PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 22, Fasc. 1 (2002), pp. 155–165

THE MAXIMAL $\mathcal{J}$-REGULAR PART OF A q-VARIATE WEAKLY STATIONARY PROCESS

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Abstract. Let x be a q-variate (weakly) stationary process over a locally compact Abelian group G, and $\mathcal{J}$ a family of subsets of G invariant under translation. We show that the set of all regular non-negative Hermitian matrix-valued measures M not exceeding the (non-stochastic) spectral measure of x and such that the Hilbert space $L^2(M)$ is $\mathcal{J}$-regular contains a unique maximal element. Moreover, this maximal element coincides with the spectral measure of the $\mathcal{J}$-regular part of the Wold decomposition of x.

1991 Mathematics Subject Classification: Primary 60G25; Secondary 15A57.

1. INTRODUCTION

Let N be the set of positive integers and $q \in N$. By M_q we denote the algebra of $q \times q$-matrices with entries from the field of complex numbers C and by $M_q^{\geqslant}$ the subset of non-negative Hermitian matrices. The symbol I stands for the unit matrix of M_q.

Let G be a locally compact Abelian group, Γ its dual, and $\langle g, \gamma \rangle$ the value of a character $\gamma \in \Gamma$ on $g \in G$. If J is a subset of G, then a (finite) M_q-linear combination of functions $\langle g, \cdot \rangle I$, $g \in J$, is called a *trigonometric polynomial with frequencies from* J.

Let x be a q-variate (weakly) stationary process over G, and H_x its time domain, i.e. the left Hilbert-M_q-module spanned by the values of x. If $\mathcal{J}$ is a family of subsets of G invariant under translation, then there exists a unique Wold decomposition of x into an orthogonal sum of q-variate stationary processes y and z such that y is $\mathcal{J}$-regular and z is $\mathcal{J}$-singular (cf. [12], Theorem 2.13). It could be expected that, in a certain sense, the process y is the "maximal $\mathcal{J}$-regular part of x". The aim of this note is to specify this statement. To do this it is more convenient to work with the spectral domain instead of the time domain of x.

Let $\mathcal{B}(\Gamma)$ be the σ-algebra of Borel sets of Γ. The (non-stochastic) spectral measure M_x of x (cf. [12], Definition 3.5) is a regular $M_q^{\geqslant}$-valued measure on

$\mathscr{B}(\Gamma)$. Loewner's partial ordering of $M_q^{\geq}$ induces a partial ordering on the set of all regular $M_q^{\geq}$-valued measures on $\mathscr{B}(\Gamma)$. We will show (see Theorem 3.3) that among all regular $M_q^{\geq}$-valued measures M on $\mathscr{B}(\Gamma)$, which do not exceed M_x and for which the space $L^2(M)$ is $\mathscr{J}$-regular, there exists a maximal measure. Moreover, in Section 4 it will be shown that this maximal measure coincides with the spectral measure of the $\mathscr{J}$-regular part y of the Wold decomposition of x. Section 5 deals with an application of our results to the case where $\mathscr{J}$ is the family $\mathscr{J}_0$ of complements of all singletons of G. Using Makagon and Weron's characterization of $\mathscr{J}_0$-regular processes (see [7], Theorem 5.3), we compute the spectral measures of the $\mathscr{J}_0$-regular and $\mathscr{J}_0$-singular parts of the Wold decomposition of x.

2. PRELIMINARIES

For any matrix B with complex entries, denote by B^* its adjoint and by $\mathscr{R}(B)$ its range. For $A \in M_q$, let $\ker A$, $\operatorname{tr} A$, and A^+ be the kernel, trace, and Moore–Penrose inverse of A, respectively. Let P_A be the orthoprojector in the left Hilbert-M_q-module C^q of column vectors of length q onto $\mathscr{R}(A)$. If $A \in M_q^{\geq}$, we denote by $A^{1/2}$ the unique non-negative Hermitian square root of A. We equip $M_q^{\geq}$ with Loewner's partial ordering, i.e. we write $A \leq B$ if and only if $B-A$ is a non-negative Hermitian, $A, B \in M_q^{\geq}$.

We give some more or less known results on $M_q^{\geq}$ and the measurability of M_q-valued functions, which for ease of reference will be stated as lemmas.

