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INDIVIDUAL ERGODIC THEOREM FOR NON-CONTRACTIVE NORMAL OPERATORS

BY

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Abstract. A condition implying the strong law of large numbers for trajectories of a normal non-contractive operator is given. The condition has been described in terms of a spectral measure, in the spirit of the well-known theorem of V. F. Gaposhkin. To embrace the non-contractive operators we pass from the classical arithmetic (Cesàro) means to the Borel methods of summability.

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1. INTRODUCTION

It is well known that, in general, the individual ergodic theorem does not hold for an arbitrary normal (even unitary) operator u in L_2 (over a probability space). It is also well known that the asymptotic behaviour of the Cesàro means $n^{-1}\sum_{k=\infty}^{n-1} u^k$ of a normal contraction operator depends heavily on the local properties of the spectrum of u near the value one.

Gaposhkin [4], [5] proved that if E is the spectral measure of a normal contraction operator u in $L_2(\Omega, \mathscr{F}, \mu)$, then for $\xi \in L_2$ the ergodic averages

(1) $$S_n(\xi) = \frac{1}{n}\sum_{k=0}^{n-1} u^k(\xi)$$

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converge almost surely to ξ (given by the mean ergodic theorem) if and only if

(2) $$E\{z\colon 0<|1-z|<2^{-n}\}\xi \to 0 \quad \mu\text{-a.s.}$$

The Gaposhkin result just formulated gives the condition implying the strong law of large numbers for trajectories $(u^n\xi)$ of a normal operator, so, in particular, for weakly stationary sequences (being the trajectories of unitary operators). Our intention is to describe some conditions implying the convergence with probability one for linear transformations of trajectories of *non--contractive* linear operators.

The Gaposhkin proof is based on the analysis of vector-valued functions determined by the kernels $K_n(z) = n^{-1}\sum_{k=1}^{n} z^k$, $|z| \leqslant 1$, $n = 1, 2, \ldots$ It is clear that the efficiency of the arithmetic means (Cesàro averages) in this context follows from the fact that $K_n(z) \to 0$ for $|z| \leqslant 1$, $z \neq 1$. Obviously, the Cesàro means cannot be effective for the kernels $K_n(z)$ with z from beyond the disc ($|z| \leqslant 1$). But there are Borel-type methods of summability which are powerful in the theory of analytic continuations and one can expect that they may be useful if we try to extend the ergodic theorem to the case of non-contractive normal operators.

Namely, for $\alpha > 0$ and a sequence $x = (\xi_n)$ of numbers (or vectors), let us put

(3) $$B_\alpha(t, x) = \alpha e^{-t}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\xi_n, \qquad t > 0.$$

The function $B_\alpha(t, x)$ is called a B_α-transform of the sequence $x = (\xi_n)$. If $\lim_{t\to\infty} B_\alpha(t, x) = \xi$, then we say that (ξ_n) is *summable* to ξ by the method B_α, and then we write $\xi_n \to \xi\,(B_\alpha)$ or $B_\alpha - \lim \xi_n = \xi$. The results in the paper concern "discrete" Borel methods, i.e. $B_\alpha(t, x)$ is taken only for $t = 1, 2, \ldots$ though in the proofs we often consider the transform $B_\alpha(t, x)$ with continuous parameter. We take only $\alpha = 2^{-\nu}$, $\nu = 1, 2, \ldots$ In the sequel we shall need the following transformation W:

(4) $$W(f)(t) = \frac{e^{-t}}{2\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{u^2}{4t}+u\right) f(u)\,du,$$

defined for continuous functions $f\colon (0, \infty) \to \boldsymbol{R}$ (cf. [7], p. 140).

The transformation W is *regular* in the sense that $\lim_{u\to\infty} f(u) = \beta$ implies $\lim_{t\to\infty} W(f)(t) = \beta$. Moreover, for the Borel transforms $B_\alpha(t, x)$ with *continuous parameter* $t > 0$ we have

(5) $$W\big(B_{2^{-k}}(\cdot, x)\big)(t) = B_{2^{-(k+1)}}(t, x), \qquad t > 0,$$

for $k = 0, 1, \ldots$, assuming that the Borel transforms $B_{2^{-k}}$ and $B_{2^{-(k+1)}}$ are well defined for $x = (\xi_k)$ (cf. [7], p. 153).

2. MAIN RESULT

The main goal of this paper is to extend the Gaposhkin idea to non--contractive operators by passing from the Cesàro means to the Borel methods of summability.

