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EDGEWORTH EXPANSIONS FOR L-STATISTICS

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Abstract. We study the approximation by a short Edgeworth expansion of the distribution function of normalized linear combinations

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n} c_{jn} X_{j:n}$$

of order statistics of n independent random variables with common distribution function F. Under the assumptions

$$|c_{jn}| \leqslant Cn^{-p_1}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-p_2},$$

$$|c_{jn}-c_{j-1,n}| \leqslant Cn^{-q_1}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-q_2},$$

$$|c_{j+1,n}-2c_{jn}+c_{j-1,n}| \leqslant Cn^{-r_1}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-r_2},$$

$$(F^{-1})'(s) \leqslant C[s(1-s)]^{-\kappa}$$

for some $p_1, q_1, r_1 \in R$, $p_2, q_2, r_2, C \geqslant 0$, $\kappa \in [0, 5/4)$, with an appropriate balance in these parameters, and under additional moment conditions, the rate of uniform convergence is shown to be of order n^{-1}. Moreover, a special case is considered where the c_{jn} are generated by a sequence of weight functions of a special structure.

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1. INTRODUCTION AND RESULTS

Let $X, X_1, \ldots, X_n$ be i.i.d. random variables with a common distribution function F. We put $\beta_s := E|X|^s$ for all $s \geqslant 0$ and suppose throughout the paper that $\beta_2 < +\infty$. We shall consider the statistic

$$T := \frac{1}{\sqrt{n}}\sum_{j=1}^{n} c_{jn} X_{j:n},$$

a linear combination of order statistics. Here $X_{j:n}$ denotes the j-th order statistic of $X_1, \ldots, X_n$ and $c_{1n}, \ldots, c_{nn}$ are given constants. We will assume that in all cases $E|T| < +\infty$.

For any symmetric statistic $T = T(X_1, \ldots, X_n)$ with $E|T| < +\infty$, let

$$T_1 := E(T \mid X_1) - ET, \quad T_2 := E(T \mid X_2) - ET,$$
$$T_{12} := E(T \mid X_1, X_2) - E(T \mid X_1) - E(T \mid X_2) + ET$$

and for $1 \leqslant i, j \leqslant n$ write

$$E_i T := E(T \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \quad \text{and} \quad E_{ij} T := E_i E_j T.$$

In addition, write

$$D_i T := T - E_i T, \quad i = 1, \ldots, n,$$

and

$$\beta_s := E|n^{1/2} T_1|^s, \quad \gamma_s := E|n^{3/2} T_{12}|^s, \quad \Delta_2^s := E|n^{5/2} D_1 D_2 D_3 T|^s, \quad s \geqslant 0.$$

Finally, for $\hat{\sigma} := \sqrt{\operatorname{var} T} > 0$, let

$$q := 1 - \sup_{|t| \in [\hat{\sigma}^2/2\beta_3, \sqrt{n}/\hat{\sigma}]} |E \exp\{itn^{1/2} T_1\}|$$

and

$$\eta := E(n^{1/2} T_1)^3 + 3E\, n^{5/2} T_1 T_2 T_{12}.$$

We shall estimate

(1) $$\delta := \sup_{x \in \mathbf{R}} \left| P\left(\frac{T - E(T)}{\hat{\sigma}} \leqslant x\right) - \left(\Phi(x) - \frac{\eta}{6\hat{\sigma}^3 \sqrt{n}} \Phi'''(x)\right) \right|.$$

From now on, by c and C we shall denote absolute generic constants: if such a c or C depends on, say, α, we will write $c(\alpha)$ or $C(\alpha)$. By Φ we shall mean the standard normal distribution function. Moreover, $I\{A\}$ will always denote the indicator function of event A.

Recently, a short Edgeworth expansion for symmetric statistics has been obtained in Bentkus et al. [2]:

(2) $$\delta \leqslant \frac{C}{q^2 n} \left(\frac{\beta_4}{\hat{\sigma}^4} + \frac{\gamma_3}{\hat{\sigma}^3} + \frac{\Delta_3^2}{\hat{\sigma}^2} \right).$$

In Lemmas 1, 2 and 3 of Section 2 we will derive explicit expressions for β_4, γ_3 and Δ_3^2 in the special case of linear combinations of order statistics. These lead to precise upper bounds for these quantities in terms of moments of the underlying distribution function F, and hence to a short Edgeworth expansion of order n^{-1} for T, where the upper bound is given again in terms of the moments of F. The proofs are given in Sections 3, 4 and 5. Note that the results of Helmers [3] are not applicable because here the weights are assumed to be of the form

$$c_{jn} = J\left(\frac{j}{n+1}\right) \quad \text{or} \quad c_{jn} = n \int_{(j-1)/n}^{j/n} J(t)\,dt$$

with a single weight function $J\colon (0,1) \to \mathbf{R}$. In Section 2.7 of Bentkus et al. [2] this same structure is used, whereas it is also assumed that $\sup_x |J'(x)|$ is bounded.

We assume the quantile function F^{-1} of the population to be differentiable and for $\kappa \geqslant 0$ we set

$$K = K(F, \kappa) := \sup_{s \in (0,1)} [s(1-s)]^{\kappa} (F^{-1})'(s).$$

For

$$d_1 := \max_{1 \leqslant j \leqslant n} n^{p_1} \left[\frac{j}{n}\left(1 - \frac{j-1}{n}\right)\right]^{p_2} |c_{jn}|,$$

(3) $$d_2 := \max_{2 \leqslant j \leqslant n} n^{q_1} \left[\frac{j}{n}\left(1 - \frac{j-1}{n}\right)\right]^{q_2} |c_{jn} - c_{j-1,n}|,$$

(4) $$d_3 := \max_{2 \leqslant j \leqslant n-1} n^{r_1} \left[\frac{j}{n}\left(1 - \frac{j-1}{n}\right)\right]^{r_2} |c_{j+1,n} - 2c_{jn} + c_{j-1,n}|,$$

we have the following theorem:

THEOREM 1. *Let $\kappa \in [0, 5/4)$, and $p_1, p_2, q_1, q_2, r_1, r_2$ be real numbers satisfying $p_2, q_2, r_2 \geqslant 0$. Then there exist constants C and $c = c(p_2, q_2, r_2, \kappa)$ (independent of n) such that for any n we have*

$$\delta \leqslant \frac{C}{q^2 n}\left(\frac{d_1^4}{\hat{\sigma}^4}(A_n^4 \beta_4 + c\tilde{A}_n^4 K^4) + \frac{d_2^3}{\hat{\sigma}^3}(B_n^3 \beta_3 + c\tilde{B}_n^3 K^3) + \frac{d_3^2}{\hat{\sigma}^2}(C_n^2 \beta_2 + c\tilde{C}_n^2 K^2)\right),$$

where

$$A_n = n^{p_2 - p_1} I\{p_1 \geqslant p_2\},$$

$$\tilde{A}_n = n^{-p_1} I\{p_1 < p_2\}$$
$$\times (I\{\kappa + p_2 < 5/4\} + I\{\kappa + p_2 = 5/4\}\log n + I\{\kappa + p_2 > 5/4\} n^{\kappa + p_2 - 5/4}),$$

$$B_n = n^{q_2 + 1 - q_1} I\{q_1 \geqslant q_2 + 1\},$$

$$\tilde{B}_n = n^{1 - q_1} I\{q_1 < q_2 + 1\}$$
$$\times (I\{\kappa + q_2 < 5/3\} + I\{\kappa + q_2 = 5/3\}\log n + I\{\kappa + q_2 > 5/3\} n^{\kappa + q_2 - 5/3}),$$

$$C_n = n^{r_2 + 2 - r_1} I\{r_1 \geqslant r_2 + 2\},$$

$$\tilde{C}_n = n^{2 - r_1} I\{r_1 < r_2 + 2\}$$
$$\times (I\{\kappa + r_2 < 5/2\} + I\{\kappa + r_2 = 5/2\}\log n + I\{\kappa + r_2 > 5/2\} n^{\kappa + r_2 - 5/2}).$$

The proof of Theorem 1 is based on the fact that

$$(5)\qquad \begin{cases} \hat{\beta}_4 \leqslant C d_1^4 (A_n^4 \beta_4 + c\tilde{A}_n^4 K^4), \\ \gamma_3 \leqslant C d_2^3 (B_n^3 \beta_3 + c\tilde{B}_n^3 K^3), \\ \Delta_3^2 \leqslant C d_3^2 (C_n^2 \beta_2 + c\tilde{C}_n^2 K^2), \end{cases}$$

where $c = c(p_2, q_2, r_2, \kappa)$, which follows from Lemmas 1, 2 and 3 in combination with Lemmas 4, 5 and 6 (Sections 6, 7 and 8). (From Lemmas 4–6 it also follows that we may take $C = 27$.) By (2), Theorem 1 then follows immediately. Note that $X_1, \ldots, X_n$, T, $\hat{\beta}_s$, γ_s, Δ_3^s, q, η, d_1, d_2 and d_3 all may depend on n.

The following corollary is a direct consequence of Theorem 1. It is the analogue of Corollary 4.2 of van Zwet [7].

COROLLARY 1. *In the special case where $p_1 = p_2 = q_2 = r_2 = 0$, $q_1 = 1$ and $r_1 = 2$ we have under the conditions of the theorem:*

$$\delta \leqslant \frac{C}{q^2 n}\left(\frac{d_1^4 \beta_4}{\hat{\sigma}^4} + \frac{d_2^3 \beta_3}{\hat{\sigma}^3} + \frac{d_3^2 \beta_2}{\hat{\sigma}^2}\right),$$

where C denotes a universal constant. If $\beta_4 < +\infty$, both $\hat{\sigma}^2$ and q are uniformly bounded from below and d_1, d_2 and d_3 are uniformly bounded from above, this provides an Edgeworth expansion of order n^{-1} for T.

