdot, i)} \quad \textit{and} \quad \int_0^t R_2 \,\sharp dS_2 \in \bigoplus_{i \leqslant j} \mathscr{A}^{(\cdot, i)};$$

6. *if* $S_2 = \Lambda_\nu$, *then there exists* j *such that for each* $t \in \boldsymbol{R}_+$ *we have*

$$\int_0^t R_1 \,\sharp dS_1 \in \bigoplus_{|i| \leqslant j} \mathscr{A}^{[i]};$$

7. *if* $S_2 = A^*$, *then there exists* j *such that for each* $t \in \boldsymbol{R}_+$ *we have*

$$R_2(t) \in \bigoplus_{i \leqslant j} \mathscr{A}^{(i, \cdot)} \otimes \mathscr{A} \quad \textit{and} \quad \int_0^t R_1 \,\sharp dS_1 \in \bigoplus_{i \leqslant j} \mathscr{A}^{(i, \cdot)}.$$

Then

$$\int_0^\infty R_1(t) \Big[\int_0^t R_2(s) \,\sharp dS_2(s)\Big] \sharp dS_1(t) = \lim_{n \to \infty} \int_0^\infty R_1(t) \Big[\int_0^{\tau_n(t)} R_2(s) \,\sharp dS_2(s)\Big] \sharp dS_1(t),$$

$$\int_0^\infty \Big[\int_0^s R_1(t) \,\sharp dS_1(t)\Big] R_2(s) \,\sharp dS_2(s) = \lim_{n \to \infty} \int_0^\infty \Big[\int_0^{\tau_n(s)} R_1(t) \,\sharp dS_1(t)\Big] R_2(s) \,\sharp dS_2(s).$$

Proof. For $S_2 \neq \Lambda_\nu$, functions $\|\int_{\tau_n(t)}^t R_2 \,\sharp dS_2(s)\|$ tend uniformly to 0. Therefore by preceding lemmas appropriate norms of biprocesses $R_1(t)[\int_{\tau_n(t)}^t R_2(s) \,\sharp dS_2(s)]$ tend to 0, which proves that the limit in the first equation holds in the operator norm.

For $S_2 = \Lambda_\nu$ and $S_1 \in \{T, A^*\}$ for each $\Psi \in \Gamma$, functions $\|\int_{\tau_n(t)}^t R_2(s) \,\sharp dS_2(s) \Psi\|$ by Theorem 3.8 tend uniformly to 0, and Theorem 3.5 shows that the limit in the first equation holds in the strong operator topology.

For $S_1 = \Lambda_\mu$ and $S_2 = A^*$, Lemma 4.3 and Theorem 3.8 assure that the limit in the second equation holds in the strong operator topology.

For $S_1 = \Lambda_\mu$, $S_2 = \Lambda_\nu$, we introduce a Hilbert space $\mathscr{H} \oplus \tilde{\mathscr{H}} \oplus \hat{\mathscr{H}}$ such that there exist operators U and V which restricted to $\tilde{\mathscr{H}} \oplus \hat{\mathscr{H}}$ are equal to 0 and which map isometrically $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ and $\hat{\mathscr{H}}$, respectively. In the following, for $I \subset \boldsymbol{R}_+$ we denote by $\Pi_{I^\sim}$ and $\Pi_{I^\wedge}$ the orthogonal projections onto $U[\mathscr{L}^2(I)]$ and $V[\mathscr{L}^2(I)]$, respectively. Furthermore $\tilde{\Lambda}_\mu(t) = \lambda_\mu(\Pi_{(0,t)^\sim})$ and $\hat{\Lambda}_\mu(t) = \lambda_\mu(\Pi_{(0,t)^\wedge})$.

We have

$$(25)\qquad \int_0^\infty \Big[\int_{\tau_n(s)}^{s} R_1(t)\,\sharp d\Lambda_\mu(t)\Big] R_2(s)\,\sharp d\Lambda_\nu(s)$$

$$= \lambda(V^*)\lambda(U^*)\Big[\int_0^\infty R_1(t)\,\sharp d\widetilde{\Lambda}_\mu(t)\Big]\int_0^\infty \lambda(U\Pi_{(\tau_n(s),s)})R_2(s)\,\sharp d\widehat{\Lambda}_\nu(s)\lambda(V).$$

Note that even though $\lambda(U)$ and $\lambda(V)$ are unbounded operators, the right-hand side of this equation is well defined on the domain $\Gamma(\mathscr{H})$ (see the proof of Theorem 3.6).

