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ON THE UNCONDITIONAL BUNDLE CONVERGENCE IN L_2-SPACE OVER A VON NEUMANN ALGEBRA

BY

ADAM SKALSKI (ŁÓDŹ)

Abstract. The Tandori theorem concerning the sufficient condition for the unconditional a.e. convergence of orthogonal series is generalized for the bundle convergence in L_2-space over a σ-finite von Neumann algebra. The result implies a noncommutative version of the Orlicz theorem proved earlier by Hensz, Jajte and Paszkiewicz.

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1. INTRODUCTION

The paper is devoted to proving the strongest classical theorem on the unconditional almost everywhere (a.e.) convergence of the orthogonal series, namely the Tandori theorem, in the context of operator algebras. The idea of the generalization of standard results to noncommutative setting derives from the algebraic approach to quantum statistical mechanics. Obviously, such a generalization demands defining counterparts of pointwise convergence in the case of von Neumann algebras.

There are several concepts of formulating such definitions (see e.g. [6], [1], [9], [2]). Apparently the most satisfactory notion, enjoying the additivity property (and equivalent to the notion of a.e. convergence when the considered algebra is just L_∞ over some probability space), was introduced in 1996 by Hensz et al. in [4] and called the *bundle convergence*. In [4] and in later publications by other authors (see e.g. [7]) several results concerning the bundle convergence in a von Neumann algebra and in its L_2-space were obtained.

The Tandori theorem [10] gives the weakest condition sufficient for unconditional a.e. convergence of the orthogonal series in $L_2(0, 1)$. As such, it is a generalization of a much earlier result by Orlicz [8]. The Orlicz theorem for the bundle convergence was proved in the above-mentioned paper [4].

It is worth mentioning that Tandori's theorem for a.s. convergence introduced in L_2-space over a von Neumann algebra by Hensz and Jajte in [2] was proved by them jointly with Paszkiewicz in [3]. However, although both notions coincide in the classical case, generally the a.s. convergence seems to be weaker than the bundle convergence (in the sense precisely described in the next section).

The general treatment on the probability in noncommutative L_2-spaces can be found in the monograph by Jajte [5].

2. BASIC NOTIONS AND DEFINITIONS

Let M be a σ-finite von Neumann algebra with a faithful and normal state Φ. We assume that M acts in a standard way in its *GNS* representation Hilbert space, denoted further by H. This means that $H = L_2(M, \Phi)$ (the completion of M under the norm $\|x\|_2 = \Phi(x^* x)^{1/2}$, $x \in M$). Moreover, there exists a cyclic and separating vector $\Omega \in H$ such that, for all $x \in M$, $\Phi(x) = (x\Omega, \Omega)$. The set of all projections in M will be denoted by $\mathbf{Proj}M$, the set of all positive operators in M by M^+. We will also write $p^\perp = \mathbf{1} - p$ for $p \in \mathbf{Proj}M$, $|x|^2 = x^* x$ for $x \in M$ (obviously, $|x|^2 \in M^+$). Finally, the norm in H will be denoted by $\|\cdot\|$, the operator norm in M by $\|\cdot\|_\infty$.

The following definitions were introduced in [4]:

Definition 2.1. Let $(D_m)_{m=1}^\infty$ be a sequence of operators from M^+ such that $\sum_{m=1}^\infty \Phi(D_m) < \infty$. The *bundle* (determined by the sequence $(D_m)_{m=1}^\infty$) is the set

$$\mathscr{P}_D = \{p \in \mathbf{Proj}M\colon \sup_m \big\|p\big(\sum_{k=1}^m D_k\big)p\big\|_\infty < \infty \text{ and } \|pD_m p\|_\infty \xrightarrow{m\to\infty} 0\}.$$

The crucial fact concerning so defined bundles is that every bundle contains projections arbitrarily close to $\mathbf{1}$. This allowed to formulate the following equivalents of the a.e. convergence in M and in H:

Definition 2.2. Let $(x_n)_{n=1}^\infty$ be a sequence of operators from M, $x \in M$. We say that $(x_n)_{n=1}^\infty$ is *bundle convergent to* x (and write $x_n \xrightarrow{b,M} x$) if there exists a bundle $\mathscr{P}_D$ such that for each $p \in \mathscr{P}_D$ we have $\|p(x_n - x)p\|_\infty \to 0$ as $n \to \infty$.