LEMMA 2.1. *Let $\mathscr{D}$ be a directed subset of $M_q^{\geq}$, which has an upper bound. Then there exists a least upper bound C of $\mathscr{D}$ and we have*

$$u^* Cu = \sup\{u^* Du:\ D \in \mathscr{D}\}, \qquad u \in C^q.$$

Proof. For $u \in C^q$, set $t(u) := \sup\{u^* Du:\ D \in \mathscr{D}\}$. Obviously, if $\lambda \in C$, we have

$$t(\lambda u) = |\lambda|^2 t(u), \tag{2.1}$$

and if $u, v \in C^q$, we obtain

$$\sup\{u^* Du + v^* Dv:\ D \in \mathscr{D}\} \leq t(u) + t(v). \tag{2.2}$$

Since $\mathscr{D}$ is directed, for $D_1, D_2 \in \mathscr{D}$ there exists $D_3 \in \mathscr{D}$ such that

$$u^* D_1 u + v^* D_2 v \leq u^* D_3 u + v^* D_3 v.$$

This yields

$$t(u) + t(v) \leq \sup\{u^* Du + v^* Dv:\ D \in \mathscr{D}\}. \tag{2.3}$$

The parallelogram identity implies that

$$\text{(2.4)}\qquad \sup\{(u+v)^* D(u+v)+(u-v)^* D(u-v)\colon\ D\in\mathscr{D}\} = 2\sup\{u^* Du+v^* Dv\colon\ D\in\mathscr{D}\}.$$

Combining (2.4), (2.2), and (2.3), we get

$$\text{(2.5)}\qquad t(u+v)+t(u-v) = 2t(u)+2t(v).$$

From (2.1) and (2.5) it follows that there exists $C\in M_q^{\geqslant}$ such that $t(u) = u^* Cu$, $u\in C^q$. From the definition of t it is clear that C is the least upper bound of $\mathscr{D}$.

LEMMA 2.2 (cf. [1], Theorem 1). *Let $p, q\in N$. A block matrix*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \in M_{p+q}$$

belongs to $M_{p+q}^{\geqslant}$ if and only if

(i) $\mathscr{R}(X_{12}^*)\subseteq \mathscr{R}(X_{22})$,

(ii) $X_{22}\in M_q^{\geqslant}$,

(iii) $X_{11}-X_{12}X_{22}^{+}X_{12}^* =: (X/X_{22})$ *is a non-negative Hermitian.*

LEMMA 2.3 (cf. [3], p. 391). *If F is a (Borel) measurable M_q-valued function on Γ, then P_F is measurable. If W is a measurable $M_q^{\geqslant}$-valued function on Γ, then $W^{1/2}$ and W^+ are measurable.*

Let M be a regular $M_q^{\geqslant}$-valued measure on $\mathscr{B}(\Gamma)$ and τ a regular non--negative σ-finite measure on $\mathscr{B}(\Gamma)$ such that M is absolutely continuous with respect to τ. For example, one can take $\tau = \operatorname{tr} M$. Let $W := dM/d\tau$ be the Radon–Nikodym derivative of M with respect to (abbreviated to "w.r.t.") τ. By definition, the left Hilbert-M_q-module $L^2(M)$ consists of (equivalence classes of) measurable M_q-valued functions F on Γ such that

$$\operatorname{tr}\int_\Gamma F(\gamma)\, W(\gamma)\, F(\gamma)^*\, \tau(d\gamma) < \infty.$$

The corresponding scalar product of $L^2(M)$ is defined by

$$\operatorname{tr}\int_\Gamma F(\gamma)\, W(\gamma)\, G(\gamma)^*\, \tau(d\gamma), \qquad F, G\in L^2(M).$$

The definition does not depend on the choice of τ (cf. [10]).

LEMMA 2.4. *Let $F\in L^2(M)$. Then $F = 0$ in $L^2(M)$ if and only if $\mathscr{R}(W)\subseteq \ker F$ τ-a.e.*

Proof. Since $FWF^* = FW^{1/2}(FW^{1/2})^*$, we have $F = 0$ in $L^2(M)$ if and only if $\mathscr{R}(W^{1/2})\subseteq \ker F$ τ-a.e. Since $\mathscr{R}(W^{1/2}) = \mathscr{R}(W)$, the result follows.