For each n we consider a sequence $t_{n,k} = k/n$ and a bounded operator

$$\Xi_n\colon \lambda(U)\lambda(V)\Gamma(\mathscr{H}) \to \lambda(U)\lambda(V)\Gamma(\mathscr{H})$$

defined as

$$\Xi_n = \sum_i \lambda(\Pi_{(u_{n,i},t_{n,i+1})^\sim})\,\lambda(\Pi_{(t_{n,i},t_{n,i+1})^\wedge}),$$

where $u_{n,i} = \inf_{x\in(t_{n,i},t_{n,i+1})}\tau_n(x)$. It is easy to see that the sequence (Ξ_n) tends strongly to $\mathbf{0}$, and since

$$\int_0^\infty \lambda(\Pi_{(\tau_n(s),s)^\sim}U)R_2(s)\,\sharp d\widehat{\Lambda}_\nu(s)\lambda(V) = \Xi_n\int_0^\infty \lambda(\Pi_{(\tau_n(s),s)^\sim}U)R_2(s)\,\sharp d\widehat{\Lambda}_\nu(s)\lambda(V),$$

the expression (25) tends strongly to $\mathbf{0}$, which proves the first equation.

All the other cases can be obtained by taking the adjoint of considered cases. ∎

5. ITÔ'S FORMULA

5.1. Properties of P_0. We introduce a map $P_0\colon \mathscr{A}_{\mathscr{K}}(\mathscr{H}) \to \mathscr{A}_{\mathscr{K}}(\mathscr{H})$ as follows. Let ϕ be a unital vector perpendicular to $\mathscr{K}$. The map $P_0(R)$ is an operator defined by

$$P_0(R)\Psi = a(\phi)Ra^*(\phi)\Psi \qquad \text{for all } \Psi\in\Gamma_{\mathscr{K}}.$$

It is easy to see that for

$$R = a^*(\phi_1)\ldots a^*(\phi_i)\lambda_{\nu_1}(T_1)\ldots\lambda_{\nu_l}(T_l)\gamma_\mu[a(\psi_1)\ldots a(\psi_j)]$$

we have

$$P_0(R) = q^{i+j}\nu_1\ldots\nu_l\,\mu R,$$

and therefore $P_0(R)\in\mathscr{A}_{\mathscr{K}}(\mathscr{H})$.

5.2. Itô's formula.

THEOREM 5.1. *If assumptions 2–7 of Theorem 4.4 are fulfilled, then*

$$[\int R_1(s)\sharp dS_1(s)][\int R_2(t)\sharp dS_2(t)] = \int [R_1(u)\sharp dS_1(u)][R_2(u)\sharp dS_2(u)] \tag{26}$$
$$+\int R_1(s)[\int_0^s R_2(t)\sharp dS_2(t)]\sharp dS_1(s)+\int [\int_0^t R_1(s)\sharp dS_1(s)] R_2(t)\sharp dS_2(t),$$

where the first summand on the right-hand side is defined as follows:

$$\int [R_1(u)\sharp dA(u)][R_2(u)\sharp dA^*(u)] = \int (1\otimes P_0\otimes 1)(R_1(u)R_2(u))\,du, \tag{27}$$

$$\int [R_1(u)\sharp dA_\mu(u)][R_2(u)\sharp dA^*(u)] = \int [R_1(u)\sharp\gamma_\mu]R_2(u)\sharp dA^*(u), \tag{28}$$

$$\int [R_1(u)\sharp dA(u)][R_2(u)\sharp dA_\nu(u)] = \int [R_1(u)[R_2(u)\sharp\gamma_\nu]]\sharp dA(u), \tag{29}$$

$$\int [R_1(u)\sharp dA_\mu(u)][R_2(u)\sharp dA_\nu(u)] = \int [R_1(u)\sharp 1]R_2(u)\sharp dA_{\mu\nu}(u), \tag{30}$$

$$\int [R_1(u)\sharp dS_1(u)][R_2(u)\sharp dS_2(u)] = 0 \quad \textit{for other values of } S_1, S_2. \tag{31}$$