Let us begin with a few general remarks. In comparison with the Cesàro averages $(C, 1)$, the Borel methods of summability are very efficient for *rapidly* divergent sequences, like (z^n) with large $|z|$. This is not the case when we apply the Borel methods to sequences *slowly* divergent (cf. Hardy [6], p. 364). Hardy writes in his *Divergent Series*: "usually, the delicacy of a method decreases as its power increases, and that very powerful methods, adapted to the summation of rapidly divergent series, are apt to fail with divergent series of a less violent kind (such as we encounter, for example, in the theory of Fourier series)". It is worth noting here that, for a sequence of i.i.d. random variables (X_j), the limit $B_1 - \lim X_n = EX_1$ exists almost surely if and only if $E|X_1|^2 < \infty$, so, in the classical context of the SLLN, the Borel method is *less efficient* than the Cesàro means (cf. [2] and [3]).

When we apply the Borel summability methods to non-contractive normal operators we can go far away from the unit disc (support of the spectrum of a normal contraction) but we have to pay for that by setting some additional condition on the spectrum of the operator near the value one.

Let us now formulate our main result. In the sequel we consider a probability space $(\Omega, \mathscr{F}, \mu)$ and a bounded normal operator u acting in $L_2(\Omega, \mathscr{F}, u)$, with a spectral measure E, i.e.

$$u = \int_{\Delta} zE(dz) \tag{6}$$

for some Borel bounded set $\Delta \subset \boldsymbol{C}$.

Our purpose is to prove the following theorem:

THEOREM. *For* $\alpha = 2^{-k}$ *and* $0 < d < 1$, *let us define a set* $D_{\alpha,d} \subset \boldsymbol{C}$ *by putting*

$$D_{\alpha,d} = \{z;\ \operatorname{Re} z < 0\} \cup \{z;\ |z| \leqslant 1\} \cup \{z;\ \operatorname{Re} z^{2^k} \leqslant 1-d\}. \tag{7}$$

Assume that for the operator (6) *we have the inclusion* $\Delta \subset D_{\alpha,d}$ *and that, for* $\xi \in L_2$,

$$\int_{\Delta \cap \{|z|<1\}} \frac{(E(dz)\xi, \xi)}{|1-z|^2} < \infty. \tag{8}$$

Then the following two conditions:

$$u^n \xi \to E\{1\}\xi(B_\alpha) \ \ \mu\text{-a.s.} \tag{9}$$

and

$$\lim_{n\to\infty} E\{z;\ 0 < |1-z| < 2^{-n}\}\xi = 0 \ \ \mu\text{-a.s.} \tag{10}$$

are equivalent.

3. AUXILIARY RESULTS

Before starting the proof of the theorem, let us begin with some estimations concerning the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \quad \alpha > 0,$$

which will be a crucial point in all our considerations. The asymptotic behaviour of E_α was investigated in the years 1889–1904 by Mittag-Leffler in connection with analytic continuations. It has been well known for a long time that

$$(11) \qquad E_\alpha(z) \sim \alpha^{-1} \exp(z^{1/\alpha})$$

when $z \to \infty$ in the angle $|\theta| < (\pi/2)\alpha$ (cf. [6]). From (11) it follows that, at least in the angle $|\theta| < \pi/2^{k+1}$,

$$B_{2^{-k}}(t, \zeta) \sim \exp\left(-t(1-z^{2^k})\right), \quad k = 1, 2, \ldots$$

Here and throughout the paper, $\zeta = (z^n)$, $z \in C$. We shall find some more specific connections between the functions $E_\alpha(z)$ and $\exp(z^{1/\alpha})$, which will make it possible to get the estimations of $|B_\alpha(t, \zeta)|$, good enough for our purpose.

We follow a rather elementary way indicated by Włodarski [7]. First, we note that, for a fixed $\alpha = 2^{-k}$, the function

$$f(t) = E_\alpha(t^\alpha z) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} z^n$$

(as a function of $t > 0$) satisfies the differential equation

$$f'(t) = g(t) + z^{2^k} f(t) \quad \text{with } g(t) = \sum_{\nu=1}^{2^k-1} \frac{t^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} z^\nu.$$

Consequently, we have

$$f(t) = \exp(z^{2^k} t)\left[1 + \int_0^t \exp(-z^{2^k} t) \sum_{\nu=1}^{2^k-1} \frac{t^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} z^\nu \, dt\right].$$