If M_x is the spectral measure of a q-variate stationary process x over G, the corresponding space $L^2(M_x)$ is called the *spectral domain* of x. There exists an isometric and isomorphic map V_x of H_x onto $L^2(M_x)$ such that $V_x x_g = \langle g, \cdot\rangle I$,

$g \in G$. The map V_x is called *Kolmogorov's isomorphism*. It enables us to formulate $\mathscr{J}$-regularity and $\mathscr{J}$-singularity of x in terms of $L^2(M_x)$. According to this we call a space $L^2(M)$ *$\mathscr{J}$-regular* or *$\mathscr{J}$-singular* if and only if

$$\bigcap_{J \in \mathscr{J}} \bigvee_M \{\langle g, \cdot \rangle I\colon\ g \in J\} = \{0\} \quad \text{or} \quad \bigvee_M \{\langle g, \cdot \rangle I\colon\ g \in J\} = L^2(M)$$

for all $J \in \mathscr{J}$, respectively. The symbol $\bigvee_M$ stands for the closed M_q-linear hull in $L^2(M)$. We simply write $\bigvee$ if $M = M_x$ is the spectral measure of the process x.

3. THE MAXIMAL $\mathscr{J}$-REGULAR PART

Let M_x be the spectral measure of a q-variate stationary process over G, $\tau_x := \operatorname{tr} M_x$, and $W_x := dM_x/d\tau_x$. In the sequel, all relations between measurable functions on Γ are to be understood as relations which hold true τ_x-a.e.

Let $\mathscr{W}_x$ be the set of all measurable $M_q^{\geqslant}$-valued functions W on Γ such that $W \leqslant W_x$ and let $\tilde{\mathscr{W}}_x$ be the set of all $M_q^{\geqslant}$-valued measures of the form $W d\tau_x$, $W \in \mathscr{W}_x$. The partial ordering on $\mathscr{W}_x$ induces a partial ordering on $\tilde{\mathscr{W}}_x$: define $W_1 d\tau_x \leqslant W_2 d\tau_x$ if and only if $W_1 \leqslant W_2$, $W_1, W_2 \in \mathscr{W}_x$. Note that for $M_1, M_2 \in \tilde{\mathscr{W}}_x$ we have $M_1 \leqslant M_2$ if and only if $M_1(\Delta) \leqslant M_2(\Delta)$, $\Delta \in \mathscr{B}(\Gamma)$.

LEMMA 3.1. *For any directed subset $\mathscr{D}$ of $\mathscr{W}_x$, there exists a least upper bound.*

Proof. According to the remarks preceding the lemma it is enough to show that the subset $\tilde{\mathscr{D}} := \{W d\tau_x\colon\ W \in \mathscr{D}\}$ of $\tilde{\mathscr{W}}_x$ has the least upper bound. For $\Delta \in \mathscr{B}(\Gamma)$, let $\mathscr{D}_\Delta$ be the set of matrices of the form

$$\sum_{j=1}^{n} M_j(\Delta_j), \tag{3.1}$$

where $M_1, \ldots, M_n \in \tilde{\mathscr{D}}$, and $\{\Delta_1, \ldots, \Delta_n\}$ is a partition of Δ, $n \in N$. The matrix $M_x(\Delta)$ is an upper bound of $\mathscr{D}_\Delta$. Moreover, $\mathscr{D}_\Delta$ is a directed set. In fact, if (3.1) and

$$\sum_{k=1}^{m} M'_k(\Delta'_k) \tag{3.2}$$

are two elements of $\mathscr{D}_\Delta$, consider $M_{jk} \in \tilde{\mathscr{D}}$ such that $M_j \leqslant M_{jk}$, $M'_k \leqslant M_{jk}$, $j = 1, \ldots, n$, $k = 1, \ldots, m$. Then $\sum_{j=1}^{n} \sum_{k=1}^{m} M_{jk}(\Delta_j \cap \Delta'_k)$ belongs to $\mathscr{D}_\Delta$ and exceeds both matrices (3.1) and (3.2). From Lemma 2.1 it follows that $\mathscr{D}_\Delta$ has the least upper bound $N(\Delta)$ and that

$$u^* N(\Delta) u = \sup\{u^* D u\colon\ D \in \mathscr{D}_\Delta\}, \qquad u \in C^q. \tag{3.3}$$

Standard measure-theoretic arguments (cf. the proof of Theorem 5 of Section III.7 of [2]) show that, for $u \in C^q$, $u^* N u$ is an additive function on $\mathscr{B}(\Gamma)$. Hence

N is additive. Since $N \leqslant M_x$, it even belongs to $\tilde{\mathscr{W}}_x$. Finally, from (3.3) it follows easily that N is the least upper bound of $\tilde{\mathscr{D}}$.