Next we state the analogue of Theorem 3 from Pap and van Zuijlen [5]. Let $\psi\colon (0, 1) \to \boldsymbol{R}$ be a Lebesgue measurable real-valued function on (0, 1) and γ a real number. Taking $J\colon t \mapsto \psi(t)[t(1-t)]^{-\gamma}$, we consider the weights

$$(6)\qquad c_{jn} := n \int_{(j-1)/n}^{j/n} J(t)\,dt$$

and

$$(7)\qquad c_{jn} := J\left(\frac{j}{n+1}\right).$$

We start by quoting Theorem 2 of Pap and van Zuijlen [5], a Central Limit Theorem. Assume that the weights c_{jn} satisfy (6).

THEOREM 2. *Suppose that $0 \leqslant \gamma < \frac{1}{2}$ and that there exist numbers $\Lambda \geqslant 0$ and $\lambda > \frac{1}{2}$ such that $|\psi(t) - \psi(s)| \leqslant \Lambda |t-s|^{\lambda}$ for all $s, t \in (0, 1)$. If $\beta_m < +\infty$ for some $m > (\frac{1}{2} - \gamma)^{-1}$, then*

$$T - ET \xrightarrow{d} N(0, \hat{\sigma}^2(\psi, F)) \quad \text{and} \quad \hat{\sigma}^2(T) \to \hat{\sigma}^2(\psi, F),$$

where

$$\hat{\sigma}^2(\psi, F) = \int_0^1 \int_0^1 [s(1-s)t(1-t)]^{-\gamma} \psi(s)\psi(t)\big(\min(s, t) - st\big)\,dF^{-1}(s)\,dF^{-1}(t).$$

In the case of weights (7), we have the same results.

Assume we take our weights of the form (7). The announced Theorem 3 reads as follows:

THEOREM 3. *Suppose that* $\kappa \in [0, 5/4)$, $\gamma > 0$, $\kappa + \gamma < \frac{1}{2}$ *and that* ψ *is twice boundedly differentiable. Then there is a constant* $c = c(\kappa, \gamma)$ *such that*

$$\delta \leqslant \frac{c}{q^2 n}\left(\frac{K^4 \|\psi\|_\infty^4}{\hat{\sigma}^4} + \frac{K^3 (\|\psi'\|_\infty + \|\psi\|_\infty)^3}{\hat{\sigma}^3} + \frac{K^2 (\|\psi''\|_\infty + \|\psi'\|_\infty + \|\psi\|_\infty)^2}{\hat{\sigma}^2}\right).$$

A theorem similar to Theorem 3 can be proved in the case of weights of the form (6). The proof of Theorem 3 will be given in Section 9.

Remark. Suppose that, instead of (6), for $\delta_1 = 0$, $\gamma_1, \gamma_2 \geqslant 0$, $\delta_2 > 0$ we consider weights of the form

$$c_{jn} := n \int_{(j-1)/n}^{j/n} J_n(t)\,dt, \quad \text{where } J_n(t) := \sum_{i=1}^{2} \psi_i(t)\,[t(1-t)]^{-\gamma_i} n^{-\delta_i}.$$

Using the same techniques as in the proof of Theorem 3, it is not too difficult to formulate a counterpart of the theorem. Of course, all expressions get more notationally involved. Naturally, we can go on in this way.

2. THE BETA DENSITY AND SOME FUNDAMENTAL LEMMAS

From now on we pretend that $X_j = F^{-1}(U_j)$, where U_j, $j = 1, \ldots, n$, are i.i.d. random variables such that all U_j have the uniform distribution on the interval (0, 1). As usual, for any sequence $S_1, \ldots, S_r$ of random variables the order statistics $S_{1:r}, \ldots, S_{r:r}$ denote a reordering of that sequence such that $S_{1:r} \leqslant \ldots \leqslant S_{r:r}$. For any subsequence $S_1, \ldots, S_r$ of $U_1, \ldots, U_n$, by convention, $S_{-1:r} = S_{0:r} := 0$ and $S_{r+1:r} = S_{r+2:r} := 1$; for any subsequence $S_1, \ldots, S_r$ of $X_1, \ldots, X_n$, by convention, $S_{-1:r} = S_{0:r} := -\infty$ and $S_{r+1:r} = S_{r+2:r} := +\infty$.
The beta density will play an important role when we examine γ_3. For $1 \leqslant k \leqslant l$ it is defined by

$$b_{k,l}(s) := \frac{l!}{(k-1)!\,(l-k)!} s^{k-1}(1-s)^{l-k} = l\binom{l-1}{k-1} s^{k-1}(1-s)^{l-k} \quad (s \in [0, 1]).$$

By convention, $b_{-1,l} := b_{0,l} := b_{l+1,l} := b_{l+2,l} \equiv 0$. We note that $b_{j,n}$ is in fact the probability density of $U_{j:n}$. Furthermore, we set $P_l^s(k) := P(X = k)$ for a random variable X which is binomially distributed with parameters l and s, that is, for $s \in (0, 1)$ we set

$$(8) \qquad P_l^s(k) := \begin{cases} \binom{l}{k} s^k (1-s)^{l-k} & \text{for } k = 0, \ldots, l, \\ 0 & \text{for } k \notin \{0, \ldots, l\}. \end{cases}$$

The following simple equalities will be used in the sequel: for $0 \leqslant k \leqslant l+1$ we have

$$b_{k-1,l-1}(s)-b_{k,l-1}(s)=\frac{1}{l}b'_{k,l}(s) \quad \text{and} \quad (l-k)\,b_{k,l}(s)+kb_{k+1,l}(s)=lb_{k,l-1}(s),$$

and hence

$$b_{k,l}(s)-b_{k,l-1}(s)=\frac{k}{l(l+1)}b'_{k+1,l+1}(s); \tag{9}$$

for all $1 \leqslant k \leqslant l$ we have

$$b_{k,l}(s)=lP^s_{l-1}(k-1) \quad \text{and} \quad \int_0^s [b_{k,l}-b_{k,l-1}](t)\,dt=sP^s_{l-1}(k-1). \tag{10}$$

Note that for each $l \in \{1, 2, \ldots\}$ we have

$$\sum_{k=1}^{l} b_{k,l}(s)=\sum_{k=1}^{l} lP^s_{l-1}(k-1)=l. \tag{11}$$

Moreover, for all $0 \leqslant k \leqslant l$

$$\begin{aligned} P^s_l(k) &= sP^s_{l-1}(k-1)+(1-s)P^s_{l-1}(k) \\ &= P^s_{l-1}(k)+s\left[P^s_{l-1}(k-1)-P^s_{l-1}(k)\right]. \end{aligned} \tag{12}$$

We also have (by application of (9)) for $1 < k \leqslant l$:

$$\int_0^s [b_{k-1,l-1}-b_{k,l}](t)\,dt=(1-s)P^s_{l-1}(k-1). \tag{13}$$

The next three lemmas are crucial for the analysis of $\beta_4 = E|n^{1/2}T_1|^4$, $\gamma_3 = E|n^{3/2}T_{12}|^3$ and $\Delta_3^2 = E|n^{5/2}D_1D_2D_3T|^2$. The first one has already been mentioned in van Zwet [7]. The second and the third one will be shown to be correct in Sections 4 and 5. Some preparations concerning conditional distributions of order statistics are made in Section 3.

LEMMA 1. *We have*:

$$n^{1/2}T_1=\frac{1}{n}\sum_{j=1}^{n} c_{jn}\Big\{\int_0^{U_1} sb_{j,n}(s)\,dF^{-1}(s)-\int_{U_1}^{1}(1-s)\,b_{j,n}(s)\,dF^{-1}(s)\Big\}.$$

LEMMA 2. *We have*:

$$n^{3/2}T_{12}=\frac{n}{n-1}\sum_{j=1}^{n-1}(c_{jn}-c_{j+1,n})\Big\{\sum_{i=0}^{2}(-1)^i\int_{U_{i:2}}^{U_{i+1:2}} s^{2-i}(1-s)^i\,b_{j,n-1}(s)\,dF^{-1}(s)\Big\}.$$

Next we set $K_0 := 0$, $K_4 := n+1$ and define $K_1 < K_2 < K_3$ as the ordered ranks of X_1, X_2 and X_3 among $X_1, \ldots, X_n$.

LEMMA 3. *We have*:

$$(14)\quad n^{1/2}(D_1 D_2 D_3 T) = \sum_{i=0}^{3}(-1)^i \sum_{j=K_i+2-i}^{K_{i+1}+1-i} (c_{j+1,n}-2c_{j,n}+c_{j-1,n}) \int_{U_{j-2+i:n}}^{U_{j-1+i:n}} s^{3-i}(1-s)^i\, dF^{-1}(s).$$

3. CONDITIONAL DISTRIBUTIONS OF $U_{j:n}$

3.1. The conditional distribution given U_1 and/or U_2. In order to analyse γ_3 we clearly need the conditional distribution of $U_{j:n}$ given U_1 and/or U_2, since

$$\gamma_3 = E|n^{3/2} T_{12}|^3 = E\Big|n \sum_{j=1}^{n} c_{jn} H_j\Big|^3$$

with

$$(15)\quad \begin{aligned} H_j &= E(X_{j:n}|X_1, X_2) - E(X_{j:n}|X_1) - E(X_{j:n}|X_2) + EX_{j:n} \\ &= E\big(F^{-1}(U_{j:n})\,|\,U_1, U_2\big) - E\big(F^{-1}(U_{j:n})\,|\,U_1\big) \\ &\quad - E\big(F^{-1}(U_{j:n})\,|\,U_2\big) + EF^{-1}(U_{j:n}). \end{aligned}$$

From elementary considerations the following results can be deduced. The conditional distribution of $U_{j:n}$ given U_1 is given by

$$P_{U_{j:n}|U_1} = b_{j,n-1} 1_{[0,U_1]}\lambda + P(U_{j:n} = U_1)\,\delta_{U_1} + b_{j-1,n-1} 1_{[U_1,1]}\lambda,$$

where λ denotes the Lebesgue measure on $\boldsymbol{R}$, δ_{U_1} is the Dirac measure in U_1, and

$$P(U_{j:n} = U_1) = \begin{cases} \int_0^{U_1} [b_{j-1,n-1}(s) - b_{j,n-1}(s)]\, ds & \text{for } j = 2, \ldots, n, \\ \int_{U_1}^{1} [b_{j,n-1}(s) - b_{j-1,n-1}(s)]\, ds & \text{for } j = 1, \ldots, n-1. \end{cases}$$

Of course, we obtain the conditional distribution of $U_{j:n}$ given U_2 after substituting U_2 for U_1 in these results.

