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ON THE GEOMETRIC COMPOUNDING MODEL WITH APPLICATIONS

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Abstract. Under the geometric compounding model, we first investigate the relationship between the compound geometric distribution and the underlying distribution, including the preservation of the infinite divisibility property. An interesting upper bound for the tail probability of the compound geometric distribution is provided by using only the mean of the underlying distribution. Secondly, we apply the obtained results to understand better the $\mathscr{L}$-class of life distributions. In particular, we strengthen a surprising result of Bhattacharjee and Sengupta [5] and show that there are life distributions $F \in \mathscr{L}$ with the following properties:

(i) the support of F consists of countably infinite points,

(ii) the coefficient of variation of F is equal to one, and

(iii) F is not in the HNBUE class (the harmonic new better than used in expectation class).

Finally, we apply geometric compounds to characterize the semi-Mittag–Leffler distribution and extend a known result about the exponential distribution.

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1. Introduction. Consider a sequence of independent non-negative random variables X_n, $n = 1, 2, \ldots$, with common distribution F. Assume N_p, independent of $\{X_n\}_{n=1}^{\infty}$, is a geometric random variable with parameter $p \in (0, 1)$, namely, $P_r\{N_p = n\} = p(1-p)^{n-1}$ for $n = 1, 2, \ldots$ Then we say that the random summation $S_{N_p} \equiv \sum_{n=1}^{N_p} X_n$ is a *geometric compound* of the sequence $\{X_n\}_{n=1}^{\infty}$, and that its distribution, denoted by F_S, is a *compound geometric distribution.* Such a compounding mechanism is closely related to the rarefaction (or the thinning version) of a renewal process, and has practical applications to the traffic theory, reliability and ecology problems involving rare events (see, e.g., Kovalenko [15], [16] and Gertsbakh [8]). In actuarial applications, the compounding variable N_p is interpreted as the number of claims, and the sequence $\{X_n\}_{n=1}^{\infty}$ as the costs of the claims. The summation S_{N_p} is the total amount of claims (or the aggregate claim amount) paid by a company (see Rolski et al. [26]). For more applications of geometric compounds, see Szekli [29], pp. 31–33, and the references therein.

In the next section we investigate the relationship between the compound geometric distribution F_S and the underlying distribution F. An interesting upper bound for the tail probability of the compound geometric distribution is provided by using only the mean of the underlying distribution. We introduce the geometric compounding operator on the class of all Laplace transforms of non-negative random variables, and investigate the fundamental properties of this operator, including the preservation of the Laplace transform ordering. In Section 3 we apply the obtained results to understand better the $\mathscr{L}$-class of life distributions. In particular, we strengthen a surprising result of Bhattacharjee and Sengupta [5] and show that there are life distributions $F \in \mathscr{L}$ with the following properties:

(i) the support of F consists of countably infinite points,

(ii) the coefficient of variation (CV) of F is equal to one, and

(iii) F is not in the HNBUE class (the harmonic new better than used in expectation class).

An *absolutely continuous* distribution $F \in \mathscr{L}$, which shares the properties (ii) and (iii), is also given. Finally, in Section 4 we apply geometric compounds to characterize the semi-Mittag–Leffler distribution and extend a known result about the exponential distribution.

2. Relationship between the compound geometric distribution and the underlying distribution. Consider the rescaled geometric compound $S_p \equiv pS_{N_p}$, $p \in (0, 1)$. Let X be a non-negative random variable distributed as X_1, that is, $F(x) = P_r(X \leqslant x)$ for $x \geqslant 0$. Then S_p has the same mean as X, namely, $E(S_p) = EX$. As for higher-order moments of S_p, we have the following result, where $\mathrm{Var}(X)$ and $\mathrm{CV}(X) = (\mathrm{Var}(X))^{1/2}/(EX)$ denote the variance and the coefficient of variation of X, respectively.

LEMMA 1. *Let $p \in (0, 1)$. Then, under the geometric compounding model,*

(i) $p^{k-1} E(X^k) \leqslant E(S_p^k) \leqslant p^k E(N_p^k) E(X^k)$ *for each integer* $k > 0$.

Assume, in addition, $E(X^2) < \infty$. *Then*

(ii) $\mathrm{Var}(S_p) = p\mathrm{Var}(X) + (1-p)(EX)^2$;

(iii) $E(S_p^2) - 2(E(S_p))^2 = p[E(X^2) - 2(EX)^2]$;

(iv) $\mathrm{Var}(S_p) \geqslant \mathrm{Var}(X)$ *if and only if* $\mathrm{CV}(X) \leqslant 1$, *provided* $EX > 0$;

(v) $\mathrm{CV}(S_p) = 1$ *if and only if* $\mathrm{CV}(X) = 1$.