If $W \in \mathscr{W}_x$, set $L^2(W) := L^2(W d\tau_x)$. Moreover, we define

$$\mathscr{W}_x^{(r)} := \{W \in \mathscr{W}_x \colon L^2(W) \text{ is } \mathscr{J}\text{-regular}\}.$$

LEMMA 3.2. *The set $\mathscr{W}_x^{(r)}$ is directed.*

Proof. Let $W_1, W_2 \in \mathscr{W}_x^{(r)}$ and let $Q(\gamma)$ be the orthogonal projection in C^q onto the algebraic sum $\mathscr{R}(W_1(\gamma)) + \mathscr{R}(W_2(\gamma))$, $\gamma \in \Gamma$. From von Neumann's alternating projections theorem (cf. [4], Problem 96) we can conclude the measurability of the function Q. Let

$$W_x = \begin{pmatrix} W_{x,11} & W_{x,12} \\ W_{x,12}^* & W_{x,22} \end{pmatrix}$$

be the block partition of W_x w.r.t. the orthogonal decomposition

$$C^q = QC^q \oplus (I-Q)C^q.$$

Let us set

$$W_3 := \begin{pmatrix} W_{x,11} - W_{x,12} W_{x,22}^+ W_{x,12}^* & 0 \\ 0 & 0 \end{pmatrix}.$$

The measurability of Q and Lemmas 2.3 and 2.2 imply that $W_3 \in \mathscr{W}_x$. Moreover, from Lemma 2.2 it follows that $W_1 \leqslant W_3$ and $W_2 \leqslant W_3$. To complete the proof it is enough to show that $L^2(W_3)$ is $\mathscr{J}$-regular. Let $F \in L^2(W_3)$ be such that for each $J \in \mathscr{J}$ it can be approximated by trigonometric polynomials with frequencies from J in $L^2(W_3)$. Since $W_1 \leqslant W_3$, an analogous approximation exists in $L^2(W_1)$. The $\mathscr{J}$-regularity of $L^2(W_1)$ yields $F = 0$ in $L^2(W_1)$. Similarly, $F = 0$ in $L^2(W_2)$. Using Lemma 2.4, we can conclude that $\mathscr{R}(W_1) + \mathscr{R}(W_2) \subseteq \ker F$. Since $\mathscr{R}(W_3) \subseteq \mathscr{R}(W_1) + \mathscr{R}(W_2)$, it follows that $F = 0$ in $L^2(W_3)$.

THEOREM 3.3. *The set $\mathscr{W}_x^{(r)}$ has a unique maximal element.*

Proof. By Lemmas 3.1 and 3.2, the set $\mathscr{W}_x^{(r)}$ has the least upper bound $W^{(r)} \in \mathscr{W}_x$. Assume that $L^2(W^{(r)})$ is not $\mathscr{J}$-regular. Then there exists $F \in L^2(W^{(r)})$, $F \neq 0$, such that, for each $J \in \mathscr{J}$, F can be approximated by trigonometric polynomials with frequencies from J. Let $W \in \mathscr{W}_x^{(r)}$. Then, in particular, $W \leqslant W^{(r)}$, and similar arguments to those in the proof of Lemma 3.2 show that

$$\mathscr{R}(W) \subseteq \ker F. \tag{3.4}$$

Let

$$W^{(r)} = \begin{pmatrix} W_{11}^{(r)} & W_{12}^{(r)} \\ W_{12}^{(r)*} & W_{22}^{(r)} \end{pmatrix}$$

be the block partition of $W^{(r)}$ w.r.t. the orthogonal decomposition $C^q = \mathscr{R}(F^*) \oplus \ker F$. Let us set

$$W^{(\varrho)} := \begin{pmatrix} W_{11}^{(r)} - W_{12}^{(r)} W_{22}^{(r)+} W_{12}^{(r)*} & 0 \\ 0 & 0 \end{pmatrix}.$$

It is not hard to see (cf. the proof of Lemma 3.2) that

$$W^{(\varrho)} \in \mathscr{W}_x, \quad W^{(\varrho)} \leqslant W^{(r)}, \quad \text{and} \quad W \leqslant W^{(\varrho)}. \tag{3.5}$$

On the other hand, since $F \neq 0$ in $L^2(W^{(r)})$, Lemma 2.4 implies that there exists $\Delta \in \mathscr{B}(\Gamma)$ such that $\tau_x(\Delta) > 0$ and $\mathscr{R}(W^{(r)})$ is not a subspace of $\ker F$ on Δ. It follows that $W_{22}^{(r)} \neq 0$ on Δ, and hence $W^{(r)} - W^{(\varrho)} \neq 0$ on Δ. Combining this with (3.5), we obtain a contradiction to the definition of a least upper bound. Thus, $L^2(W^{(r)})$ is $\mathscr{I}$-regular and $W^{(r)}$ is a maximal element of $\mathscr{W}_x^{(r)}$. Its uniqueness follows from Lemma 3.2.