The conditional distribution of $U_{j:n}$ given U_1, U_2 in turn is given by

$$\begin{aligned} P_{U_{j:n}|U_1,U_2} &= b_{j,n-2} 1_{[0,U_{1:2}]}\lambda + P(U_{j:n} = U_{1:2})\,\delta_{U_{1:2}} + b_{j-1,n-2} 1_{[U_{1:2},U_{2:2}]}\lambda \\ &\quad + P(U_{j:n} = U_{2:2})\,\delta_{U_{2:2}} + b_{j-2,n-2} 1_{[U_{2:2},1]}\lambda, \end{aligned}$$

where

$$P(U_{j:n} = U_{1:2}) = \int_0^{U_{1:2}} [b_{j-1,n-2}(s) - b_{j,n-2}(s)]\, ds \qquad \text{for } j = 2, \ldots, n,$$

$$P(U_{j:n} = U_{2:2}) = \int_{U_{2:2}}^{1} [b_{j-1,n-2}(s) - b_{j-2,n-2}(s)]\, ds \qquad \text{for } j = 1, \ldots, n-1,$$

and

$$P(U_{1:n}=U_{1:2})=1-\int_0^{U_{1:2}} b_{1,n-2}(s)\,ds, \qquad P(U_{n:n}=U_{2:2})=1-\int_{U_{2:2}}^{1} b_{n-2,n-2}(s)\,ds.$$

3.2. The conditional distribution given $n-3$, $n-2$ or $n-1$ of the U_j's. In order to analyze Δ_3^2 we need the conditional distribution of $U_{j:n}$ given $n-3$, $n-2$ or $n-1$ of the U_j's, since $\Delta_3^2=E|n^{5/2}D_1D_2D_3T|^2$, where

$$D_1D_2D_3T=T-E_1T-E_2T-E_3T+E_{12}T+E_{13}T+E_{23}T-E_{123}T,$$

and hence

$$D_1D_2D_3T=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}c_{jn}M_j$$

with

$$M_j:=D_1D_2D_3X_{j:n}=X_{j:n}-E_1(X_{j:n})-E_2(X_{j:n})-E_3(X_{j:n})$$
$$+E_{12}(X_{j:n})+E_{13}(X_{j:n})+E_{23}(X_{j:n})-E_{123}(X_{j:n}).$$

To obtain the conditional distribution of $U_{j:n}$ given $U_2,\dots,U_n$, given $U_1,U_3,\dots,U_n$, given $U_1,U_2,U_4,\dots,U_n$, given $U_3,\dots,U_n$, given $U_2,U_4,\dots,U_n$, given $U_1,U_4,\dots,U_n$, and given $U_4,\dots,U_n$, respectively, we define subsequences of $X_1,\dots,X_n$ and $U_1,\dots,U_n$, respectively, by

$$\begin{cases}(A_1,\dots,A_{n-1}):=(X_2,X_3,X_4,\dots,X_n),\\(B_1,\dots,B_{n-1}):=(X_1,X_3,X_4,\dots,X_n),\\(C_1,\dots,C_{n-1}):=(X_1,X_2,X_4,\dots,X_n),\end{cases}$$

$$\begin{cases}(A_1^*,\dots,A_{n-1}^*):=(U_2,U_3,U_4,\dots,U_n),\\(B_1^*,\dots,B_{n-1}^*):=(U_1,U_3,U_4,\dots,U_n),\\(C_1^*,\dots,C_{n-1}^*):=(U_1,U_2,U_4,\dots,U_n),\end{cases}$$

and

$$\begin{cases}(P_1,\dots,P_{n-2}):=(X_3,X_4,\dots,X_n),\\(Q_1,\dots,Q_{n-2}):=(X_2,X_4,\dots,X_n),\\(R_1,\dots,R_{n-2}):=(X_1,X_4,\dots,X_n),\end{cases}\qquad\begin{cases}(P_1^*,\dots,P_{n-2}^*):=(U_3,U_4,\dots,U_n),\\(Q_1^*,\dots,Q_{n-2}^*):=(U_2,U_4,\dots,U_n),\\(R_1^*,\dots,R_{n-2}^*):=(U_1,U_4,\dots,U_n),\end{cases}$$

and, finally,

$$(T_1,\dots,T_{n-3}):=(X_4,\dots,X_n),\qquad(T_1^*,\dots,T_{n-3}^*):=(U_4,\dots,U_n).$$

Remember that $X_j = F^{-1}(U_j)$, so that

$$E_1 X_{j:n} = E\big(F^{-1}(U_{j:n}) \mid U_2, \ldots, U_n\big) \quad \text{for } j = 1, \ldots, n,$$

and so on.

As can be checked easily: the conditional distribution of $U_{j:n}$ given $A_1^*, \ldots, A_{n-1}^*$ is determined by

$$P_{U_{j:n}|A_1^*,\ldots,A_{n-1}^*} = A_{j-1:n-1}^* \delta_{A_{j-1:n-1}^*} + 1_{[A_{j-1:n-1}^*, A_{j,n-1}^*]} \lambda + (1 - A_{j:n-1}^*) \delta_{A_{j:n-1}^*},$$

and hence

$$\begin{aligned} &E\big(F^{-1}(U_{j:n}) \mid A_1^*, \ldots, A_{n-1}^*\big) \\ &= F^{-1}(A_{j-1:n-1}^*) A_{j-1:n-1}^* + \int_{A_{j-1:n-1}^*}^{A_{j:n-1}^*} F^{-1}(s)\, ds + F^{-1}(A_{j:n-1}^*)(1 - A_{j:n-1}^*). \end{aligned}$$

Therefore, by partial integration we obtain

$$E(X_{j:n} \mid A_1, \ldots, A_{n-1}) = A_{j-1:n-1} + \int_{A_{j-1:n-1}}^{A_{j:n-1}} [1 - F(t)]\, dt. \tag{16}$$

The conditional distribution of $U_{j:n}$ given either $B_1^*, \ldots, B_{n-1}^*$ or $C_1^*, \ldots, C_{n-1}^*$ can be dealt with in exactly the same way.

Next we look at the probability distribution of $U_{j:n}$ given $P_1^*, \ldots, P_{n-2}^*$ or $Q_1^*, \ldots, Q_{n-2}^*$ or $R_1^*, \ldots, R_{n-2}^*$. We obtain:

$$\begin{aligned} P_{U_{j:n}|P_1^*,\ldots,P_{n-2}^*} &= \sum_{l=1}^{2} g_l 1_{[P_{j-3+l:n-2}^*, P_{j-2+l:n-2}^*]} \lambda \\ &\quad + \sum_{k=0}^{2} \binom{2}{k} [P_{j-2+k:n-2}^*]^{2-k} (1 - P_{j-2+k:n-2}^*)^k \delta_{P_{j-2+k:n-2}^*}, \end{aligned}$$

where $g_1\colon s \mapsto 2s$ and $g_2\colon s \mapsto 2(1-s)$. Therefore

$$\begin{aligned} E\big(F^{-1}(U_{j:n}) \mid P_1^*, \ldots, P_{n-2}^*\big) &= \sum_{l=1}^{2} \int_{P_{j-3+l:n-2}^*}^{P_{j-2+l:n-2}^*} F^{-1}(s) g_l(s)\, ds \\ &\quad + \sum_{k=0}^{2} F^{-1}(P_{j-2+k:n-2}^*) \binom{2}{k} [P_{j-2+k:n-2}^*]^{2-k} (1 - P_{j-2+k:n-2}^*)^k. \end{aligned}$$

Partial integration leads to

$$\begin{aligned} E(X_{j:n} \mid P_1, \ldots, P_{n-2}) = P_{j-2:n-2} &+ \int_{P_{j-2:n-2}}^{P_{j-1:n-2}} [1 - F^2(t)]\, dt \\ &+ \int_{P_{j-1:n-2}}^{P_{j:n-2}} [1 - F(t)]^2\, dt. \end{aligned} \tag{17}$$

Again, the other sequences can be dealt with in the same way.