Definition 2.3. Let $(\xi_n)_{n=1}^\infty$ be a sequence of vectors from H, $\xi \in H$. We say that $(\xi_n)_{n=1}^\infty$ is *bundle convergent to* ξ (and write $\xi_n \xrightarrow{b} \xi$) if there exists a sequence $(x_n)_{n=1}^\infty$ of operators from M such that $\sum_{n=1}^\infty \|\xi_n - \xi - x_n\Omega\|^2 < \infty$ and $x_n \xrightarrow{b,M} 0$.

As the intersection of two bundles is a bundle again, introduced above convergences are additive. The proof of the equivalence of the bundle convergence and a.e. convergence in the commutative case ($M = L_\infty(X, \mathscr{F}, \mu)$) may be found in [4].

Now we recall, mentioned in the Introduction, notion of the a.s. convergence in H, proposed in [2]. For each $\xi \in H$ and $p \in \mathbf{Proj} M$ we put

$$\|\xi\|_p = \inf\{\|\sum_{k=1}^{\infty} x_k p\|_{\infty}\colon (x_k)_{k=1}^{\infty} \text{ is a sequence of operators from } M,$$
$$\xi = \sum_{k=1}^{\infty} x_k \Omega \text{ in } H,\ \sum_{k=1}^{\infty} x_k p \text{ converges in norm in } M\}$$

(with the natural convention that $\inf \emptyset = \infty$).

DEFINITION 2.4. Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of vectors from H, $\xi \in H$. We say that $(\xi_n)_{n=1}^{\infty}$ is *almost surely convergent to* ξ (and write $\xi_n \to \xi$ a.s.) if for every $\varepsilon > 0$ there is $p \in \mathbf{Proj} M$ such that $\Phi(p^{\perp}) < \varepsilon$ and $\|\xi_n - \xi\|_p \to 0$ as $n \to \infty$.

As already mentioned, in the noncommutative setting the a.s. convergence seems to be weaker than the bundle convergence. More precisely, the bundle convergence of any sequence of vectors from H to 0 implies its a.s. convergence to 0 (see [4]).

We say that the series $\sum_{n=1}^{\infty} \xi_n$ of vectors from H is *unconditionally bundle convergent* if the series $\sum_{n=1}^{\infty} \xi_{\pi(n)}$ is bundle convergent for each $\pi\colon N \xrightarrow{1\text{-}1} N$. All of the logarithms in the paper are to the base 2. We will frequently write N_0 for $N \cup \{0\}$ and use the notation $I_k = \{2^{2^k}+1, \ldots, 2^{2^{k+1}}\}$ for $k \in N$, $I_0 = \{1, 2, 3, 4\}$.

3. MAIN RESULT

In this section we formulate the main theorem of the paper:

THEOREM 3.1. *Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of pairwise orthogonal vectors in $H = L^2(M, \Phi)$ and*

$$\sum_{k=0}^{\infty} \Big(\sum_{n \in I_k} \|\xi_n\|^2 \log^2 (n+1)\Big)^{1/2} < \infty. \tag{1}$$

Then the series $\sum_{n=1}^{\infty} \xi_n$ is unconditionally bundle convergent.