Proof. For convenience, let us put $q = 1-p$, $p_n = pq^{n-1}$ and $N = N_p$.

(i) For integer $k > 0$, we have

$$(1) \qquad E[(\sum_{n=1}^{N} X_n)^k] \geqslant E(\sum_{n=1}^{N} X_n^k) = E(N)E(X^k).$$

On the other hand, by the moment inequality: $E(X^r) \leqslant (E(X^k))^{r/k}$ for $0 < r < k$, we have $E[(\sum_{j=1}^{n} X_j)^k] \leqslant n^k E(X^k)$, and hence

$$(2) \quad E[(\sum_{n=1}^{N} X_n)^k] = \sum_{n=1}^{\infty} p_n E[(\sum_{j=1}^{n} X_j)^k] \leqslant \sum_{n=1}^{\infty} p_n n^k E(X^k) = E(N^k)E(X^k).$$

Part (i) follows from the inequalities (1), (2) and the fact that $EN = 1/p$.

(ii) Note that $\mathrm{Var}(N) = q/p^2$. Then part (ii) is an immediate consequence of the following identity:

$$\mathrm{Var}(\sum_{n=1}^{N} X_n) = (EN)\,\mathrm{Var}(X) + (\mathrm{Var}(N))(EX)^2,$$

which can be verified by a straightforward calculation.

(iii) Part (iii) is equivalent to part (ii) because $E(S_p) = EX$.

The remaining parts (iv) and (v) follow immediately from part (iii). Thus the proof is complete.

Remark 1. (a) Lemma 1 (i) implies that $E(S_p^k) < \infty$ if and only if $E(X^k) < \infty$.

(b) In view of the proof above, the inequalities (1) and (2) hold true not only for a geometric random variable but also for any positive integer-valued random variable N.

(c) Lemma 1 (iii) implies that $\mathrm{CV}(S_p) - 1 = p[\mathrm{CV}(X) - 1]$ if $EX > 0$, and hence $\mathrm{CV}(S_p) \leqslant 1$ or $\mathrm{CV}(S_p) \geqslant 1$ according to whether $\mathrm{CV}(X) \leqslant 1$ or $\mathrm{CV}(X) \geqslant 1$.

To estimate the tail probability of the compound geometric distribution F_S, Rolski et al. [26], Theorem 6.2.4, suggested an upper bound for $1 - F_S(x)$ in terms of the Laplace transform of the underlying distribution. However, this bound, as remarked by Rolski et al., is hard to handle, and is too complicated to be restated here. In the next theorem we shall give another upper bound which involves only the mean of the underlying distribution F. On the other hand, let $N_p^* = N_p - 1$; then the random summation $S_{N_p^*}$ is also called a geometric compound in the literature (where $S_{N_p^*} = 0$ if $N_p^* = 0$). For comparison, we shall give the corresponding result of $S_{N_p^*}$ as well. For each $\mu > 0$, define the distribution H_μ by

$$H_\mu(x) = 1 - e\left\{1 - \frac{x}{\mu}(1 - \exp(-\mu/x))\right\}, \qquad x \geqslant \mu.$$

Then we have the following result (for extensions see Remark 3 (b) below):

THEOREM 1. *Let* $p \in (0, 1)$, *and let* F *be a distribution with support* $S_F \subset [0, \infty)$ *and mean* $\mu \in (0, \infty)$. *Then*

(i) *the compound geometric distribution* F_S *of* S_{N_p} *satisfies*

$$1-F_S(x) \leqslant 1-H_{\mu/p}(x) \quad \text{for } x \geqslant \mu/p; \tag{3}$$

(ii) *the distribution* F_{S^*} *of* $S_{N_p^*}$ *satisfies*

$$1-F_{S^*}(x) \leqslant 1-H_{\mu^*}(x) \quad \text{for } x \geqslant \mu^* \equiv \mu(1-p)/p. \tag{4}$$

Proof. Recall that $E(S_{N_p}) = \mu/p$ and $E(S_{N_p^*}) = \mu(1-p)/p$. Then the inequalities (3) and (4) follow immediately from Lemma 2 below.