4. CONCORDANCE OF THE MAXIMAL REGULAR PART AND THE REGULAR PART OF THE WOLD DECOMPOSITION

Let x be a q-variate stationary process over G and $\mathscr{I}$ a family of subsets of G invariant under translation. Let $x_g = y_g + z_g$, $g \in G$, be the Wold decomposition of x, where y is $\mathscr{I}$-regular and z is $\mathscr{I}$-singular. If we set $\tilde{W}_y := dM_y/d\tau_x$ and $\tilde{W}_z := dM_z/d\tau_x$, we have (cf. [9], Lemmas 4.3 and 4.4)

$$\tilde{W}_y + \tilde{W}_z = W_x, \quad \mathscr{R}(\tilde{W}_y) \cap \mathscr{R}(\tilde{W}_z) = \{0\}. \tag{4.1}$$

Let V_x be Kolmogorov's isomorphism of H_x onto $L^2(W_x)$ and set

$$V_x y_0 =: F_y \quad \text{and} \quad V_x z_0 =: F_z,$$

where 0 is the neutral element of G. It is not hard to see that

$$F_y W_x F_y^* = \tilde{W}_y \quad \text{and} \quad F_z W_x F_z^* = \tilde{W}_z, \tag{4.2}$$

$$V_x H_y = \bigvee \{\langle g, \cdot \rangle F_y \colon g \in G\} \quad \text{and} \quad V_x H_z = \bigvee \{\langle g, \cdot \rangle F_z \colon g \in G\}, \tag{4.3}$$

and hence

$$\bigvee \{\langle g, \cdot \rangle F_y \colon g \in G\} \oplus \bigvee \{\langle g, \cdot \rangle F_z \colon g \in G\} = L^2(W_x). \tag{4.4}$$

From the relation (4.4) it follows that

$$\int_\Gamma \langle g, \gamma \rangle F_y(\gamma) W_x(\gamma) F_z(\gamma)^* \tau_x(d\gamma) = 0, \quad g \in G,$$

which yields

$$F_y W_x F_z^* = 0. \tag{4.5}$$

We can assume and we will do so in the sequel that

$$\ker W_x \subseteq (\ker F_y \cap \ker F_z). \tag{4.6}$$

Then we have

$$P_{W_x} = F_y + F_z \tag{4.7}$$

as well as

$$\mathscr{R}(F_y W_x) = \mathscr{R}(F_y W_x^{1/2}) = \mathscr{R}(F_y) \text{ and } \mathscr{R}(F_z W_x) = \mathscr{R}(F_z W_x^{1/2}) = \mathscr{R}(F_z). \tag{4.8}$$

Moreover, from (4.2) it follows that $\mathscr{R}(\tilde{W}_y^{1/2}) = \mathscr{R}(F_y W_x^{1/2})$ and $\mathscr{R}(\tilde{W}_z^{1/2}) = = \mathscr{R}(F_z W_x^{1/2})$. Combining this with (4.8), we obtain

$$\mathscr{R}(\tilde{W}_y) = \mathscr{R}(F_y) \quad \text{and} \quad \mathscr{R}(\tilde{W}_z) = \mathscr{R}(F_z). \tag{4.9}$$

Let $W^{(r)}$ be the maximal element of $\mathscr{W}_x^{(r)}$. We wish to show that $W^{(r)}$ coincides with $\tilde{W}_y$. In order to prove this we first derive some properties of F_z, which eventually lead to the conclusion that $P_{F_z^*} = 0$ in $L^2(W^{(r)})$. Then we will see that the assumption $W^{(r)} \neq \tilde{W}_y$ would imply that $P_{F_z^*} \neq 0$ in $L^2(W^{(r)})$.