This theorem, exactly as in the classical case, implies the following version of the theorem by Orlicz, formulated and proved in [4]:

THEOREM 3.2. *Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of pairwise orthogonal vectors in $H = L^2(M, \Phi)$. Moreover, let $(w_n)_{n=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that for some constant $c > 1$ and some strictly increasing sequence of positive integers $(\nu_m)_{m=1}^{\infty}$ satisfying*

$$\log \nu_{m+1} \leqslant c \log \nu_m \qquad (m = 1, 2, \ldots)$$

the condition

$$\sum_{m=1}^{\infty} \frac{1}{w_{\nu_m}} < \infty$$

holds. If

$$\sum_{n=1}^{\infty} w_n \log^2(n+1) \|\xi_n\|^2 < \infty,$$

then the series $\sum_{n=1}^{\infty} \xi_n$ *is unconditionally bundle convergent.*

For the sake of completeness we include in the Appendix a simple combinatorial lemma showing the implication mentioned before the last theorem.

4. AUXILIARY LEMMAS

For the proof of the main theorems some known lemmas describing some of the properties of operators in a von Neumann algebra M and vectors in its L_2-space will be helpful. The first lemma of Rademacher–Menchoff type originates from [3] (it was proved earlier in a slightly different version in [2]):

LEMMA 4.1. *Let* $J \subset N$ *and* $\mu = \# J$. *Let* $(\eta_i)_{i=1}^{\infty}$ *be a sequence of pairwise orthogonal elements in* H *such that* $\eta_i = 0$ *for* $i \notin J$. *Let* $(\varepsilon_i)_{i=1}^{\infty}$ *be a sequence of positive numbers. Then there exist operators* $B \in M^+$ *and* $y_i \in M$ $(i \in N)$ *such that* $y_i = 0$ *when* $\eta_i = 0$ *and*

$$\|\eta_i - y_i \Omega\| < \varepsilon_i, \ i \in N, \quad \Big|\sum_{i=1}^{n} y_i\Big|^2 \leqslant B, \ n \in N,$$

$$\Phi(B) \leqslant 2(1+\log\mu)^2 \sum_{i=1}^{\infty} \|\eta_i\|^2.$$

The next lemma, being actually a generalization of the standard Schwarz inequality for operators, was published by Hensz et al. [3].

LEMMA 4.2. *Let* $\varepsilon_k > 0$, $x_k \in M$, $E_k \in M^+$, *and* $|x_k|^2 \leqslant \varepsilon_k E_k$ *for each* $k \in \{1, \ldots, n\}$. *Then*

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{\infty}^2 \leqslant \Big\|\sum_{k=1}^{n} E_k\Big\|_{\infty} \Big(\sum_{k=1}^{n} \varepsilon_k\Big).$$

We will also use the following properties of bundle convergence noticed in [4]:

PROPERTY 4.3. *Let* $(x_n)_{n=1}^{\infty}$ *be a sequence of operators from* M. *Then* $x_n \xrightarrow{b,M} 0$ *if and only if* $x_n \Omega \xrightarrow{b} 0$.

PROPERTY 4.4. *Let* $(\xi_n)_{n=1}^{\infty}$ *be a sequence of vectors from* H. *Then* $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ *implies* $\xi_n \Omega \xrightarrow{b} 0$.

5. PROOF

Proof of Theorem 3.1. Let us fix the permutation $\pi\colon N \xrightarrow{1\text{-}1} N$. As the condition (1) implies the absolute convergence of the series $\sum_{k=0}^{\infty} \xi_k$, the series $\sum_{k=0}^{\infty} \xi_{\pi(k)}$ is norm convergent. The following notation will be used ($n, i \in N$, $k \in N_0$):

$$\sigma_n = \sum_{j=1}^{n} \xi_{\pi(j)}, \quad \sigma = \sum_{j=1}^{\infty} \xi_{\pi(j)}, \quad J_k = \pi^{-1}(I_k), \quad \alpha_k = \Big(\sum_{n \in I_k} \|\xi_n\|^2 \log^2(n+1)\Big)^{1/2},$$

$$\varepsilon_i^k = \begin{cases} 1 \text{ when } i \in J_k, \\ 0 \text{ when } i \notin J_k, \end{cases} \quad \eta_i^k = \alpha_k^{-1/2} \varepsilon_i^k \xi_{\pi(i)},$$

$$j(n) = \min\{j \in N\colon \{\pi(1), \ldots, \pi(n)\} \subset I_0 \cup \ldots \cup I_j\},$$

$$k(n) = \max\big(\{k \in N_0\colon I_0 \cup \ldots \cup I_k \subset \{\pi(1), \ldots, \pi(n)\}\} \cup \{-1\}\big)$$