LEMMA 2. *If* F *is a distribution with support* $S_F \subset [0, \infty)$ *and mean* $\mu \in (0, \infty)$, *then*

$$1-F(x) \leqslant e\left\{1-\frac{x}{\mu}\left(1-\exp(-\mu/x)\right)\right\} = 1-H_\mu(x) \quad \text{for } x \geqslant \mu. \tag{5}$$

Proof. Denote by L the Laplace(–Stieltjes) transform of F, and recall the truncation inequality:

$$1-F(1/x) \leqslant \frac{e}{x}\int_0^x \left(1-L(s)\right) ds \quad \text{for } x > 0 \tag{6}$$

(see Rossberg et al. [27], p. 62). Here is a detailed proof for this inequality (for its extension see Remark 3 (b) below). Using Fubini's theorem, we write, for $x > 0$,

$$\frac{1}{x}\int_0^x \left(1-L(s)\right) ds = \frac{1}{x}\int_0^x\int_0^\infty (1-e^{-sy})\, dF(y)\, ds = \frac{1}{x}\int_0^\infty\int_0^x (1-e^{-sy})\, ds\, dF(y)$$

$$= \int_{0+}^\infty \left(1+\frac{1}{xy}e^{-xy}-\frac{1}{xy}\right) dF(y) \geqslant \int_{xy>1} \left(1+\frac{1}{xy}e^{-xy}-\frac{1}{xy}\right) dF(y) \geqslant \frac{1}{e}\left(1-F(1/x)\right),$$

in which the integrand of the last integral is a positive and increasing function in xy. Then applying the truncation inequality (6) and the fact that $L(s) \geqslant \exp(-\mu s)$ for $s \geqslant 0$ (see Lemma 3 (i) below), we obtain the required result (5). The proof is complete.

Remark 2. In Lemma 2, $1-F(x) \leqslant \mu/x \leqslant 1-H_\mu(x)$ for sufficiently large x, and, as shown below, the distribution H_μ is interesting in itself. Let X be a non-negative random variable with finite mean $\mu > 0$, and let Z_μ be another random variable obeying the distribution H_μ. Then Lemma 2 asserts that X is smaller than Z_μ in the stochastic order, and hence $E(X^\beta) \leqslant E[(Z_\mu)^\beta]$ for all $\beta > 0$. On the other hand, H_μ is heavy-tailed and the tail $1-H_\mu(x)$ is regularly varying at infinity of index $\alpha = -1$. Namely, for each $y > 0$,

$$\frac{1-H_\mu(xy)}{1-H_\mu(x)} \to y^{-1} \quad \text{as } x \to \infty.$$

In particular, $E[(Z_\mu)^\beta] < \infty$ or $E[(Z_\mu)^\beta] = \infty$ according to whether $\beta < 1$ or $\beta \geqslant 1$. Set $K = 1/Z_1$; then its density function is of the form $k(x) = ex^{-2}(1-e^{-x}-xe^{-x})$, $x \in (0, 1)$. Therefore

$$e^{-1} = \int_0^1 x^{-2}(1-e^{-x}-xe^{-x})\,dx,$$

an interesting identity for the transcendental number e.

Let us compare Theorem 1 with some known results. For the distribution F_{S^*} of $S_{N_p^*}$, Brown [6] obtained the following bound for $1-F_{S^*}(x)$:

$$1-F_{S^*}(x) \leqslant \exp\left(-\frac{px}{\mu(1-p)}\right)+\frac{\gamma p}{1-p}, \quad x \geqslant 0, \qquad \text{if } E(X^2) < \infty,$$

where $\gamma = E(X^2)/(2\mu^2)$ and $\mu = EX$. As $x \to \infty$, the bound above tends to $\gamma p/(1-p) > 0$ while ours in (4) tends to zero. Therefore our bound is better than Brown's for estimating the tail probability of F_{S^*}. As for the tail probability of the compound geometric distribution F_S, Brown [6] considered some *special* classes of underlying distributions (e.g., the NBUE class) and gave upper bounds sharper than ours in (3). It is worth noting that our bounds in Theorem 1 above are valid as long as the underlying distribution F has finite mean. On the other hand, if F is subexponential (a heavy-tailed distribution which may have an infinite mean), then

$$1-F_S(x) \sim (1-F(x))/p \quad \text{and} \quad 1-F_{S^*}(x) \sim (1/p-1)(1-F(x)) \quad \text{as } x \to \infty$$

(see, e.g., Embrechts et al. [7], p. 580).

To extend the result of Lemma 2 (and hence of Theorem 1), we need the following bounds for Laplace transforms. To the knowledge of the authors, Lemmas 2 and 3 (ii), although simple, appear for the first time. More general results are given in Remark 3 below.