LEMMA 4.1. *The values of F_z are diagonalizable matrices.*

Proof. From (4.5) and (4.7) it follows that

$$F_z W_x F_z^* = W_x F_z^*. \tag{4.10}$$

Since $\mathscr{R}(F_z W_x F_z^*) = \mathscr{R}(F_z W_x^{1/2})$, from (4.8) and (4.10) we obtain

$$\mathscr{R}(F_z) = \mathscr{R}(W_x F_z^*) \subseteq \mathscr{R}(W_x). \tag{4.11}$$

On the other hand, (4.6) gives

$$\mathscr{R}(F_z^*) \subseteq \mathscr{R}(W_x). \tag{4.12}$$

The relations (4.11) and (4.12) show that it is enough to prove that the restrictions $\bar{F}_z$ of F_z to $\mathscr{R}(W_x)$ are diagonalizable. Denoting by $\bar{W}_x$ the restrictions of W_x to $\mathscr{R}(W_x)$, from (4.10)–(4.12) we get $\bar{F}_z \bar{W}_x \bar{F}_z^* = \bar{W}_x \bar{F}_z^*$, which yields

$$\bar{W}_x^{-1/2} \bar{F}_z \bar{W}_x \bar{F}_z^* \bar{W}_x^{-1/2} = \bar{W}_x^{1/2} \bar{F}_z^* \bar{W}_x^{-1/2}.$$

This shows that the values of $\bar{F}_z^*$, and hence of $\bar{F}_z$, are similar to self-adjoint matrices, which implies that they are diagonalizable.

LEMMA 4.2. *We have* $\ker P_{F_z^*} \cap \mathscr{R}(F_z) = \{0\}$.

Proof. Let $\gamma \in \Gamma$ and $u \in (\ker P_{F_z(\gamma)^*}) \cap \mathscr{R}(F_z(\gamma))$. Then $u \in \ker F_z(\gamma)$, $u = F_z(\gamma) v$ for some $v \in C^q$, and hence $F_z(\gamma)^2 v = 0$. If $u \neq 0$ were true, this would contradict Lemma 4.1.

LEMMA 4.3. *We have* $P_{F_z^*} = 0$ *in* $L^2(W^{(r)})$.

Proof. From (4.5) we get $F_y W_x P_{F_z^*} = 0$. This implies that the function $P_{F_z^*}$ is orthogonal (in $L^2(W_x)$) to $\bigvee\{\langle g, \cdot\rangle F_y\colon g \in G\}$. Examining the proof of the Wold decomposition (cf. the proof of Theorem 2.13 of [12]) and taking into account Kolmogorov's isomorphism, we obtain

$$\bigvee\{\langle g, \cdot\rangle F_z\colon g \in G\} = \bigcap_{J \in \mathscr{J}} \bigvee\{\langle g, \cdot\rangle I\colon g \in J\}.$$

It follows that, for $J \in \mathscr{J}$, $P_{F_z^*}$ can be approximated by trigonometric polynomials with frequencies from J in $L^2(W_x)$. Since $W^{(r)} \leqslant W_x$, an analogous result is true for $L^2(W^{(r)})$. But since $L^2(W^{(r)})$ is $\mathscr{J}$-regular, we conclude that $P_{F_z^*} = 0$ in $L^2(W^{(r)})$.

THEOREM 4.4. *The functions $W^{(r)}$ and $\tilde{W}_y$ coincide.*

Proof. Since $\tilde{W}_y \in \mathscr{W}_x^{(r)}$, it follows that $\tilde{W}_y \leqslant W^{(r)}$. Assume that $\tilde{W}_y \neq W^{(r)}$ on a set $\Delta \in \mathscr{B}(\Gamma)$ such that $\tau_x(\Delta) > 0$. First note that $\mathscr{R}(\tilde{W}_y) \neq \mathscr{R}(W^{(r)})$ on Δ. For if $\mathscr{R}(\tilde{W}_y) = \mathscr{R}(W^{(r)})$ and $\tilde{W}_y \neq W^{(r)}$ were true on a set of positive measure τ_x, we would get $\mathscr{R}(W^{(r)} - \tilde{W}_y) \cap \mathscr{R}(\tilde{W}_y) \neq \{0\}$, and because of $\mathscr{R}(\tilde{W}_z) = \mathscr{R}(W_x - \tilde{W}_y) \supseteq \mathscr{R}(W^{(r)} - \tilde{W}_y)$ also $\mathscr{R}(\tilde{W}_z) \cap \mathscr{R}(\tilde{W}_y) \neq \{0\}$, which contradicts (4.1). Thus, $\mathscr{R}(\tilde{W}_y)$ is a proper subspace of $\mathscr{R}(W^{(r)})$ on Δ. Then from (4.1) and (4.9) it follows that $\mathscr{R}(W^{(r)}) \cap \mathscr{R}(F_z) \neq \{0\}$ on Δ. Combining this with Lemma 4.2, we infer that $\mathscr{R}(W^{(r)})$ is not a subspace of $\ker P_{F_z^*}$ on Δ. Applying Lemma 2.4, we conclude that $P_{F_z^*} \neq 0$ in $L^2(W^{(r)})$, which is a contradiction to Lemma 4.3.