(without loss of generality we can assume that, for all $k \in N_0$, $\alpha_k \neq 0$). By a proper choice of the strictly increasing sequence of positive integers $(m_n)_{n=1}^{\infty}$ we can obtain the condition $\sum_{n=1}^{\infty} \|\sigma - \sigma_{m_n}\|^2 < \infty$, which by Property 4.4 implies

$$\sigma - \sigma_{m_n} \xrightarrow{b} 0. \tag{2}$$

Further, the modification of the idea of proof from [3] will be employed. We can consequently apply Lemma 4.1 to all sequences of vectors $(\eta_i^k)_{i=1}^{\infty}$. Then there exist sequences of operators: $(D_k)_{k=0}^{\infty}$ in M^+ and $(y_i^k)_{i=1}^{\infty}$ (for each $k \in N_0$) in M, such that

$$\|\eta_i^k - y_i^k \Omega\| \leqslant \frac{1}{2^{i+1}} \cdot \alpha_k^{-1/2} \text{ for } i \in J_k, \quad y_i^k = 0 \text{ for } i \notin J_k, \tag{3}$$

$$|s_l^k|^2 \leqslant D_k \text{ for } l \in N \quad \Big(s_l^k = \sum_{j=1}^{l} y_j^k\Big) \tag{4}$$

and

$$\Phi(D_k) \leqslant 2\big(1 + \log(\# J_k)\big)^2 \sum_{i=1}^{\infty} \|\eta_i^k\|^2.$$

The last inequality gives

$$\Phi(D_k) \leqslant 2\alpha_k^{-1}\big(1 + \log(2^{2^{k+1}})\big)^2 \sum_{i \in J_k} \|\xi_{\pi(i)}\|^2 = 2(1 + 2^{k+1})^2 \alpha_k^{-1} \sum_{n \in I_k} \|\xi_n\|^2.$$

Hence for $k \neq 0$ we obtain

$$\Phi(D_k) \leqslant 2\alpha_k^{-1} \sum_{n \in I_k} (1 + 2\log n)^2 \|\xi_n\|^2 = 8\alpha_k^{-1} \sum_{n \in I_k} (\log n + \tfrac{1}{2})^2 \|\xi_n\|^2$$

$$\leqslant 32\alpha_k^{-1} \sum_{n \in I_k} \log^2(n+1) \|\xi_n\|^2 = 32\alpha_k.$$

The condition (1) implies that $\sum_{k=0}^{\infty} \Phi(D_k) < \infty$, and therefore the sequence $(D_k)_{k=0}^{\infty}$ determines a bundle $\mathscr{P}_D$.

For each $j \in N$ there exists an operator $x_j \in M$ such that

$$\|x_j \Omega - \xi_{\pi(j)}\| \leqslant 1/2^{j+1}. \tag{5}$$

Then for every $n \in N$ we have

$$\Big\|\sigma_{m_n} - \sigma_n - \sum_{j=n+1}^{m_n} x_j \Omega\Big\|^2 = \Big\|\sum_{j=n+1}^{m_n} (\xi_{\pi(j)} - x_j \Omega)\Big\|^2 \leqslant \Big(\sum_{j=n+1}^{m_n} \|\xi_{\pi(j)} - x_j \Omega\|\Big)^2$$
$$\leqslant \Big(\sum_{j=n+1}^{\infty} \|\xi_{\pi(j)} - x_j \Omega\|\Big)^2 \leqslant 1/2^n,$$

so $\sum_{n=1}^{\infty} \big\|\sigma_{m_n} - \sigma_n - \sum_{j=n+1}^{m_n} x_j \Omega\big\|^2 < \infty$, which by Property 4.4 implies

$$\sigma_{m_n} - \sigma_n - \sum_{j=n+1}^{m_n} x_j \Omega \xrightarrow{b} 0. \tag{6}$$