Finally, for $T_1^*, \ldots, T_{n-3}^*$ we note that

$$P_{U_{j:n}|T_1^*,\ldots,T_{n-3}^*} = \sum_{l=1}^{3} h_l 1_{[T^*_{j-4+l:n-3}, T^*_{j-3+l:n-3}]} \lambda$$
$$+ \sum_{k=0}^{3} \binom{3}{k} [T^*_{j-3+k:n-3}]^{3-k} (1-T^*_{j-3+k:n-3})^k \delta_{T^*_{j-3+k:n-3}},$$

where $h_1\colon s \mapsto 3s^2$, $h_2\colon s \mapsto 6s(1-s)$, and $h_3\colon s \mapsto 3(1-s)^2$. This leads to

$$E(F^{-1}(U_{j:n}) \mid T_1^*, \ldots, T_{n-3}^*) = \sum_{l=1}^{3} \int_{T^*_{j-4+l:n-3}}^{T^*_{j-3+l:n-3}} F^{-1}(s) h_l(s)\, ds$$
$$+ \sum_{k=0}^{3} F^{-1}(T^*_{j-3+k:n-3}) \binom{3}{k} [T^*_{j-3+k:n-3}]^{3-k} (1-T^*_{j-3+k:n-3})^k,$$

which in turn leads to:

$$\text{(18)} \quad E(X_{j:n} \mid T_1, \ldots, T_{n-3})$$
$$= T_{j-3:n-3} + \sum_{m=0}^{2} \int_{T_{j-3+m:n-3}}^{T_{j-2+m:n-3}} \left\{ \sum_{k=0}^{2-m} \binom{3}{k} F(t)^k [1-F(t)]^{3-k} \right\} dt.$$

4. ANALYSIS OF γ_3: PROOF OF LEMMA 2

Recall that $n^{3/2} T_{12} = n \sum_{j=1}^{n} c_{jn} H_j$, with H_j as in (15). With the results of Section 3, for all j we are able to give the following explicit formula for H_j:

$$H_j = \int_0^{U_{1:2}} F^{-1}(s) [b_{j,n-2} - 2b_{j,n-1} + b_{j,n}](s)\, ds$$
$$+ F^{-1}(U_{1:2}) \int_0^{U_{1:2}} [b_{j-1,n-2} - b_{j,n-2} - b_{j-1,n-1} + b_{j,n-1}](s)\, ds$$
$$+ \int_{U_{1:2}}^{U_{2:2}} F^{-1}(s) [b_{j-1,n-2} - b_{j-1,n-1} - b_{j,n-1} + b_{j,n}](s)\, ds$$
$$+ F^{-1}(U_{2:2}) \int_{U_{2:2}}^{1} [b_{j-1,n-2} - b_{j-2,n-2} - b_{j,n-1} + b_{j-1,n-1}](s)\, ds$$
$$+ \int_{U_{2:2}}^{1} F^{-1}(s) [b_{j-2,n-2} - 2b_{j-1,n-1} + b_{j,n}](s)\, ds.$$

We are looking for an alternative form for H_j.

We use partial integration on the first, third and fifth term of this expression in order to obtain this nicer form. For this purpose we define the following three indefinite integrals:

$$I_1(s) := \int_0^s [b_{j,n-2} - 2b_{j,n-1} + b_{j,n}](t)\,dt,$$

$$I_2(s) := \int_0^s [b_{j-1,n-2} - b_{j-1,n-1} - b_{j,n-1} + b_{j,n}](t)\,dt$$

and

$$I_3(s) := \int_0^s [b_{j-2,n-2} - 2b_{j-1,n-1} + b_{j,n}](t)\,dt.$$

Application of (10) and (13) leads to the equalities

$$I_1(s) = -s\{P^s_{n-2}(j-1) - P^s_{n-1}(j-1)\},$$
$$I_2(s) = -s\{P^s_{n-2}(j-2) - P^s_{n-1}(j-1)\}$$

and

$$I_3(s) = -(1-s)\{P^s_{n-1}(j-1) - P^s_{n-2}(j-2)\}.$$

As $E|X| < +\infty$, we have

$$\lim_{s\downarrow 0} F^{-1}(s) I_1(s) = 0 \quad \text{and} \quad \lim_{s\uparrow 1} F^{-1}(s) I_3(s) = 0.$$

Now the first term of the expression for H_j equals

$$[F^{-1}(s) I_1(s)]_0^{U_{1:2}} - \int_0^{U_{1:2}} I_1(s)\,dF^{-1}(s),$$

and so on. Substituting these forms in the expression for H_j we see that the second and fourth term cancel and we find that

$$H_j = -\int_0^{U_{1:2}} I_1(s)\,dF^{-1}(s) - \int_{U_{1:2}}^{U_{2:2}} I_2(s)\,dF^{-1}(s) - \int_{U_{2:2}}^{1} I_3(s)\,dF^{-1}(s).$$

Finally, application of (12) shows that

$$H_j = \sum_{i=0}^{2} (-1)^i \int_{U_{i:2}}^{U_{i+1:2}} s^{2-i}(1-s)^i \{P^s_{n-2}(j-1) - P^s_{n-2}(j-2)\}\,dF^{-1}(s).$$

Consequently,

$$\sum_{j=1}^{n} c_{jn} H_j = \sum_{j=1}^{n-1} (c_{jn} - c_{j+1,n}) \{\sum_{i=0}^{2} (-1)^i \int_{U_{i:2}}^{U_{i+1:2}} s^{2-i}(1-s)^i P^s_{n-2}(j-1)\,dF^{-1}(s)\},$$

and hence the statement of Lemma 2 follows readily.

We remark that a proof of Lemma 1 can be easily constructed along the lines of the proof of Lemma 2.

5. ANALYSIS OF Δ_3^2: PROOF OF LEMMA 3

Summarizing the results of Section 3 we see that

$$D_1 D_2 D_3 T = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_{jn} M_j,$$

where (see (16)–(18))

$$M_j = X_{j:n} - \sum_{E\in\{A,B,C\}} \Big(E_{j-1:n-1} + \int_{E_{j-1:n-1}}^{E_{j:n-1}} \{1-F(t)\}\,dt\Big)$$
$$+ \sum_{F\in\{P,Q,R\}} \Big(F_{j-2:n-2} + \int_{F_{j-2:n-2}}^{F_{j-1:n-2}} \{1-F^2(t)\}\,dt + \int_{F_{j-1:n-2}}^{F_{j:n-2}} \{1-F(t)\}^2\,dt\Big)$$
$$-\left(T_{j-3:n-3} + \sum_{m=0}^{2} \int_{T_{j-3+m:n-3}}^{T_{j-2+m:n-3}} \left\{\sum_{k=0}^{2-m} \binom{3}{k} F(t)^k \big(1-F(t)\big)^{3-k}\right\} dt\right).$$

As mentioned in Lemma 3, we denote the ranks of X_1, X_2 and X_3 in increasing order by K_1, K_2 and K_3. With the aid of the given ordered ranks of X_1, X_2 and X_3, we are able to reconstruct the order statistics of the X's from the ordered A's, and so on. For example, for $X_1 \leqslant X_2 \leqslant X_3$ we see that

$$(A_{1:n-1}, \ldots, A_{K_1-1:n-1}, A_{K_1:n-1}, \ldots, A_{n-1:n-1})$$
$$= (X_{1:n}, \ldots, X_{K_1-1:n}, X_{K_1+1:n}, \ldots, X_{n:n}).$$

From this point on it is a matter of careful bookkeeping to find out that (14) is correct, which completes the proof of Lemma 3.

6. AN UPPER BOUND FOR $\hat{\beta}_4$

In the next three sections we will prove (5), from which our main theorem follows. The three lemmas that will follow, Lemmas 4–6, precisely state what we need.

First we prove a lemma concerning $\hat{\beta}_4$. In the following we repeatedly use the L^p-norm $\|T\|_p := \{E|T|^p\}^{1/p}$ ($p \geqslant 1$). For the following three sections, let A_n, $\tilde{A}_n$, B_n, $\tilde{B}_n$, C_n, $\tilde{C}_n$ be as defined in Theorem 1.

LEMMA 4. *There exists a* $c = c(p_2, \kappa)$ *for which*

$$\hat{\beta}_4^{1/4} \leqslant 2d_1 A_n \beta_4^{1/4} + cKd_1 \tilde{A}_n.$$

Proof. First we note that

$$E|X| = \int_0^{+\infty} (1-F(s))\,ds + \int_{-\infty}^{0} F(s)\,ds \tag{19}$$

and

$$d_1 \geqslant n^{p_1-p_2}|c_{jn}|, \quad d_2 \geqslant n^{q_1-q_2}|c_{jn}-c_{j-1,n}|, \tag{20}$$
$$d_3 \geqslant n^{r_1-r_2}|c_{j+1,n}-2c_{jn}+c_{j-1,n}|,$$

since

$$\frac{1}{n} \leqslant \frac{j}{n}\left(1-\frac{j-1}{n}\right) \quad \text{for } 1 \leqslant j \leqslant n.$$

Hence Lemma 1 leads us to

$$\begin{aligned} d_1^{-1} n^{p_1-p_2}|n^{1/2}\, T_1| &\leqslant \frac{1}{n}\sum_{j=1}^{n}\Big\{\int_0^{U_1} s b_{j,n}(s)\,dF^{-1}(s)+\int_{U_1}^{1}(1-s)\,b_{j,n}(s)\,dF^{-1}(s)\Big\} \\ &= \int_0^{U_1} s\,dF^{-1}(s)+\int_{U_1}^{1}(1-s)\,dF^{-1}(s) \qquad \text{(see (11))} \\ &= \int_{-\infty}^{0} F(t)\,dt+\int_0^{X_1} F(t)\,dt+\int_{X_1}^{0}\big(1-F(t)\big)\,dt+\int_0^{+\infty}\big(1-F(t)\big)\,dt \\ &\leqslant E|X_1|+|X_1| \qquad \text{(see (19))}. \end{aligned}$$

In the case where $p_1 \geqslant p_2$, this shows us that

$$\beta_4^{1/4} = \|n^{1/2}\, T_1\|_4 \leqslant d_1\, n^{p_2-p_1}(\|E|X_1|\|_4+\|X_1\|_4) \leqslant 2d_1\, A_n \beta_4^{1/4}, \tag{21}$$

which completes the proof for $p_1 \geqslant p_2$.