Let us mention the following consequence of Theorem 4.4.

COROLLARY 4.5. *If $L^2(W_x)$ is $\mathscr{J}$-singular, then for $W \in \mathscr{W}_x$ so is $L^2(W)$.*

Proof. The $\mathscr{J}$-singularity of $L^2(W_x)$ and Theorem 4.4 imply that $\mathscr{W}_x^{(r)} = \{0\}$. For $W \in W_x$, consider the Wold decomposition of the corresponding stationary process over G. Since the spectral measure of its $\mathscr{J}$-regular part belongs to $\mathscr{W}_x^{(r)}$, it is zero measure. Thus, $L^2(W)$ is $\mathscr{J}$-singular.

Remark 4.6. It would be of interest to have generalizations of Theorem 4.4 to the infinite-variate case. Treil' ([13], Theorem 3.1) gave such a result if G is the group of integers and $\mathscr{J}$ is the family of translates of the set of non-negative integers.

5. THE MAXIMAL $\mathscr{J}_0$-REGULAR PART

Let G be a discrete Abelian group, $\mathscr{J}_0$ the family of complements of all singletons of G, and σ the normalized Haar measure of Γ. Let M_x be the spectral measure of a q-variate stationary process over G.

THEOREM 5.1 ([7], Theorem 5.3). *The space $L^2(M_x)$ is $\mathscr{J}_0$-regular if and only if*

(i) *M_x is absolutely continuous w.r.t. σ,*

(ii) $\mathscr{R}(dM_x/d\sigma) = \text{const}$ *σ-a.e.*,
(iii) $(dM_x/d\sigma)^+$ *is integrable w.r.t.* σ.

It follows that the maximal $\mathscr{J}_0$-regular parts of M_x and of the absolutely continuous part of M_x coincide. Thus we can assume that M_x is absolutely continuous w.r.t. σ and replace the measure τ_x of the preceding sections by σ. For simplicity, now denote by W_x the function $W_x = dM_x/d\sigma$ and according to this notation define the corresponding objects $\mathscr{W}_x$ etc. of Sections 3 and 4.

Let us set

$$L_1 := \{u \in C^q\colon\ u^* W_x^+ u \text{ is integrable w.r.t. } \sigma\},$$

$$L_2 := \{u \in C^q\colon\ u \in \mathscr{R}(W_x)\ \sigma\text{-a.e.}\},$$

$$L := L_1 \cap L_2 .$$

Remark 5.2. Note that the space L coincides with the space $\mathscr{M}$ which appeared in Theorem 4.5 of [6] and was identified there as the range of the Grammian interpolation error matrix. Note further that L is the orthogonal complement of the space H of Lemma 9 of [5].

Let

$$W_x = \begin{pmatrix} W_{x,11} & W_{x,12} \\ W_{x,12}^* & W_{x,22} \end{pmatrix}$$

be the block representation of W_x w.r.t. the orthogonal decomposition $C^q = L \oplus L^\perp$. Set

$$W^{(r)} := \begin{pmatrix} (W_x/W_{x,22}) & 0 \\ 0 & 0 \end{pmatrix}, \qquad W^{(s)} := \begin{pmatrix} W_{x,12} W_{x,22}^+ W_{x,12}^* & W_{x,12} \\ W_{x,12}^* & W_{x,22} \end{pmatrix}.$$

Using Theorem 4.4 we will show that $W^{(r)}\,d\sigma$ is the spectral measure of the $\mathscr{J}_0$-regular part of the Wold decomposition of x, and hence $W^{(s)} = W_x - W^{(r)}$ is the spectral measure of the $\mathscr{J}_0$-singular part.