LEMMA 3. *Let F be the distribution of a non-negative random variable X with finite mean $\mu > 0$. Further, assume that L is the Laplace(–Stieltjes) transform of F. Then*

(i) $L(s) \geqslant \exp(-\mu s)$ *for* $s \geqslant 0$.

If, in addition, $E(X^2) < \infty$, *then*

(ii) $L(s) \leqslant 1-s\mu\exp(-sE(X^2)/(2\mu))$ *for* $s \geqslant 0$.

Proof. Part (i) is an application of Jensen's inequality because, for each $s > 0$, the function $g_s(x) = \exp(-sx)$ is convex on $[0, \infty)$. To prove part (ii), we recall the equilibrium distribution $F_{(1)}$ of F:

$$F_{(1)}(x) = \mu^{-1}\int_0^x (1-F(t))\,dt \qquad \text{for } x \geqslant 0.$$

Suppose $X_{(1)}$ is a random variable with distribution $F_{(1)}$ and denote by $L_{(1)}$ the Laplace transform of $F_{(1)}$. Then we have $E(X_{(1)}) = E(X^2)/(2\mu)$ and

$L_{(1)}(s) = (1-L(s))/(s\mu)$ for $s > 0$. Applying Jensen's inequality to the equilibrium distribution $F_{(1)}$ yields $L_{(1)}(s) \geqslant \exp(-sE(X_{(1)}))$ for $s \geqslant 0$ or, equivalently, $L(s) \leqslant 1 - s\mu\exp(-sE(X^2)/(2\mu))$ for $s \geqslant 0$. This completes the proof.

Remark 3. (a) We now give two more functional bounds for the Laplace transform $L(s)$. For $s \geqslant 0$ and for integer $m \geqslant 0$, define the function $g_{s,m}$ by

$$(7) \qquad g_{s,m}(x) = \sum_{k=0}^{m} \frac{(-sx)^k}{k!} - e^{-sx} \quad \text{for } x \geqslant 0$$

(where $0^0 = 1$). It is known that, for even integer $m = 2n \geqslant 2$, $g_{s,2n} \geqslant 0$ is a convex function on $[0, \infty)$ (see, e.g., Hardy et al. [10], p. 104). Therefore, if $E(X^{2n}) < \infty$, we have, by Jensen's inequality, $E(g_{s,2n}(X)) \geqslant g_{s,2n}(EX) \equiv g_{s,2n}(\mu)$ for $s \geqslant 0$. Equivalently, for integer $n \geqslant 1$,

$$L(s) \leqslant e^{-\mu s} + \sum_{k=2}^{2n} \frac{(-s)^k}{k!}(E(X^k) - \mu^k), \; s \geqslant 0, \quad \text{if } E(X^{2n}) < \infty.$$

Clearly, both upper bounds for $L(s)$ are equal to $1 - \mu s + \frac{1}{2}s^2 E(X^2) + o(s^2)$ as $s \to 0^+$. The first bound (Lemma 3 (ii)) involves only the first two moments of F, while the second bound involves higher-order moments. On the other hand, for odd integer $m = 2n+1 \geqslant 1$, $g_{s,2n+1} \leqslant 0$ is a concave function on $[0, \infty)$. This implies that $E(g_{s,2n+1}(X)) \leqslant g_{s,2n+1}(\mu)$ if $E(X^{2n+1}) < \infty$. Equivalently, for integer $n \geqslant 0$,

$$(8) \qquad L(s) \geqslant e^{-\mu s} + \sum_{k=2}^{2n+1} \frac{(-s)^k}{k!}(E(X^k) - \mu^k), \; s \geqslant 0, \quad \text{if } E(X^{2n+1}) < \infty.$$

(b) We are ready to extend the inequality (5). Let X be a non-negative random variable with distribution F and finite mean $\mu > 0$. If, in addition, $E(X^{2n+1}) < \infty$ for some integer $n \geqslant 0$, then by (6) and (8) we have

$$(9) \quad 1 - F(x) \leqslant e\left\{1 - \frac{x}{\mu}(1 - e^{-\mu/x}) + \sum_{k=2}^{2n+1} \frac{(-1)^{k+1}(E(X^k) - \mu^k)}{(k+1)!\,x^k}\right\} \quad \text{for } x > 0.$$