Next we consider the case where $p_1 < p_2$. Note that

$$\begin{aligned} \beta_4^{1/4} &= \|n^{1/2}\, T_1\|_4 \\ &\leqslant \frac{d_1}{n^{p_1+1}}\sum_{j=1}^{n}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-p_2} \\ &\quad \times\Big\{\Big\|\int_0^{U_1} s b_{j,n}(s)\,dF^{-1}(s)\Big\|_4+\Big\|\int_{U_1}^{1}(1-s)\,b_{j,n}(s)\,dF^{-1}(s)\Big\|_4\Big\}. \end{aligned}$$

A little later we will show that for $j = 1, \ldots, n$ and some $c = c(\kappa)$

$$\Big\|\int_0^{U_1} s b_{j,n}(s)\,dF^{-1}(s)\Big\|_4 \leqslant cK\left(\frac{j}{n}\right)^{1-\kappa}\left(1-\frac{j-1}{n}\right)^{1/4-\kappa}. \tag{22}$$

By symmetry arguments we have

$$\Big\|\int_{U_1}^{1}(1-s)\,b_{j,n}(s)\,dF^{-1}(s)\Big\|_4 \leqslant cK\left(\frac{j}{n}\right)^{1/4-\kappa}\left(1-\frac{j-1}{n}\right)^{1-\kappa},$$

so that

$$\Big\|\int_0^{U_1} s b_{j,n}(s)\,dF^{-1}(s)\Big\|_4+\Big\|\int_{U_1}^{1}(1-s)\,b_{j,n}(s)\,dF^{-1}(s)\Big\|_4 \leqslant 2cK\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{1/4-\kappa},$$

and therefore

$$\beta_4^{1/4} \leqslant 2cKd_1 n^{-p_1} \frac{1}{n} \sum_{j=1}^{n} \left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{1/4-\kappa-p_2}.$$

In order to study the behavior of this expression we approximate

$$\frac{1}{n} \sum_{j=1}^{n} \left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{1/4-\kappa-p_2}$$

by integrals of the form

$$\int_{1/n}^{1-1/n} [s(1-s)]^{1/4-\kappa-p_2}\, ds.$$

Constants which appear over here depend on $\kappa+p_2$, so at the end we have constants depending both on κ and on p_2. Doing this it follows easily that also in this case the result of Lemma 4 applies, which completes its proof, provided that (22) is correct.

We turn to the proof of (22). We remind the reader of the gamma function

$$\Gamma\colon s \mapsto \int_0^{+\infty} t^{s-1} e^{-t}\, dt$$

and the beta function

$$B(u, v) := \int_0^1 t^{u-1} (1-t)^{v-1}\, dt,$$

satisfying

$$\Gamma(k+1) = k! \quad \text{and} \quad B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad \text{for all } k \in N,\ u, v > 0.$$

It is known (see, for example, Lemma 2 in Pap and van Zuijlen [5]) that

$$C_1(y) \leqslant \frac{\Gamma(k+y)}{\Gamma(k)} / k^y \leqslant C_2(y) \quad \text{for } k > -y. \tag{23}$$

Suppose that $j \in \{1, \ldots, n-1\}$ or $\kappa < 1$. As

$$sb_{j,n}(s) = \frac{j}{n+1} b_{j+1,n+1}(s),$$

we have

$$\left\|\int_0^{U_1} sb_{j,n}(s)\, dF^{-1}(s)\right\|_4^4 = \left(\frac{j}{n+1}\right)^4 E\left(\int_0^{U_1} b_{j+1,n+1}(s)\, dF^{-1}(s)\right)^4$$

$$\leqslant \left(\frac{j}{n+1}\right)^4 \left(\int_0^1 b_{j+1,n+1}(s)\, dF^{-1}(s)\right)^3 E \int_0^{U_1} b_{j+1,n+1}(s)\, dF^{-1}(s).$$

Moreover,

$$\int_0^1 b_{j+1,n+1}(s)\,dF^{-1}(s) \leqslant K\frac{(n+1)!}{j!\,(n-j)!}B(j+1-\kappa,\, n-j+1-\kappa)$$

$$\leqslant K\frac{\Gamma(n+2)}{\Gamma(n+2-2\kappa)}\frac{\Gamma(j+1-\kappa)\,\Gamma(n-j+1-\kappa)}{\Gamma(j+1)\quad\Gamma(n-j+1)}$$

$$\leqslant c_1 K(n+2)^{2\kappa}(j+1)^{-\kappa}(n-j+1)^{-\kappa} \qquad \text{(see (23))}$$

$$\leqslant c_1 K\left(\frac{j+1}{n+2}\right)^{-\kappa}\left(\frac{n-(j-1)}{n+2}\right)^{-\kappa} \leqslant c_2 K\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\kappa}$$

for constants c_1 and c_2 depending on κ, and

$$E\int_0^{U_1} b_{j+1,n+1}(s)\,dF^{-1}(s) = \int_0^1\int_0^t b_{j+1,n+1}(s)(F^{-1})'(s)\,ds\,dt$$

$$= \int_0^1 b_{j+1,n+1}(s)(F^{-1})'(s)\left\{\int_s^1 dt\right\}ds \leqslant K\frac{(n+1)!}{j!\,(n-j)!}\int_0^1 s^{j-\kappa}(1-s)^{n-j+1-\kappa}\,ds$$

$$= K\frac{\Gamma(n+2)}{\Gamma(n+2+1-2\kappa)}\frac{\Gamma(j+1-\kappa)\,\Gamma(n-j+1+1-\kappa)}{\Gamma(j+1)\quad\Gamma(n-j+1)}$$

$$\leqslant c_3 K\left(\frac{j}{n}\right)^{-\kappa}\left(1-\frac{j-1}{n}\right)^{1-\kappa}$$

for some constant $c_3 = c_3(\kappa)$. Thus

$$\left\|\int_0^{U_1} sb_{j,n}(s)\,dF^{-1}(s)\right\|_4^4 \leqslant \left(\frac{j}{n+1}\right)^4 c_2^3 c_3 K^4\left(\frac{j}{n}\right)^{-4\kappa}\left(1-\frac{j-1}{n}\right)^{1-4\kappa}$$

$$\leqslant c_4 K^4\left(\frac{j}{n}\right)^{4(1-\kappa)}\left(1-\frac{j-1}{n}\right)^{4(1/4-\kappa)}$$

for some $c_4 = c_4(\kappa)$, which proves (22) in the case where $j \in \{1, \ldots, n-1\}$ or $\kappa < 1$.

The cases which remain are more difficult to handle, as they imply that

$$\int_0^1 b_{j+1,n+1}(s)\,dF^{-1}(s) = +\infty.$$

This is why we use a different approach. For $j = n$ and $\kappa \in (1, 5/4)$ we will prove that for some $c = c(\kappa)$

$$\left\|\int_0^{U_1} sb_{nn}(s)\,dF^{-1}(s)\right\|_4 \leqslant cKn^{\kappa-1/4}.$$

Namely, we have

$$\begin{aligned}
\Big\|\int_0^{U_1} sb_{nn}(s)\,dF^{-1}(s)\Big\|_4^4 &\leqslant \Big\|\int_0^{U_1} ns^n K[s(1-s)]^{-\kappa}\,ds\Big\|_4^4 \\
&= K^4 n^4 E\Big(\int_0^{U_1} s^{n-\kappa}(1-s)^{-\kappa}\,ds\Big)^4 = K^4 n^4 \int_0^1\Big(\int_0^t s^{n-\kappa}(1-s)^{-\kappa}\,ds\Big)^4 dt \\
&\leqslant K^4 n^4 \int_0^1 t^{4(n-\kappa)}\bigg(\bigg[\frac{(1-s)^{-\kappa+1}}{\kappa-1}\bigg]_0^t\bigg)^4 dt \\
&= \Big(\frac{Kn}{\kappa-1}\Big)^4 B\big(4(n-\kappa)+1,\ -4(\kappa-1)+1\big) \\
&= \Big(\frac{Kn}{\kappa-1}\Big)^4 \Gamma(5-4\kappa)\frac{\Gamma(4n-4\kappa+1)}{\Gamma(4n-4\kappa+1+5-4\kappa)} \\
&\leqslant c_1 K^4 n^4 (4n-4\kappa+1)^{-(5-4\kappa)} \leqslant c_2 K^4 n^{4\kappa-1}
\end{aligned}$$

for some c_1, c_2, depending on κ. This again proves the point.

Finally, we take up the case in which $j=n$ and $\kappa=1$. The previous argument does not work as we divided by $\kappa-1$. We show that for some C

$$\Big\|\int_0^{U_1} sb_{nn}(s)\,dF^{-1}(s)\Big\|_4^4 \leqslant CK^4 n^3 \quad \text{or} \quad \Big\|\int_0^{U_1} s^{n-1}(1-s)^{-1}\,ds\Big\|_4^4 \leqslant Cn^{-1}.$$

Ronald Kortram (personal communication) provided us with the following proof. The function $s\mapsto s^{n-1}(1-s)^{1/4}$ is increasing on $[0, 1-1/(4n-3)]$ and decreasing on $[1-1/(4n-3), 1]$. So

$$\begin{aligned}
&\Big\|\int_0^{U_1} s^{n-1}(1-s)^{-1}\,ds\Big\|_4^4 \\
&= \int_0^{1-1/(4n-3)}\bigg(\int_0^t \frac{s^{n-1}(1-s)^{1/4}}{(1-s)^{5/4}}\,ds\bigg)^4 dt + \int_{1-1/(4n-3)}^1\bigg(\int_0^t \frac{s^{n-1}(1-s)^{1/4}}{(1-s)^{5/4}}\,ds\bigg)^4 dt \\
&\leqslant \int_0^{1-1/(4n-3)} t^{4n-4}(1-t)\bigg(\bigg[\frac{(1-s)^{-1/4}}{1/4}\bigg]_0^t\bigg)^4 dt \\
&\quad + \int_{1-1/(4n-3)}^1 \frac{1}{4n-3}\bigg(\bigg[\frac{(1-s)^{-1/4}}{1/4}\bigg]_0^t\bigg)^4 dt \\
&\leqslant 4^4\int_0^1 t^{4n-4}\,dt + \frac{4^4}{4n-3}\int_0^1 (1-t)^{-1/4}\,dt \leqslant Cn^{-1}
\end{aligned}$$

for some constant C. This completes the proof. ∎

7. AN UPPER BOUND FOR γ_3

The aim of this section is to prove the following lemma:

LEMMA 5. *There exists a* $c = c(q_2, \kappa)$ *for which*

$$\gamma_3^{1/3} \leqslant 3d_2 B_n \beta_3^{1/3} + cKd_2 \tilde{B}_n.$$

Proof. First we consider the case in which $q_1 \geqslant q_2+1$. By Lemma 2, (11), (19) and (20), the method we have used to prove Lemma 4 in the case where $p_1 \geqslant p_2$ yields

$$d_2^{-1} n^{q_1-q_2-1} |n^{3/2} T_{12}| \leqslant E|X_1|+|X_1|+|X_2|.$$

(Here we also used the inequality $F^{-1}(U_{2:2}) = \max(X_1, X_2) \leqslant |X_1|+|X_2|$.) As a consequence

$$\gamma_3^{1/3} = \|n^{3/2} T_{12}\|_3 \leqslant 3d_2 B_n \beta_3^{1/3},$$

which completes the proof for $q_1 \geqslant q_2+1$.