The substitution $z^{2^k} t = v$ leads to the formula

$$(12) \quad \sum_{n=0}^{\infty} \frac{t^{n2^{-k}}}{\Gamma(n2^{-k}+1)} z^n = \exp(tz^{2^k})\left[1 + \sum_{\nu=1}^{2^k-1} \alpha_\nu^{(k)}(z) \int_{[0, z^{2^k} t]} \frac{e^{-v} v^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} dv\right],$$

where

$$\alpha_\nu^{(k)}(z) = \frac{z^\nu}{[z^{2^k}\nu]^{1/2^k}} = \exp\left(i\theta_\nu^{(k)}(z)\right).$$

The functions $\alpha_\nu^{(k)}(z)$ are determined by fixing the rational power $w \to w^{1/2^k}$ as taking its values in the angle $\{z = re^{i\theta};\ r \geqslant 0,\ -\pi/2^{k-1} < \theta \leqslant \pi/2^{k-1}\}$. In particular, $\alpha_1^{(1)}(z) = -1$ for $\operatorname{Re} z < 0$, and $\alpha_\nu^{(k)}(1) = 1$ for $1 \leqslant \nu \leqslant 2^k - 1$, $k = 1, 2, \ldots$

For $z = 1$, the formula (12) gives

$$\sum_{n=0}^{\infty} \frac{t^{n2^{-k}}}{\Gamma(n2^{-k}+1)} = e^t\left[1 + \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int_{[0,t]} e^{-u} u^{\nu 2^{-k}-1} du\right]$$

(cf. [7], p. 144).

We will show using (12) that for $\mathbb{1} = (1, 1, \ldots)$ we have

$$0 \leqslant 1 - B_{2^{-k}}(t, \mathbb{1}) \leqslant Ce^{-t}, \quad t > 0. \tag{13}$$

Indeed,

$$0 < 1 - B_{2^{-k}}(t, \mathbb{1}) = 1 - 2^{-k}\left[1 + \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int_0^t e^{-u} u^{\nu 2^{-k}-1} du\right]$$

$$= 2^{-k} \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int_t^\infty e^{-u} u^{\nu 2^{-k}-1} du \leqslant C \int_t^\infty e^{-u} du \leqslant Ce^{-t}.$$

Let us now take the function $|B_{2^{-k}}(t, \zeta - \mathbb{1})|$. We shall need its estimation only in the set

$$Z_0 = \{|z| \leqslant 1\} \cap \{|1 - z| < \sigma\},$$

where σ can be taken small enough to have $\alpha_\nu^{(k)}(z) = 1$, $\nu = 1, \ldots, 2^k - 1$, and $\operatorname{Re} z^{2^k} > 0$ for $z \in Z_0$. For $z \in Z_0$, by (12) applied both to z^k and $\mathbb{1}$, we obtain

(14) $\quad 2^k B_{2^{-k}}(t, \zeta - \mathbb{1})$

$$= \left[\exp\left(-t(1 - z^{2^k})\right) - 1\right]\left[1 + \sum_{\nu=1}^{2^k-1} \int_{[0, z^{2^k} t]} e^{-u} u^{\nu 2^{-k}-1} du / \Gamma(\nu 2^{-k})\right]$$

$$+ \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})}\left[\int_{[0, z^{2^k} t]} e^{-v} u^{\nu 2^{-k}-1} du - \int_{[0,t]} e^{-u} u^{\nu 2^{-k}-1} du\right].$$

We have

$$|\exp(-t(1-z^{2^k}))-1| \leqslant Ct|1-z^{2^k}| \leqslant Ct|1-z|.$$

Indeed, since $\operatorname{Re} z^{2^k} < 1$,

$$|\exp(t(z^{2^k}-1))-1| \leqslant |z^{2^k}-1| \int_0^t \exp(u(\operatorname{Re} z^{2^k}-1))\,du \leqslant C|1-z|\,t.$$

Moreover, if z is a complex number with $0 < \operatorname{Arg} z < \pi/2$, $\frac{1}{2} \leqslant |z| \leqslant 1$, and $\alpha > -1$, then

$$\Big|\int_{[0,zt]} e^{-u} u^\alpha\,du - \int_{[0,|z|t]} e^{-u} u^\alpha\,du\Big| \leqslant C\exp(-\gamma_0 t) \tag{15}$$

with some $0 < \gamma_0 < \frac{1}{2}$ and $C > 0$. We omit a standard proof. It can also be found in [7].

In consequence, since for $1 \leqslant v < 2^{-k}$ and $t > 2$

$$\int_{|z|t}^{t} e^{-u}\,du \leqslant \int_{t/2}^{\infty} e^{-u}\,du \leqslant Ce^{-t/2},$$

by (14) and (15), we get

$$|B_{2^{-k}}(t, \zeta-1)| \leqslant C[|1-z|\,t+e^{-\gamma t}] \tag{16}$$

for some $0 < \gamma < \frac{1}{2}$ and $C > 0$.