When $n \geqslant 1$, the bound above is sharper than that of (5) for sufficiently large x. On the other hand, if $E(X^{2n}) < \infty$ instead, proceeding as in the proof of Lemma 2 we extend (6) to the following:

$$(10) \quad 1 - F(1/x) \leqslant \frac{1}{x}\left(e^{-1} - \sum_{k=0}^{2n+1} \frac{(-1)^k}{k!}\right)^{-1} \int_0^x E(g_{s,2n}(X))\,ds \quad \text{for } x > 0,$$

where $g_{s,2n}$ is defined by (7) and $E(g_{s,2n}(X)) = \sum_{k=0}^{2n} \{(-s)^k E(X^k)/k!\} - L(s)$ (which reduces to $1 - L(s)$ when $n = 0$). Then applying Lemma 3 (i) to (10) yields for $x > 0$

$$(11)\quad 1-F(x) \leqslant \left(e^{-1} - \sum_{k=0}^{2n+1} \frac{(-1)^k}{k!}\right)^{-1} \left\{1 - \frac{x}{\mu}(1-e^{-\mu/x}) + \sum_{k=1}^{2n} \frac{(-1)^k E(X^k)}{(k+1)!\,x^k}\right\}.$$

Both (9) and (11) reduce to the inequality (5) when $n = 0$.

Denote the Laplace transform of S_p by

$$(12)\quad L_{p,F}(s) \equiv E(\exp(-sS_p)) = G_{N_p}(L_F(ps)) = \frac{pL_F(ps)}{1-(1-p)L_F(ps)} \quad \text{for } s \geqslant 0,$$

where $G_{N_p}(z) = pz/(1-(1-p)z)$, $z \in [0, 1)$, is the probability generating function of N_p. We say that $L_{p,F}$ is a *geometric compounding transform* of F (or of X), comparing to the Laplace transform L_F of F. The mapping which maps L_F to $L_{p,F}$ will be denoted by $\mathscr{G}_p(L_F) = L_{p,F}$, and is called a *geometric compounding operator* (GC operator). We shall further investigate the properties of the compound geometric distribution through the GC operator $\mathscr{G}_p$. Let $\mathscr{C}$ denote the class of all Laplace transforms of non-negative random variables. Namely, the class $\mathscr{C}$ consists of all L_F for which $L_F(s) = E(\exp(-sX))$ for $s \geqslant 0$, where X is a non-negative random variable with distribution F. The following result summarizes the properties of GC operator $\mathscr{G}_p$, which can be verified by using the relation (12).

PROPOSITION 1. *Let $p, p_1, p_2 \in (0, 1)$ be three constants. Then*

(i) *the geometric compounding operator $\mathscr{G}_p: \mathscr{C} \to \mathscr{C}$ is one-to-one;*

(ii) *the composite operator $\mathscr{G}_{p_2} \circ \mathscr{G}_{p_1} = \mathscr{G}_{p_2 p_1}$;*

(iii) *$\mathscr{G}_p(L_F) \leqslant \mathscr{G}_p(L_G)$ if and only if $L_F \leqslant L_G$;*

(iv) *$\lim_{p\to 0}(\mathscr{G}_p(L))(s) = 1/(1+\mu s)$ for $s \geqslant 0$ if and only if $L(s) = 1-\mu s + o(s)$ as $s \to 0^+$, where $\mu \geqslant 0$ is a constant.*

Remark 4. (a) Proposition 1 (i) means that the geometric compounding transform $L_{p,F}$ uniquely determines the distribution F, a property also possessed by the Laplace transform L_F.

(b) Proposition 1 (ii) means that successive application of the GC operators $\mathscr{G}_{p_1}$ and $\mathscr{G}_{p_2}$ is equivalent to the operator $\mathscr{G}_{p_2 p_1}$. (In particular, $\mathscr{G}_p \circ \mathscr{G}_p = \mathscr{G}_{p^2}$.) This is exactly the well-known result: two successive rarefactions of a renewal process can be replaced by a single rarefaction procedure.

(c) Proposition 1 (iii) shows the preservation of the Laplace transform ordering under GC operation, a property to be used in the next section.

(d) The sufficiency part of Proposition 1 (iv) is exactly the Rényi limit theorem. The latter asserts that if the underlying distribution F (fixed, independent of p) has finite mean μ, then, as $p \to 0^+$, S_p converges weakly to an exponential random variable with mean μ (see, e.g., Kalashnikov [12], p. 3).

Recall that $S_p(= pS_{N_p})$ and X are equally distributed if and only if X has an exponential distribution (including the degenerate distribution at zero) (see, e.g., Azlarov et al. [2], Arnold [1], and Azlarov and Volodin [3], p. 79). We rewrite this result in terms of the fixed point of GC operator, and at the same time offer a somewhat simpler proof (for related results see Theorem 2 below and Lin and Hu [22]).