For every $j \in N$ we put

$$\zeta_j = x_j \Omega - \alpha_k^{1/2} y_j^k \Omega,$$

where $j \in J_k$. Applying the definitions of $k(n)$ and $j(n)$ we easily obtain

$$\sum_{j=n+1}^{m_n} x_j \Omega = \sum_{j=n+1}^{m_n} \sum_{k=0}^{\infty} \varepsilon_j^k x_j \Omega = \sum_{j=n+1}^{m_n} \sum_{k=k(n)+1}^{j(m_n)} \varepsilon_j^k x_j \Omega \tag{7}$$
$$= \sum_{j=n+1}^{m_n} \sum_{k=k(n)+1}^{j(m_n)} \varepsilon_j^k (\zeta_j + \alpha_k^{1/2} y_j^k \Omega) = \sum_{j=n+1}^{m_n} \zeta_j + \sum_{k=k(n)+1}^{j(m_n)} \alpha_k^{1/2} (s_{m_n}^k - s_n^k) \Omega.$$

The estimates (3) and (5) give for every $j \in N$ (where again k is such that $j \in J_k$)

$$\|\zeta_j\|^2 = \|x_j \Omega - \alpha_k^{1/2} y_j^k \Omega\|^2 = \alpha_k \|\alpha_k^{-1/2} x_j \Omega - y_j^k \Omega\|^2$$
$$= \alpha_k \|\alpha_k^{-1/2} x_j \Omega - \alpha_k^{-1/2} \xi_{\pi(j)} + \eta_j^k - y_j^k \Omega\|^2$$
$$\leqslant \alpha_k (\alpha_k^{-1/2} \|x_j \Omega - \xi_{\pi(j)}\| + \|\eta_j^k - y_j^k \Omega\|)^2 \leqslant (1/2^j)^2,$$

and further

$$\sum_{n=1}^{\infty} \Big\|\sum_{j=n+1}^{m_n} \zeta_j\Big\|^2 \leqslant \sum_{n=1}^{\infty} \Big(\sum_{j=n+1}^{m_n} \|\zeta_j\|\Big)^2 \leqslant \sum_{n=1}^{\infty} \Big(\sum_{j=n+1}^{\infty} \|\zeta_j\|\Big)^2 \leqslant \sum_{n=1}^{\infty} (1/2^n)^2 < \infty,$$

which implies the convergence

$$\sum_{j=n+1}^{m_n} \zeta_j \xrightarrow{b} 0. \tag{8}$$

Finally, for any projection $p \in \mathbf{Proj}\, M$, $n \in N$, $k \in N_0$, the condition (4) gives

$$|\alpha_k^{1/2} s_n^k p|^2 = \alpha_k p |s_n^k|^2 p \leqslant \alpha_k (p D_k p). \tag{9}$$

If moreover $p \in \mathscr{P}_D$, then there exists a constant $C_p > 0$ such that for all $n \in N$

$$\left\| p\Big(\sum_{k=k(n)+1}^{j(m_n)} D_k\Big) p \right\|_\infty < C_p.$$

Using (9) and Lemma 4.2 we obtain for any $p \in \mathscr{P}_D$ the estimation

$$\begin{aligned}\Big\|\Big(\sum_{k=k(n)+1}^{j(m_n)} \alpha_k^{1/2} s_n^k\Big) p\Big\|_\infty &\leqslant \Big\| p\Big(\sum_{k=k(n)+1}^{j(m_n)} D_k\Big) p\Big\|_\infty^{1/2} \cdot \Big(\sum_{k=k(n)+1}^{j(m_n)} \alpha_k\Big)^{1/2} \\ &\leqslant C_p^{1/2} \cdot \Big(\sum_{k=k(n)+1}^{j(m_n)} \alpha_k\Big)^{1/2} \xrightarrow{n\to\infty} 0,\end{aligned}$$