Now suppose that $q_1 < q_2+1$. By Lemma 2 and (3) we have

$$\gamma_3^{1/3} = \|n^{3/2} T_{12}\|_3 \leqslant \frac{n}{n-1} \sum_{j=1}^{n-1} d_2 n^{-q_1} \left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-q_2} \{\|\Gamma_{1j}\|_3+\|\Gamma_{2j}\|_3+\|\Gamma_{3j}\|_3\},$$

where for $i = 1, 2, 3$

$$\Gamma_{ij} := \int_{U_{i-1:2}}^{U_{i:2}} s^{3-i}(1-s)^{i-1}\, dF^{-1}(s). \tag{24}$$

First we determine the order of $\|\Gamma_{1j}\|_3$. As

$$s^2 b_{j,n-1}(s) = \left(\frac{j+1}{n}\right)^2 b_{j+2,n+1}(s),$$

we have

$$\|\Gamma_{1j}\|_3^3 = \left(\frac{j}{n+1}\right)^6 E\Big(\int_0^{U_{1:2}} b_{j+2,n+1}(s)\, dF^{-1}(s)\Big)^3$$

and we can find an upper bound in the same way as we did for $\left\|\int_0^{U_1} sb_{j,n}(s)\, dF^{-1}(s)\right\|_4$ in Section 6. Again we have the following three cases:

(i) $j = 1, \ldots, n-1$ or $\kappa < 1$, (ii) $j = n$ and $\kappa \in (1, 5/4)$, (iii) $j = n$ and $\kappa = 1$;

again in each of them the result is the same and the methods to prove them differ considerably. We confine ourselves to the first case. We have

$$\int_0^1 b_{j+2,n+1}(s)\, dF^{-1}(s) \leqslant K \frac{\Gamma(n+2)}{\Gamma(j+2)\Gamma(n-j)} \frac{\Gamma(j+2-\kappa)\Gamma(n-j-\kappa)}{\Gamma(n+2-2\kappa)}$$

and

$$E\int_0^{U_{1:2}} b_{j+2,n+1}(s)\,dF^{-1}(s) \leqslant K\frac{\Gamma(n+2)}{\Gamma(j+2)\Gamma(n-j)}\frac{\Gamma(j+2-\kappa)\Gamma(n-j+2-\kappa)}{\Gamma(n+4-2\kappa)},$$

which for some $c = c(\kappa)$ leads to

$$\|\Gamma_{1j}\|_3 \leqslant cK\left(\frac{j}{n}\right)^{2-\kappa}\left(1-\frac{j-1}{n}\right)^{2/3-\kappa}.$$

By symmetry arguments we see that

$$\|\Gamma_{3j}\|_3 \leqslant cK\left(\frac{j}{n}\right)^{2/3-\kappa}\left(1-\frac{j-1}{n}\right)^{2-\kappa}.$$

Moreover, we find that for some $c = c(\kappa)$

$$\|\Gamma_{2j}\|_3 \leqslant cK\left(\frac{j}{n}\right)^{1-\kappa}\left(1-\frac{j-1}{n}\right)^{4/3-\kappa}.$$

In conclusion, for some $c = c(\kappa)$ we have

$$(25)\qquad \|\Gamma_{1j}\|_3+\|\Gamma_{2j}\|_3+\|\Gamma_{3j}\|_3 \leqslant cK\left(\frac{j}{n}\right)^{2/3-\kappa}\left(1-\frac{j-1}{n}\right)^{2/3-\kappa},$$

so that

$$\gamma_3^{1/3} \leqslant \frac{n}{n-1}d_2\,cKn^{1-q_1}\frac{1}{n}\sum_{j=1}^{n-1}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{2/3-\kappa-q_2}.$$

In the same way as in Section 6 this leads to the result mentioned in Lemma 5. The second and the third case can also be handled with the approach of Section 6. ∎

8. AN UPPER BOUND FOR Δ_3^2

We will prove the following lemma:

LEMMA 6. *There exists a* $c = c(r_2, \kappa)$ *for which*

$$(\Delta_3^2)^{1/2} \leqslant 4d_3\,C_n\,\beta_2^{1/2}+cKd_3\,\tilde{C}_n.$$

Proof. In the case where $r_1 \geqslant r_2+2$, like before we deduce that

$$d_3^{-1}\,n^{r_1-r_2-2}\,|n^{5/2}D_1D_2D_3T| \leqslant E|X_1|+|X_1|+|X_2|+|X_3|,$$

so $(\Delta_3^2)^{1/2} = \|n^{5/2}D_1D_2D_3T\|_2 \leqslant 4d_3\,C_n\,\beta_2^{1/2}$.

Now suppose that $r_1 < r_2+2$. By Lemma 3 and (4) we have

$$(26)\quad (\Delta_3^2)^{1/2} = \|n^{5/2}D_1D_2D_3T\|_2 \leqslant d_3\,n^{2-r_1}(\|\Delta_1\|_2+\|\Delta_2\|_2+\|\Delta_3\|_2+\|\Delta_4\|_2)$$

with

$$\Delta_{i+1} := \sum_{j=K_i+2-i}^{K_{i+1}+1-i} \left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-r_2} \int_{U_{j-2+i:n}}^{U_{j-1+i:n}} s^{3-i}(1-s)^i \, dF^{-1}(s) \quad \text{for } i=0, 1, 2, 3.$$

As we use the inequality $|(F^{-1})'(s)| \leqslant K[s(1-s)]^{-\kappa}$, by symmetry arguments it can easily be shown that the upper bounds for Δ_1 and Δ_4 are of the same order. The same applies for Δ_2 and Δ_3, so that we can concentrate on finding orders for $\|\Delta_1\|_2$ and $\|\Delta_2\|_2$.

First we remark that, for each combination (k_1, k_2, k_3) for which $1 \leqslant k_1 < k_2 < k_3 \leqslant n$,

$$P[(K_1, K_2, K_3) = (k_1, k_2, k_3)] = \binom{n}{3}^{-1};$$

consequently,

$$P[(K_1, K_2) = (k_1, k_2)] = \frac{n-k_2}{\binom{n}{3}} \quad \text{for } 1 \leqslant k_1 < k_2 \leqslant n-1 \tag{27}$$

and

$$P[K_1 = k_1] = \sum_{k_2=k_1+1}^{n-1} \frac{n-k_2}{\binom{n}{3}} = \frac{\frac{1}{2}(n-k_1)\big(n-(k_1+1)\big)}{\binom{n}{3}} \quad \text{for } 1 \leqslant k_1 \leqslant n-2.$$

Since K_1 and $U_{0:n}, \ldots, U_{n+1:n}$ are independent, we have

$$E|\Delta_1|^2 = E_{K_1}(E_{(U_{0:n},\ldots,U_{n+1:n})}\{|\Delta_1|^2 \mid K_1\}),$$

so that

$$\|\Delta_1\|_2^2 = \sum_{k_1=1}^{n-2} P[K_1 = k_1] \left\| \sum_{j=2}^{k_1+1} \left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-r_2} \int_{U_{j-2:n}}^{U_{j-1:n}} s^3 \, dF^{-1}(s) \right\|_2^2.$$

Furthermore, Lemma 4 of Pap and van Zuijlen [5] states: for each fixed pair $\varepsilon_1, \varepsilon_2 \in \boldsymbol{R}$ there exists (under some conditions on the triple $(\varepsilon_1, \varepsilon_2, j)$, which for our purposes are always satisfied) a constant $c = c(\varepsilon_1, \varepsilon_2)$ such that

$$\left\| \int_{U_{j-1:n}}^{U_{j:n}} s^{\varepsilon_1}(1-s)^{\varepsilon_2} \, ds \right\|_2 \leqslant c\frac{1}{n}\left(\frac{j}{n}\right)^{\varepsilon_1}\left(1-\frac{j-2}{n}\right)^{\varepsilon_2}.$$

Therefore it follows easily that for $j = 2, \ldots, n-1$

$$\left\| \int_{U_{j-2:n}}^{U_{j-1:n}} s^3 \, dF^{-1}(s) \right\|_2 \leqslant K \left\| \int_{U_{j-2:n}}^{U_{j-1:n}} s^{3-\kappa}(1-s)^{-\kappa} \, ds \right\|_2 \leqslant cK\frac{1}{n}\left(\frac{j}{n}\right)^{3-\kappa}\left(1-\frac{j-1}{n}\right)^{-\kappa}.$$