PROPOSITION 2. *Let* $p \in (0, 1)$ *and let* $\mathscr{G}_p: \mathscr{C} \to \mathscr{C}$ *be the geometric compounding operator defined by* $\mathscr{G}_p(L_F) = L_{p,F}$. *Then* $\mathscr{G}_p(L_F) = L_F$ *if and only if* $L_F(s) = 1/(1+\mu s)$ *for* $s \geqslant 0$ *and for some fixed constant* $\mu \geqslant 0$.

Proof. The sufficiency part is trivial. To prove the necessity part, suppose that $\mathscr{G}_p(L_F) = L_F$. Then we have

$$\frac{pL_F(ps)}{1-(1-p)L_F(ps)} = L_F(s) \quad \text{for } s \geqslant 0. \tag{13}$$

Define the function $g(s) = 1/L_F(s) - 1$ for $s \geqslant 0$, and rewrite the equality (13) as $g(s) = g(ps)/p$ for $s \geqslant 0$. Then it implies that for $s > 0$ and $n \geqslant 1$

$$\frac{g(s)}{s} = \frac{g(ps)}{ps} = \frac{g(ps)-g(0)}{ps} = \ldots = \frac{g(p^n s)-g(0)}{p^n s}. \tag{14}$$

Note that, by definition of the function g, the limit $g'(0^+) \equiv \lim_{s\to 0^+} g'(s)$ exists and has a non-negative value $-L_F'(0^+)$ (being finite or infinite). On the other hand, applying the Mean Value Theorem and letting $n \to \infty$ in the equality (14), we have $g(s)/s = g'(0^+)$ for $s > 0$. This implies that the limit $g'(0^+)$, denoted by μ, should be finite, and hence $L_F(s) = 1/(1+\mu s)$ for $s \geqslant 0$. This completes the proof.

Although the class of self-decomposable distributions is not closed under geometric compounding (Szekli [28]), in the next result we are able to show the preservation of infinite divisibility property under geometric compounding. Note also that there is no restriction on the support of the underlying distribution F.

PROPOSITION 3. *Let* $p \in (0, 1)$ *and let* F *be an infinitely divisible distribution. Then the compound geometric distribution* F_S *of* S_{N_p} *is infinitely divisible.*

Proof. Let f and g denote the respective characteristic functions of F and F_S. Then

$$g(t) = G_{N_p}(f(t)) = \frac{pf(t)}{1-(1-p)f(t)} \equiv f(t)h(t) \quad \text{for all real } t,$$

where $h(t) = p/\{1-(1-p)f(t)\}$ is an infinitely divisible characteristic function (see Lukacs [23], p. 320). Therefore the product function $g = fh$ is also infinitely divisible (see, e.g., Lukacs [23], p. 109). The proof is complete.

3. Applications to the $\mathscr{L}$-class of life distributions. In this section we shall apply the above results to the $\mathscr{L}$-class of life distributions. Let F be the distribution of a non-negative random variable X with finite mean μ_F. Then we say that the life distribution F *belongs to the $\mathscr{L}$-class* if

$$\int_0^\infty e^{-sx}\big(1-F(x)\big)\,dx \geqslant \frac{\mu_F}{1+s\mu_F} \quad \text{for } s \geqslant 0 \tag{15}$$

and that F *belongs to the HNBUE class* if

$$\int_t^\infty \big(1-F(x)\big)\,dx \leqslant \mu_F \exp(-t/\mu_F) \quad \text{for } t \geqslant 0. \tag{16}$$

For convenience, the degenerate distribution at the point zero is also said to be *HNBUE*. Recall that the HNBUE class is a subfamily of the $\mathscr{L}$-class and that the relations (15) and (16) are equivalent to the following relations (17) and (18), respectively:

$$L_F(s) \leqslant \frac{1}{1+\mu s} = L_{G_\mu}(s) \quad \text{for } s \geqslant 0, \tag{17}$$

$$\int_t^\infty \big(1-F(x)\big)\,dx \leqslant \int_t^\infty \big(1-G_\mu(x)\big)\,dx \quad \text{for } t \geqslant 0, \tag{18}$$

in which G_μ denotes the exponential distribution with mean $\mu = \mu_F$, and G_0 denotes the degenerate distribution at zero. For other properties of the $\mathscr{L}$-class and of the HNBUE class, see, e.g., Klefsjö [13], [14] and Lin [20].