as, by (1), $\sum_{k=0}^{\infty} \alpha_k < \infty$, and obviously $j(m_n) \xrightarrow{n\to\infty} \infty$ and $k(n) \xrightarrow{n\to\infty} \infty$. Therefore

$$\sum_{k=k(n)+1}^{j(m_n)} \alpha_k^{1/2} s_n^k \xrightarrow{b,M} 0,$$

which, by Property 4.3, is equivalent to

$$\sum_{k=k(n)+1}^{j(m_n)} \alpha_k^{1/2} s_n^k \Omega \xrightarrow{b} 0. \tag{10}$$

Similarly we can show that

$$\sum_{k=k(n)+1}^{j(m_n)} \alpha_k^{1/2} s_{m_n}^k \Omega \xrightarrow{b} 0. \tag{11}$$

The equality (7) together with (8), (10) and (11) implies that

$$\sum_{j=n+1}^{m_n} x_j \Omega \xrightarrow{b} 0.$$

The comparison of the above condition with (2) and (6) completes the proof. ∎

APPENDIX

Here we present the afore-mentioned combinatorial lemma comparing conditions for the convergence of series imposed by Theorems 3.1 and 3.2:

LEMMA. *Let us assume that $c > 1$ and $(\nu_m)_{m=1}^{\infty}$ is a strictly increasing sequence of positive integers such that for every $m \in N$*

$$\log \nu_{m+1} < c \log \nu_m. \tag{12}$$

Let $(\kappa_n)_{n=1}^{\infty}$ be a decreasing sequence of nonnegative numbers such that

$$\sum_{m=1}^{\infty} \kappa_{\nu_m} < \infty. \tag{13}$$

Then we have

$$\sum_{k=0}^{\infty}\Big(\sum_{n\in I_k}\varkappa_n a_n\Big)^{1/2}<\infty \tag{14}$$

for every sequence $(a_n)_{n=1}^{\infty}$ *of nonnegative numbers such that*

$$\sum_{n=1}^{\infty} a_n<\infty. \tag{15}$$

Proof. We will use the following notation ($m\in N$, $k\in N_0$):

$$P_k=\Big(\sum_{n\in I_k}\varkappa_n a_n\Big)^{1/2},$$

$$m(k)=\max\{m\in N\colon\ \nu_m\leqslant 2^{2^k}+1\},\qquad M(m)=\{k\in N\colon\ m(k)=m\}$$

(obviously, we can additionally assume that $\nu_0=1$). If $m=m(k)$ for some $m\in N$, $k\in N_0$, then from (12) we get

$$\nu_{m+1}<(\nu_m)^c<(2^{2^k}+1)^c\leqslant 2^{2^{k+[\log c]+2}}+1,$$

and putting $t=[\log c]+2$ we have $m(k+t)\geqslant m(k)+1$, and the inequality

$$\# M(m)\leqslant t \tag{16}$$

follows.

Further we have for each $k\in N_0$

$$P_k\leqslant\Big(\varkappa_{\nu_{m(k)}}\sum_{n\in I_k}a_n\Big)^{1/2}$$

and for every $l,j\in N_0$, $l<j$,

$$\sum_{k=l}^{j}P_k\leqslant\sum_{k=l}^{j}\varkappa_{\nu_{m(k)}}^{1/2}\Big(\sum_{n\in I_k}a_n\Big)^{1/2}=\sum_{m=m(l)}^{m(j)}\varkappa_{\nu_m}^{1/2}\Big(\sum_{\substack{k\in M(m)\\ l\leqslant k\leqslant j}}\Big(\sum_{n\in I_k}a_n\Big)^{1/2}\Big).$$