We obtain

$$(28)\quad \|\Delta_1\|_2^2 \leqslant \sum_{k_1=1}^{n-2} P[K_1 = k_1]\left(\sum_{j=2}^{k_1+1}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-r_2}\Big\|\int_{U_{j-2:n}}^{U_{j-1:n}} s^3\, dF^{-1}(s)\Big\|_2\right)^2$$

$$\leqslant \sum_{k_1=1}^{n-2}\frac{\frac{1}{2}(n-k_1)(n-(k_1+1))}{\binom{n}{3}}\left(c_1 K\frac{1}{n}\sum_{j=2}^{k_1+1}\left(\frac{j}{n}\right)^{3-\kappa-r_2}\left(1-\frac{j-1}{n}\right)^{-\kappa-r_2}\right)^2$$

$$\leqslant c_2 K^2\frac{1}{n}\sum_{k_1=1}^{n-2}\left(1-\frac{k_1}{n}\right)^2\left(\frac{1}{n}\sum_{j=2}^{k_1+1}\left(\frac{j}{n}\right)^{3-\kappa-r_2}\left(1-\frac{j-1}{n}\right)^{-\kappa-r_2}\right)^2$$

for constants c_1, c_2, depending on κ. By integral approximation we see that

$$(29)\quad \frac{1}{n}\sum_{j=2}^{k_1+1}\left(\frac{j}{n}\right)^{3-(\kappa+r_2)}\left(1-\frac{j-1}{n}\right)^{-(\kappa+r_2)}$$

$$\leqslant c\Bigg(I\{\kappa+r_2<1\}+I\{\kappa+r_2=1\}\log\frac{n}{n-k_1}+I\{\kappa+r_2>1\}\left(\frac{n}{n-k_1}\right)^{\kappa+r_2-1}$$

$$+I\{\kappa+r_2=4\}\log n+I\{\kappa+r_2>4\}\,n^{\kappa+r_2-4}\Bigg)$$

for some $c = c(\kappa+r_2)$. For $\kappa+r_2<1$ this leads to

$$\|\Delta_1\|_2^2 \leqslant c_1^2 K^2\frac{1}{n}\sum_{k_1=1}^{n-2}\left(1-\frac{k_1}{n}\right)^2 \leqslant c_1^2 K^2$$

for a certain $c_1 = c_1(\kappa, r_2)$, that is, $\|\Delta_1\|_2 \leqslant c_1 K$. For $\kappa+r_2 = 1$ this leads to

$$\|\Delta_1\|_2^2 \leqslant c_2 K^2\frac{1}{n}\sum_{k_1=1}^{n-2}\left(1-\frac{k_1}{n}\right)^2\log^2\left(\frac{n}{n-k_1}\right)\leqslant c_3^2 K^2$$

for some c_2, c_3, depending on κ and r_2, since

$$\int_0^1 (1-s)^2\log^2\left(\frac{1}{1-s}\right)ds = \int_0^1 (t\log t)^2\,dt < +\infty.$$

Hence $\|\Delta_1\|_2 \leqslant c_3 K$. For $1<\kappa+r_2<4$ we get

$$\|\Delta_1\|_2^2 \leqslant c_4 K^2\frac{1}{n}\sum_{k_1=1}^{n-2}\left(1-\frac{k_1}{n}\right)^2\left(\frac{n}{n-k_1}\right)^{2(\kappa+r_2)-2}$$

$$= c_4 K^2\frac{1}{n}\sum_{k_1=1}^{n-2}\left(1-\frac{k_1}{n}\right)^{4-2(\kappa+r_2)} \leqslant c_5 K^2\int_0^{1-1/n}(1-s)^{4-2(\kappa+r_2)}\,ds$$

for certain c_4, c_5 (depending on κ, r_2), so that in this case, for some $c_6 = c_6(\kappa, r_2)$,

$$\|\Delta_1\|_2 \leqslant c_6 K(I\{1<\kappa+r_2<5/2\}+I\{\kappa+r_2=5/2\}\sqrt{\log n}$$

$$+I\{5/2<\kappa+r_2<4\}\,n^{\kappa+r_2-5/2}).$$

For $\kappa+r_2 \geqslant 4$ we always have two terms producing two orders, of which we need the largest. The $(n/(n-k_1))^{\kappa+r_2-1}$-part in (29) will yield $\|\Delta_1\|_2 \leqslant cKn^{\kappa+r_2-5/2}$. The other parts produce orders that are dominated by the order of the first term. Hence, for some $c = c(r_2, \kappa)$ we find

$$(30)\quad \|\Delta_1\|_2 \leqslant cK\big(I\{\kappa+r_2 < 5/2\}+I\{\kappa+r_2 = 5/2\}\sqrt{\log n} \\ +I\{\kappa+r_2 > 5/2\}\, n^{\kappa+r_2-5/2}\big).$$

To determine the order of $\|\Delta_2\|_2$ we roughly proceed similarly. We shall stop noting that the constants depend on both κ and r_2, though of course they do. Like before (see (27) as well as (28))

$$\|\Delta_2\|_2^2 \leqslant \sum_{k_1=1}^{n-2}\sum_{k_2=k_1+1}^{n-1} P[(K_1, K_2) = (k_1, k_2)]$$

$$\times\Big(\sum_{j=k_1+1}^{k_2}\Big[\frac{j}{n}\Big(1-\frac{j-1}{n}\Big)\Big]^{-r_2}\Big\|\int_{U_{j-1:n}}^{U_{j:n}} s^2(1-s)\,dF^{-1}(s)\Big\|_2\Big)^2$$

$$\leqslant \sum_{k_1=1}^{n-2}\sum_{k_2=k_1+1}^{n-1}\frac{n-k_2}{\binom{n}{3}}\Big(K\frac{1}{n}\sum_{j=k_1+1}^{k_2}\Big(\frac{j}{n}\Big)^{2-(\kappa+r_2)}\Big(1-\frac{j-1}{n}\Big)^{1-(\kappa+r_2)}\Big)^2$$

$$\leqslant c_1K^2\frac{1}{n^2}\sum_{k_1=1}^{n-2}\sum_{k_2=k_1+1}^{n-1}\Big(1-\frac{k_2}{n}\Big)\Big(\frac{1}{n}\sum_{j=k_1+1}^{k_2}\Big(\frac{j}{n}\Big)^{2-(\kappa+r_2)}\Big(1-\frac{j-1}{n}\Big)^{1-(\kappa+r_2)}\Big)^2.$$

As in (29) we have

$$\frac{1}{n}\sum_{j=k_1+1}^{k_2}\Big(\frac{j}{n}\Big)^{2-(\kappa+r_2)}\Big(1-\frac{j-1}{n}\Big)^{1-(\kappa+r_2)}$$

$$\leqslant c_2\Big(I\{\kappa+r_2 < 2\}+I\{\kappa+r_2 = 2\}\log\frac{n}{n-k_1}+I\{\kappa+r_2 > 2\}\Big(\frac{n}{n-k_1}\Big)^{\kappa+r_2-2}$$

$$+I\{\kappa+r_2 = 3\}\log n+I\{\kappa+r_2 > 3\}\,n^{\kappa+r_2-3}\Big).$$

For $\kappa+r_2 < 2$ this leads to

$$\|\Delta_2\|_2^2 \leqslant c_3K^2\frac{1}{n^2}\sum_{k_1=1}^{n-2}\sum_{k_2=k_1+1}^{n-1}\Big(1-\frac{k_2}{n}\Big) \leqslant c_4^2K^2\frac{1}{n^2}n^2 = c_4^2K^2,$$

so that $\|\Delta_2\|_2 \leqslant c_4K$. For $\kappa+r_2 = 2$ we find the same order. For $2 < \kappa+r_2 < 3$

we get

$$\|\Delta_2\|_2^2 \leqslant c_5 K^2 \frac{1}{n^2} \sum_{k_2=2}^{n-1} \sum_{k_1=1}^{k_2-1} \left(1-\frac{k_2}{n}\right)\left(\frac{n}{n-k_2}\right)^{2\kappa+2r_2-4}$$

$$= c_5 K^2 \frac{1}{n^2} \sum_{k_2=2}^{n-1} (k_2-1)\left(1-\frac{k_2}{n}\right)^{5-2\kappa-2r_2}$$

$$\leqslant c_6 K^2 \int_0^1 (1-s)^{5-2\kappa-2r_2}\, ds \leqslant c_7^2 K^2,$$

as $5-2\kappa-2r_2 > -1$. Here also $\|\Delta_2\|_2 \leqslant c_7 K$.

For $\kappa+r_2 \geqslant 3$ again we have two terms playing a part. As with Δ_1, here the $(n/(n-k_1))^{\kappa+r_2-1}$-term dominates the others. This leads us to $\|\Delta_2\|_2 \leqslant cK\sqrt{\log n}$ if $\kappa+r_2 = 3$ and to $\|\Delta_2\|_2 \leqslant cKn^{\kappa+r_2-3}$ if $\kappa+r_2 > 3$.