To understand better the $\mathscr{L}$-class of life distributions, Bhattacharjee and Sengupta [5] (henceforth referred to as BS [5]) gave a surprising two-point distribution, denoted by F_{bs}, which has the following properties:

(i) the coefficient of variation of F_{bs} is equal to one, and

(ii) F_{bs} is in the $\mathscr{L}$-class but not in the HNBUE class.

The former property (i) implies that the equality CV $= 1$ is not sufficient to characterize the exponential distribution within the $\mathscr{L}$-class, and the latter property (ii) shows that the $\mathscr{L}$-class is strictly larger than the HNBUE class. In particular, consider a random variable X_{bs} defined by $P_r(X_{bs} = 0.3) = 25/29$ and $P_r(X_{bs} = 1.75) = 4/29$ (namely, take $a = 0.3$, $b = a29/6 = 1.45$ and $\alpha = 4/29$ in Example 3.1 of BS [5]). Then BS [5] proved that the two-point distribution F_{bs} of X_{bs} belongs to the $\mathscr{L}$-class and that the CV of F_{bs} is equal to 1. The latter implies that F_{bs} is not HNBUE because the equality CV $= 1$ characterizes the exponential distribution within the HNBUE class (see Basu and Bhattacharjee [4]). Note that the above-mentioned properties (i) and (ii) of F_{bs} are scale invariant in the sense that, for any constant $c > 0$, the distribution of cX_{bs} also shares the same properties with F_{bs}.

It is natural to ask the following question: Is there any *infinite-point* discrete distribution that shares the same properties (i) and (ii) with the two-point distribution F_{bs}? As stated below, with the help of the geometric compounding model the answer to this question is positive. Therefore, the life distributions $F \in \mathscr{L}$ sharing the same properties with F_{bs} are perhaps more prevalent than it is commonly expected.

COROLLARY 1. *Let $p \in (0, 1)$ and let X be the random variable X_{bs} defined as above. Then* $\mathrm{CV}(S_p) = 1$ *and the distribution of S_p is in the $\mathscr{L}$-class but not in the HNBUE class.*

Proof. The result follows immediately from Lemma 1 (v), Propositions 1 (iii) and 2 and the fact that the equality $\mathrm{CV} = 1$ characterizes the exponential distribution within the HNBUE class.

In order to construct an *absolutely continuous* distribution sharing the same properties with F_{bs}, we need the following

LEMMA 4. *Let X_1 and X_2 be two independent random variables with $EX_1 = EX_2 \neq 0$ and let $\mathrm{CV}(X_1) = \mathrm{CV}(X_2) = 1$. Assume further that B, independent of $\{X_1, X_2\}$, is a Bernoulli random variable with parameter $p \in (0, 1)$. Then the coefficient of variation of $X_3 \equiv BX_1 + (1-B)X_2$ is equal to one.*

Proof. A straightforward calculation shows that $EX_3 = EX_1$ and $E(X_3^2) = E(X_1^2)$. Therefore $\mathrm{CV}(X_3) = 1$ because $\mathrm{CV}(X_1) = 1$. The proof is complete.

COROLLARY 2. *Let F_{bs} be the distribution defined as above, and let $G_{1/2}$ be the exponential distribution with mean 1/2. Assume further that $p \in (0, 1)$ and $H = pF_{bs} + (1-p)G_{1/2}$. Then*

(i) *the coefficient of variation of the absolutely continuous distribution H is equal to one;*

(ii) *H is in the $\mathscr{L}$-class, but not in the HNBUE class.*

Proof. Part (i) follows from Lemma 4. To prove part (ii), note first that, for $F_1, F_2 \in \mathscr{L}$ with the same mean, the convex combination $pF_1 + (1-p)F_2$ of F_1 and F_2 also belongs to the $\mathscr{L}$-class. Part (ii) then follows from the fact that the equality $\mathrm{CV} = 1$ characterizes the exponential distribution within the HNBUE class.