LEMMA 5.3. *The spaces* $\mathscr{R}(W^{(r)})$ *are equal to* L *σ-a.e.*

Proof. Clearly, $\mathscr{R}(W^{(r)}) \subseteq L$. On the other hand, $L \subseteq \mathscr{R}(W_x) = \mathscr{R}(W^{(r)}) + \mathscr{R}(W^{(s)})$. Thus, if L were not a subspace of $\mathscr{R}(W^{(r)})$, we would have $\mathscr{R}(W^{(s)}) \cap L \neq \{0\}$. However, using (i) of Lemma 2.2 we easily get $\mathscr{R}(W^{(s)}) \cap L = \{0\}$. It follows that $\mathscr{R}(W^{(r)}) = L$ σ-a.e.

LEMMA 5.4. *The function* $W^{(r)+}$ *is integrable w.r.t.* σ.

Proof. Since $L \subseteq \mathscr{R}(W_x)$, we have $\ker W_x \subseteq L^\perp$, and taking into account (i) of Lemma 2.2 we easily obtain $\ker W_x = \ker W_{22}$. Thus the generalized Banachiewicz inversion formula (cf. [8], formula (3.32)) is applicable, which implies that the left upper corner of W_x^+ is equal to $(W_x/W_{x,22})^{-1}$. From the definition of L it follows that $(W_x/W_{x,22})^{-1}$ is integrable w.r.t. σ and so is $W^{(r)+}$.

LEMMA 5.5. *The space $L^2(W^{(r)})$ is $\mathcal{J}_0$-regular.*

Proof. The result follows immediately from Theorem 5.1 and Lemmas 5.3 and 5.4.

LEMMA 5.6. *Let $W \in \mathscr{W}_x^{(r)}$. Then $W \leqslant W^{(r)}$.*

Proof. According to Theorem 5.1 there exists a subspace L_0 of C^q such that $\mathscr{R}(W) = L_0$ σ-a.e. Assume that $u \in L_0 \cap L^\perp$, $u \neq 0$. Then u can be written as $u = u_1 + u_2$ for some $u_1 \in L_1^\perp$, $u_2 \in L_2^\perp$. If $u_1 = 0$, there exists $\Delta \in \mathscr{B}(\Gamma)$ such that $\sigma(\Delta) > 0$ and $u = u_2 \notin \mathscr{R}(W_x(\gamma))$ for σ-a.a. $\gamma \in \Delta$. This contradicts the inclusion $\mathscr{R}(W) \subseteq \mathscr{R}(W_x)$ σ-a.e. It follows that $u_1 \neq 0$, and hence $u \notin L_1$. From the definition of L_1 we infer that $u^* W_x^+ u$ is not integrable w.r.t. σ. Let W_0 be the restriction of W to L_0 and let

$$W_x = \begin{pmatrix} W_{x,11}^{(0)} & W_{x,12}^{(0)} \\ W_{x,12}^{(0)*} & W_{x,22}^{(0)} \end{pmatrix}$$

be the block representation of W_x w.r.t. the orthogonal decomposition $C^q = L_0 \oplus L_0^\perp$. From the definition of $\mathscr{W}_x^{(r)}$ and (iii) of Lemma 2.2 we obtain $W_0 \leqslant (W_x/W_{x,22}^{(0)})$, and hence $(W_x/W_{x,22}^{(0)})^{-1} \leqslant W_0^{-1}$. By the generalized Banachiewicz inversion formula it follows that

$$u^* W_x^+ u \leqslant u^* (W_x/W_{x,22}^{(0)})^{-1} u \leqslant u^* W_0^{-1} u = u^* W^+ u.$$

Thus $u^* W^+ u$ is not integrable w.r.t. σ, which contradicts Theorem 5.1. We conclude that $L_0 \subseteq L$. Then again the definition of $\mathscr{W}_x^{(r)}$ and (iii) of Lemma 2.2 imply that the restriction of W to L does not exceed $(W_x/W_{x,22})$, which yields $W \leqslant W^{(r)}$ σ-a.e.

Combining Lemmas 5.5 and 5.6 with Theorem 4.4 we get the following result.

THEOREM 5.7. *The measures $W^{(r)}d\sigma$ and $W^{(s)}d\sigma$ are the spectral measures of the $\mathcal{J}_0$-regular and $\mathcal{J}_0$-singular parts of the Wold decomposition of x, respectively.*

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Received on 6.12.2001