Informally, we may write this as follows:

		dS_2			
		dA	dA^*	dA_ν	dT
dS_1	dA	0	dT	$dA\gamma_\nu$	0
	dA^*	0	0	0	0
	dA_μ	0	$\gamma_\mu dA^*$	$dA_{\mu\nu}$	0
	dT	0	0	0	0

Proof. For $n = 1, 2$ let $R_n = \sum R_{ni}\chi_{I_i}$ be simple adapted biprocesses. We assume that intervals (I_i) form a partition, i.e. that they are disjoint. Note that we can replace the partition (I_i) by a refined partition $(I_i^{(\varepsilon)})$ so that $\max_i |I_i^{(\varepsilon)}| < \varepsilon$. We have

$$[\int R_1(s)\sharp dS_1(s)][\int R_2(t)\sharp dS_2(t)] \tag{32}$$

$$= \sum_i [R_{1i}\sharp S_1(I_i)][R_{2i}\sharp S_2(I_i)] \tag{33}$$

$$+\sum_j\sum_{i<j}[R_{1i}\sharp S_1(I_i)][R_{2j}\sharp S_2(I_j)]+\sum_i\sum_{j<i}[R_{1i}\sharp S_1(I_i)][R_{2j}\sharp S_2(I_j)]. \tag{34}$$

The second and the third summands tend by Theorem 4.4 to the second and the third summands of the right-hand side of (26). We shall find the weak limit of the first summand when the grid of the partition tends to 0.

If $S_1 = T$ or $S_2 = T$, then it is easy to see that the term (33) tends strongly to $\mathbf{0}$.

If $S_1 = A^*$, Theorem 3.5 gives

$$\|\sum_i [R_{1i} \# a^*(I_i)][R_{2i} \# S_2(I_i)] \Psi\| \leqslant \sum_k \left(c_{k,q} \sum_i \|Q_k(R_{1i})[R_{2i} \# S_2(I_i)] \Psi\|^2 |I_i|\right)^{1/2}$$

$$= \|R_1\|_{A^*} \sup_i \|R_{2i} \# S_2(I_i) \Psi\|,$$

which tends to 0 as the grid of the partition (I_i) tends to 0.

By taking the adjoint we see that if $S_2 = A$, then the term (33) tends weakly to 0 as the grid of the partition (I_i) tends to 0.

The cases we have already considered show that equation (31) holds.

If $S_1 = A$ and $S_2 = A^*$, then we can split the normally ordered form of the expression $[R_{1i} \# a(I_i)][R_{2i} \# a^*(I_i)]$ into two parts: the first which does not contain operators $a^*(I_i)$, $a(I_i)$ and is equal to $|I_i|(\mathbb{1} \otimes P_0 \otimes \mathbb{1})[R_{1i} R_{2i}]$, and the second, which contains these operators in this order. The sum over i of the second part tends in operator norm to 0 as the grid of the partition tends to 0 because it is of the form (33) for $S_1 = A^*$ and $S_2 = A$, which proves equation (27).

If $S_1 = \Lambda_\mu$ and $S_2 = A^*$, then we can split the normally ordered form of the expression $[R_{1i} \# \lambda_\mu(I_i)][R_{2i} \# a^*(I_i)]$ into two parts: the first part equal to $[R_{1i} \# \gamma_\mu][R_{2i} \# a^*(I_i)]$ and the second one which contains operators $a^*(I_i)$, $\lambda_\mu(I_i)$ in this order. The sum over i of the second part is of the form (33) with $S_1 = A^*$ and $S_2 = \Lambda_\mu$, so tends strongly to 0 as the grid of the partition tends to 0, which proves equation (28).

By taking the adjoint of (28) we obtain equation (29), i.e. the case $S_1 = A$, $S_2 = \Lambda_\nu$.