4. Applications to the semi-Mittag–Leffler distributions. In this section we shall apply geometric compounds to characterize the semi-Mittag–Leffler distribution defined below. Let F be the distribution of a non-negative random variable X. Then we say that F is a *semi-Mittag–Leffler distribution with exponent* $\alpha \in (0, 1]$ if its Laplace transform is of the form

$$L_F(s) = \left(1 + \eta(s)\right)^{-1}, \quad s \geqslant 0, \tag{19}$$

where η satisfies the functional equation

(20) $$\eta(s) = b^{-\alpha}\eta(bs), \quad s \geqslant 0,$$

for some constant $b \in (0, 1)$ (see Pillai [24] or Jayakumar and Pillai [11]). We first have the following observations on the Laplace transform L_F defined by (19) and (20):

(A) For the special function $\eta(s) = s^{\alpha}$, L_F reduces to $L_{F_\alpha}(s) \equiv (1+s^{\alpha})^{-1}$, $s \geqslant 0$, which is the Laplace transform of a Mittag-Leffler distribution F_α. Gnedenko [9] might be the first one who investigated the Laplace transform L_{F_α}. For properties of F_α, see Pillai [25] and Lin [21].

(B) For the special case $\alpha = 1$, L_F reduces to

(21) $$L_F(s) = (1+\mu s)^{-1}, \ s \geqslant 0, \quad \text{for some fixed constant } \mu \geqslant 0,$$

which is exactly the Laplace transform of an exponential distribution. To see this result, it suffices to note that from (20) we obtain

$$0 \leqslant \frac{\eta(s)}{s} = \frac{\eta(bs)}{bs} = \ldots = \frac{\eta(b^n s)}{b^n s} < \infty \quad \text{for each } s > 0 \text{ and for } n \geqslant 1,$$

the last term converging to $\eta'(0^+) = -L_F'(0^+) \equiv \mu$ (the mean of F) as $n \to \infty$ (see, e.g., Lin [17], Lemma 2). This implies that $\eta(s) = \mu s$ for $s \geqslant 0$, and hence the relation (21) holds.

(C) Let $h(x) = \eta(e^x)/e^{\alpha x}$ for $x \in (-\infty, \infty)$. Then we can rewrite L_F in the form

$$L_F(s) = \{1+s^{\alpha}h(\log s)\}^{-1} \quad \text{for } s > 0.$$

Note that the function h is infinitely differentiable and with *period* $-\log b$ (this clarifies the statement of Jayakumar and Pillai [11] in Lemma 2.1).

Using a stationary first-order autoregressive process Jayakumar and Pillai [11] gave a characterization theorem for the semi-Mittag-Leffler distribution. We shall now give another characterization result via the stability of geometric compound (for related results about the Linnik distributions see Lin [18], [19]). Use $\stackrel{d}{=}$ for equality in distribution.

THEOREM 2. *Let $p \in (0, 1)$. Then the stability relation*

(22) $$S_{N_p} \stackrel{d}{=} p^{-1/\alpha} X_1$$

holds for some constant $\alpha \in (0, 1]$ if and only if X_1 has a semi-Mittag-Leffler distribution with exponent $\alpha \in (0, 1]$ (and $b = p^{1/\alpha}$).

Proof. Let L_F be the Laplace transform of X_1. Then the relation (22) is equivalent to

(23) $$L_F(p^{-1/\alpha}s) = \frac{pL_F(s)}{1-(1-p)L_F(s)} \quad \text{for } s \geqslant 0.$$

Let us put $\eta(s) = (L_F(s))^{-1} - 1$ for $s \geqslant 0$, and rewrite the relation (23) as $p\eta(p^{-1/\alpha}s) = \eta(s)$ or, equivalently, $\eta(s) = p^{-1}\eta(p^{1/\alpha}s)$ for $s \geqslant 0$. This completes the proof.

Remark 5. If $\alpha = 1$, then Theorem 2 reduces to Proposition 2. We now further extend Theorem 2 through the Pascal (or negative binomial) compounding model defined below. Let κ be a positive integer, and denote by $N_{\kappa,p}$ a negative binomial random variable having mass function

$$P_r\{N_{\kappa,p} = n\} = \binom{n-1}{\kappa-1} p^{\kappa}(1-p)^{n-\kappa}, \quad n = \kappa, \kappa+1, \ldots,$$

where $p \in (0, 1)$. For the case $\kappa = 1$, $N_{\kappa,p}$ reduces to the geometric distribution. Using $N_{\kappa,p}$ instead of N_p we extend Theorem 2 to the following:

For any fixed constants $p \in (0, 1)$, $\alpha \in (0, 1]$ *and integer* $\kappa > 0$, *the stability condition*

$$S_{N_{\kappa,p}} \stackrel{d}{=} p^{-1/\alpha}(X_1 + X_2 + \cdots + X_\kappa)$$

holds if and only if X_1 *has a semi-Mittag–Leffler distribution with exponent* α *(and* $b = p^{1/\alpha}$*).*

The proof is similar to that of Theorem 2 and is omitted.

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