Collecting the results we see that for some $c = c(\kappa, r_2)$ we have

$$\|\Delta_2\|_2 \leqslant cK(I\{\kappa+r_2 < 3\}+I\{\kappa+r_2 = 3\}\sqrt{\log n}+I\{\kappa+r_2 > 3\}\, n^{\kappa+r_2-3}).$$

Since the order of $\|\Delta_1\|_2$ dominates the ones of $\|\Delta_2\|_2$, $\|\Delta_3\|_2$ and $\|\Delta_4\|_2$, it now follows from (26) and (30) that Lemma 6 is correct. ∎

9. PROOF OF THEOREM 3

Theorem 3 is proved by using (2). We need to find upper bounds for $\beta_4(T)$, $\gamma_3(T)$ and $\Delta_3^2(T)$. First we set $J\colon t \mapsto \psi(t)[t(1-t)]^{-\gamma}$. As to $\beta_4^{1/4}(T)$ we can apply Lemma 4. By taking $p_1 := 0$ and $p_2 := \gamma$ in the expression for d_1, we obtain $d_1 \leqslant 2^{2\gamma}\|\psi\|_\infty$, so that Lemma 4 implies that for some $c = c(\kappa, \gamma)$

$$\beta_4^{1/4}(T) \leqslant cK2^{2\gamma}\|\psi\|_\infty.$$

Next we determine the order of $\gamma_3(T)$. By taking

$$c_{j,n} := J\left(\frac{j}{n+1}\right) \quad \text{for } j = 1, \ldots, n$$

we have for $1 \leqslant j < n$

$$c_{j,n} - c_{j+1,n} = e_{j1} + e_{j2}$$

with

$$e_{j1} := \left[\frac{j}{n+1}\left(1-\frac{j}{n+1}\right)\right]^{-\gamma}\left(\psi\left(\frac{j}{n+1}\right)-\psi\left(\frac{j+1}{n+1}\right)\right),$$

$$e_{j2} := \psi\left(\frac{j+1}{n+1}\right)\left(\left[\frac{j}{n+1}\left(1-\frac{j}{n+1}\right)\right]^{-\gamma}-\left[\frac{j+1}{n+1}\left(1-\frac{j+1}{n+1}\right)\right]^{-\gamma}\right).$$

By Lemma 2 and (24), we obtain

$$\gamma_3^{1/3}(T) = \|n^{3/2}\,T_{12}\|_3 = \left\|\frac{n}{n-1}\sum_{j=1}^{n-1}(c_{j,n}-c_{j+1,n})\{\Gamma_{1j}-\Gamma_{2j}+\Gamma_{3j}\}\right\|_3$$
$$\leqslant \sum_{k=1}^{2}\left\|\frac{n}{n-1}\sum_{j=1}^{n-1}e_{jk}\{\Gamma_{1j}-\Gamma_{2j}+\Gamma_{3j}\}\right\|_3,$$

which leaves us two terms to estimate from above.

First we take a look at the e_{j1}-term. As

$$|e_{j1}| \leqslant 2^{2\gamma}\,\|\psi'\|_\infty \frac{1}{n}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma} \qquad \text{for } j = 1, \ldots, n-1,$$

we obtain

$$\left\|\frac{n}{n-1}\sum_{j=1}^{n-1}e_{j1}\{\Gamma_{1j}-\Gamma_{2j}+\Gamma_{3j}\}\right\|_3$$
$$\leqslant 2^{2\gamma}\,\|\psi'\|_\infty \frac{1}{n}\sum_{j=1}^{n-1}\{\|\Gamma_{1j}\|_3+\|\Gamma_{2j}\|_3+\|\Gamma_{3j}\|_3\}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma}$$
$$\leqslant cK\,\|\psi'\|_\infty \frac{1}{n-1}\sum_{j=1}^{n-1}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{2/3-\kappa-\gamma}$$

for some constant $c = c(\kappa, \gamma)$ (see (25)). Integral approximation yields

$$\left\|\frac{n}{n-1}\sum_{j=1}^{n-1}e_{j1}\{\Gamma_{1j}-\Gamma_{2j}+\Gamma_{3j}\}\right\|_3 \leqslant cK\,\|\psi'\|_\infty.$$

Regarding the second term we need to estimate $|e_{j2}|$ from above. To this end we introduce the function

$$\varphi\colon s \mapsto [s(1+1/n-s)]^{-\gamma} \qquad \text{for } s \in [1/n, 1].$$

We are mainly concerned with expressions of the form

$$\left|\varphi\left(\frac{j+1}{n}\right)-\varphi\left(\frac{j}{n}\right)\right| \qquad \text{for } j = 1, \ldots, n-1.$$

As

$$\varphi'(s) = -\gamma(1+1/n-2s)\,[s(1+1/n-s)]^{-\gamma-1},$$

the mean value theorem leads to

$$|e_{j2}| \leqslant 2^{3\gamma+1}\,\gamma\,\|\psi\|_\infty \frac{1}{n}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma-1}.$$

In the same way as with e_{j1}, for some $c=c(\kappa,\gamma)$ we obtain

$$\left\|\frac{n}{n-1}\sum_{j=1}^{n-1} e_{j2}\{\Gamma_{1j}-\Gamma_{2j}+\Gamma_{3j}\}\right\|_3 \leqslant cK\|\psi\|_\infty .$$

We may conclude that for some $c=c(\kappa,\gamma)$

$$\gamma_3^{1/3}(T)\leqslant cK(\|\psi'\|_\infty+\|\psi\|_\infty).$$

Finally, we turn to $\Delta_3^2(T)$, where $\sqrt{\Delta_3^2(T)}=\|n^{5/2}D_1D_2D_3T\|_2$ with $D_1D_2D_3T$ as in Lemma 3. Now for $j=2,\dots,n-1$

$$c_{j+1,n}-2c_{j,n}+c_{j-1,n}=f_{j1}+f_{j2}+f_{j3}$$

if for all such j we set

$$f_{j1}=\left(\frac{n+1}{n}\right)^{2\gamma}\left\{\psi\left(\frac{j+1}{n+1}\right)-2\psi\left(\frac{j}{n+1}\right)+\psi\left(\frac{j-1}{n+1}\right)\right\}\left[\frac{j+1}{n}\left(1-\frac{j}{n}\right)\right]^{-\gamma},$$

$$f_{j2}=\left(\frac{n+1}{n}\right)^{2\gamma}2\left\{\psi\left(\frac{j}{n+1}\right)-\psi\left(\frac{j-1}{n+1}\right)\right\}$$
$$\times\left(\left[\frac{j+1}{n}\left(1-\frac{j}{n}\right)\right]^{-\gamma}-\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma}\right),$$

$$f_{j3}=\left(\frac{n+1}{n}\right)^{2\gamma}\psi\left(\frac{j-1}{n+1}\right)\left(\left[\frac{j+1}{n}\left(1-\frac{j}{n}\right)\right]^{-\gamma}-2\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma}\right.$$
$$\left.+\left[\frac{j-1}{n}\left(1-\frac{j-2}{n}\right)\right]^{-\gamma}\right).$$

We proceed as with the $\gamma_3(T)$, splitting $\|n^{5/2}D_1D_2D_3T\|_2$ up into three parts, corresponding to f_{j1}, f_{j2} and f_{j3}, respectively. We start with f_{j1}.

Applying the mean value theorem for two times we see that for all j

$$|f_{j1}|\leqslant 2^{3\gamma+1}\|\psi''\|_\infty\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma}n^{-2}.$$

Now we try to find an upper bound for the expression corresponding to f_{j1}. See (14) for the parts we abbreviated to '...'. As in Section 8 we see that

$$\|n^2\{\dots(f_{j1})\dots\}\|_2\leqslant n^2 2^{3\gamma+1}\|\psi''\|_\infty n^{-2}\{\|\Delta_1\|_2+\dots+\|\Delta_4\|_2\},$$

with the $\Delta_1,\dots,\Delta_4$ as before, taking $r_2=\gamma$. So for some $c=c(\kappa,\gamma)$

$$\|n^2\{\dots(f_{j1})\dots\}\|_2\leqslant cK\|\psi''\|_\infty .$$

We turn to the second term, where we need to estimate f_{2j} from above for all j. To this we apply the same function φ that we used to estimate e_{j2}.

We obtain

$$|f_{j2}| \leqslant 2^{3\gamma+2}\gamma\,\|\psi'\|_\infty\, n^{-2}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma-1},$$

and as with f_{j1} we see that

$$\|n^2\{\ldots(f_{j2})\ldots\}\|_2 \leqslant cK\,\|\psi'\|_\infty.$$

As to the third term we need to estimate $|f_{3j}|$ from above (all j). To this end again we use the function φ, as we have to deal with expressions of the form

$$\left|\varphi\left(\frac{j+1}{n}\right)-2\varphi\left(\frac{j}{n}\right)+\varphi\left(\frac{j-1}{n}\right)\right| \quad \text{(all } j\text{)}.$$

We apply the mean value theorem two times to see that

$$\left|\varphi\left(\frac{j+1}{n}\right)-2\varphi\left(\frac{j}{n}\right)+\varphi\left(\frac{j-1}{n}\right)\right| \leqslant \frac{2}{n^2}|\varphi''(\xi)| \quad \text{for some } \xi\in\left[\frac{j-1}{n},\frac{j+1}{n}\right].$$

Moreover,

$$\varphi''(s) = \{2s(1+1/n-s)+(\gamma+1)(1+1/n-2s)^2\}\,\gamma\,[s(1+1/n-s)]^{-(\gamma+2)} \quad \text{for all } s,$$

so

$$|\varphi''(s)| \leqslant \gamma(\gamma+3)\,[s(1+1/n-s)]^{-(\gamma+2)}.$$

Restricting ourselves to $s\in[(j-1)/n,\,(j+1)/n]$ we obtain

$$|\varphi''(s)| \leqslant \gamma(\gamma+3)\,2^{2(\gamma+2)}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-(\gamma+2)}.$$

Thus we see that for all j we have

$$|f_{j3}| \leqslant 2^{4\gamma+5}\gamma(\gamma+3)\,\|\psi\|_\infty\, n^{-2}\left[\frac{j}{n}\left(1-\frac{j-1}{n}\right)\right]^{-\gamma-2}.$$

Hence for some $c = c(\kappa, \gamma)$

$$\|n^2\{\ldots(f_{j3})\ldots\}\|_2 \leqslant cK\,\|\psi\|_\infty.$$

We conclude that for some $c = c(\kappa, \gamma)$

$$\sqrt{\Delta_3^2(T)} \leqslant cK\,(\|\psi''\|_\infty+\|\psi'\|_\infty+\|\psi\|_\infty).$$

Now Theorem 3 is an easy consequence. ∎

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