Using the inequality

$$\sqrt{x_1}+\dots+\sqrt{x_r}\leqslant\sqrt{t}\sqrt{x_1+\dots+x_r},$$

which is true for all $r\in\{0,1,\dots,t\}$, $x_1,\dots,x_r\in \boldsymbol{R}_+$, and remembering about (16) we obtain the following estimation:

$$\begin{aligned}\sum_{k=l}^{j}P_k&\leqslant\sum_{m=m(l)}^{m(j)}\varkappa_{\nu_m}^{1/2}\cdot\sqrt{t}\Big(\sum_{k\in M(m)}\sum_{n\in I_k}a_n\Big)^{1/2}\\&\leqslant\sqrt{t}\Big(\sum_{m=m(l)}^{m(j)}\varkappa_{\nu_m}+\sum_{m=m(l)}^{m(j)}\sum_{k\in M(m)}\sum_{n\in I_k}a_n\Big)=\sqrt{t}\Big(\sum_{m=m(l)}^{m(j)}\varkappa_{\nu_m}+\sum_{n\in A_{l,j}}a_n\Big),\end{aligned} \tag{17}$$

where $A_{l,j} = \{2^{2^{\min M(m(l))}}+1, \ldots, 2^{2^{\max M(m(j))}}\}$. It is easy to see that

$$m(l) \xrightarrow{l\to\infty} \infty \quad \text{and} \quad \min M(m(l)) \xrightarrow{l\to\infty} \infty,$$

so the comparison of (13), (15) and (17) shows that the series in (14) is Cauchy. ∎

According to the remark in Section 3 it is enough to apply the above lemma to sequences

$$(a_n)_{n=1}^{\infty} := \left(w_n \log^2(n+1)\,\|\xi_n\|^2\right)_{n=1}^{\infty} \quad \text{and} \quad (\kappa_n)_{n=1}^{\infty} := (1/w_n)_{n=1}^{\infty}$$

to infer that the assumption in Theorem 3.1 is weaker than that in Theorem 3.2.

REFERENCES

[1] M. S. Goldstein, *Theorems in almost everywhere convergence* (in Russian), J. Operator Theory 6 (1981), pp. 233–311.

[2] E. Hensz and R. Jajte, *Pointwise convergence theorems in L_2 over a von Neumann algebra*, Math. Z. 193 (1986), pp. 413–429.

[3] E. Hensz, R. Jajte and A. Paszkiewicz, *The unconditional pointwise convergence in L_2 over a von Neumann algebra*, Colloq. Math. 69 (1995), pp. 167–178.

[4] E. Hensz, R. Jajte and A. Paszkiewicz, *The bundle convergence in von Neumann algebras and their L_2 spaces*, Studia Math. 120 (1996), pp. 23–46.

[5] R. Jajte, *Strong limit theorems in noncommutative L_2 spaces*, Lecture Notes in Math. 1477, Springer, Berlin–Heidelberg–New York 1991.

[6] E. C. Lance, *Ergodic theorem for convex sets and operator algebras*, Invent. Math. 37 (1976), pp. 201–214.

[7] B. Le Gac and F. Móricz, *On the bundle convergence of orthogonal series and SLLN in noncommutative L_2 spaces*, Acta Sci. Math. (Szeged) 64 (1998), pp. 575–599.

[8] W. Orlicz, *Zur Theorie der orthogonalen Reihen*, Bull. Internat. Acad. Polon. Sci. Sér. A (1927), pp. 81–115.

[9] D. Petz, *Quasi-uniform ergodic theorems in von Neumann algebra*, Bull. London Math. Soc. 16 (1984), pp. 151–156.

[10] K. Tandori, *Über die orthogonalen Funktionen X (unbedingte Konvergenz)*, Acta Sci. Math. (Szeged) 23 (1962), pp. 185–221.

Department of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: adskal@imul.math.uni.lodz.pl

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