If $S_1 = \Lambda_\mu$, $S_2 = \Lambda_\nu$, we introduce a Hilbert space $\mathscr{H} \oplus \tilde{\mathscr{H}} \oplus \hat{\mathscr{H}}$ such that there exist operators U, V which restricted to $\tilde{\mathscr{H}} \oplus \hat{\mathscr{H}}$ are equal to 0 and which map isometrically $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ and $\hat{\mathscr{H}}$, respectively. We have

$$\langle \Psi, \sum_i [R_{1i} \# \lambda_\mu(I_i)][R_{2i} \lambda_\nu(I_i)] \Phi \rangle$$

$$= \langle \Psi, \lambda(U^*)\lambda(V^*)[\textstyle\int R_1(t) \# d\tilde{\Lambda}_\mu(t)][\textstyle\int R_2(s)\, d\hat{\Lambda}_\nu(s)][\sum_k \lambda(U\Pi_{I_k})\lambda(V\Pi_{I_k})] \Phi \rangle$$

$$+ \langle \Psi, \sum_i [R_{1i} \# \gamma_\mu][R_{2i} \lambda_\nu(\Pi_{I_i})] \Phi \rangle.$$

It is easy to see that, as the grid of the partition (I_i) tends to 0, the operators $\sum_k \lambda(U\Pi_{I_k})\lambda(V\Pi_{I_k})$ tend strongly to 0; therefore the first summand tends strongly to 0, which proves (30).

Now it is enough to notice that any biprocesses can be approximated by simple biprocesses. ∎

Particularly, we can obtain Itô's formula for noncommutative Brownian motion:

$$B(t) = A(t) + A^*(t),$$

and Poisson process with intensity l and deformation parameter μ:

$$P_{\mu,l}(t) = \sqrt{l}\left(\gamma_\mu A(t) + A^*(t)\gamma_\mu\right) + \frac{A(t)}{\mu} + lt\gamma_\mu.$$

We have

dS_1 \ dS_2	dt	$dB(t)$
dt	0	0
$dB(t)$	0	dt

dS_1 \ dS_2	dt	$dP_{\nu,l}(t)$
dt	0	0
$dP_{\mu,l}(t)$	0	$dP_{\mu\nu,l}(t)$

6. FINAL REMARKS

In this paper we have presented foundations of q-deformed stochastic calculus. The lack of space does not allow us to present its applications, among them the connection between q-deformed stochastic integral and noncommutative local martingales. Especially interesting is the possibility of interpolation of classical Brownian motion and Poisson process by their bounded q-deformed analogues for $q \to 1$, where new tools are useful. There are also many questions concerning deformed Poisson process.

Acknowledgements. The author would like to thank Professor Marek Bożejko for many inspiring discussions. This work was partially supported by the State Committee for Scientific Research in Warsaw under grant number P03A05415.

REFERENCES

[1] B. Barnett, R. F. Streater and I. F. Wilde, *The Itô–Clifford integral*, J. Funct. Anal. 48 (1982), pp. 172–212.

[2] B. Barnett, R. F. Streater and I. F. Wilde, *Quasi-free quantum stochastic integrals for the CAR and CCR*, J. Funct. Anal. 52 (1983), pp. 19–47.

[3] P. Biane and R. Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, preprint, 1997, Probab. Theory Related Fields (to appear).

[4] M. Bożejko, *Completely positive maps on Coxeter groups and the ultracontractivity of the q-Ornstein–Uhlenbeck semigroup*, Banach Center Publ. 43 (1998), pp. 87–93.

[5] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion. II*, in: *Quantum Probability and Related Topics VII*, L. Accardi (Ed.), World Scientific, Singapore 1992, pp. 219–236.

[6] M. Bożejko and R. Speicher, *Completely positive maps on Coxeter, deformed commutation relations, and operator spaces*, Math. Ann. 300 (1994), pp. 97–120.

[7] U. Frisch and R. Bourret, *Parastochastics*, J. Math. Phys. 11 (1970), pp. 364–390.

[8] R. L. Hudson and K. R. Parthasarathy, *Quantum Itô's formula and stochastic evolution*, Comm. Math. Phys. 93 (1994), pp. 301–323.

[9] B. Kümmerer and R. Speicher, *Stochastic integration on the Cuntz algebra* O_∞, J. Funct. Anal. 103 (1992), pp. 372–408.

[10] R. Speicher, *Stochastic Integration on the Full Fock Space with the Help of a Kernel Calculus*, Publ. Res. Inst. Math. Sci., Kyoto University, 27, No. 1 (1991).

Institute of Mathematics, Wrocław University
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: psnia@math.uni.wroc.pl

Received on 13.11.2000