The function $|B_{2^{-k}}(t, \zeta)|$ will be estimated separately in several parts of the set $D_{\alpha,d}$ (defined in (7)). Take

$$G_1 = \{z;\ \operatorname{Re} z^{2^k} \leqslant 1-d\}. \tag{17}$$

Let us notice that, for $\operatorname{Re} z < 1$, $t > 1$, and $\beta > -1$, we have the inequality

$$\Big|e^{t(z-1)} \int_{[0,zt]} u^\beta e^{-u}\,du\Big| \leqslant |zt|^{\beta+1} \max(e^{-t}, e^{-t(1-\operatorname{Re} z)}) \tag{18}$$

(we omit a rather standard proof).

To avoid trivial complications in writing the formula, we estimate $|B_{2^{-k}}(t, \zeta)|$ on the set $G_1 \cap \Delta$, where Δ a fixed bounded set. Then, by (13) and (18), we get

$$|B_{2^{-k}}(t, \zeta)| \leqslant Cte^{-dt} \leqslant C\exp\left(-\frac{d}{2}t\right) \tag{19}$$

for $z \in G_1 \cap \Delta$ and t large enough (the constant C depends only on Δ).

Take

(20) $$G_2 = \{\operatorname{Re} z^{2^k} > 1-d\} \cap \{|z| \leqslant 1\} \cap \{\operatorname{Re} z > 0\}.$$

For $z \in G_2$, we have the estimation

(21) $$|B_{2^{-k}}(t, \zeta)| \leqslant C \exp\left(-\frac{t}{2}|1-z|^2\right).$$

Indeed, writing $z \in re^{i\theta}$ for $z \in G_2$, we have $|\theta| < \pi/2^{k+1}$. Thus

$$\operatorname{Re} z^{2^k} = r^{2^k} \cos 2^k\theta \leqslant r\cos\theta = \operatorname{Re} z.$$

Moreover, if $|z| \leqslant 1$, then we can see that $1-\operatorname{Re} z \geqslant \frac{1}{2}|1-z|^2$. Consequently, for $z \in G_2$ we get

$$1-\operatorname{Re} z^{2^k} \geqslant \tfrac{1}{2}|1-z|^2,$$

and (21) follows.

Take $G_3 = \{\operatorname{Re} z \leqslant 0\}$. We shall show that

(22) $$|B_{2^{-k}}(t, \zeta)| \leqslant Ce^{-t} \quad \text{for } z \in G_3.$$

Indeed, taking the Borel transform $B_1 = B_{2^0}$ for $\zeta = (z^n)$ with $\operatorname{Re} z \leqslant 0$, we have $B_1(t, \zeta) = e^{-t(1-z)}$, so

(23) $$|B_1(t, \zeta)| \leqslant e^{-t}.$$

Taking on both sides of (23) the k-th iteration of the transformation W defined in (4) and using the positivity of W and (5) we easily get (22).

4. PROOF OF THE THEOREM

Now we are in a position to prove our theorem. We split the proof in two steps.

PROPOSITION 1. *Let $0 < d < 1$ and $\alpha = 2^{-k}$. Let Δ be an arbitrary bounded Borel subset of $D_{\alpha,d}$, where $D_{\alpha,d}$ is defined by* (7). *Let us consider a normal operator of the form* (6). *Assume that a vector $\xi \in L_2$ satisfies the condition* (8), *i.e.*

(*) $$\int_{\Delta \cap \{|z|<1\}} \frac{F_\xi(dz)}{|1-z|^2} < \infty,$$

where $F_\xi(\cdot) = (E(\cdot)\xi, \xi)$.

For $x = (u^\nu \xi)_{\nu=0}^{\infty}$, let us put

$$\delta_n = B_{2^{-k}}(2^n, x) - E\{z \in \Delta \colon 0 < |1-z| < 2^{-n}\}\xi.$$

Then $\delta_n \to 0$ μ-a.s.

Proof. We shall show that $\sum_{n=1}^{\infty}\|\delta_n\|^2 < \infty$. Let us write

$$\sum_{n=1}^{\infty}\|\delta_n\|^2 = \sum_{n=1}^{\infty}\int_{\Delta\cap\{0<|1-z|<2^{-n}\}} |B_{2^{-\kappa}}(2^n, \zeta)-1|^2\, dF_\xi(z)$$

$$+\int_{\Delta\cap\{|1-z|\geqslant 2^{-n}\}} |B_{2^{-\kappa}}(2^n, \zeta)|^2\, dF_\xi(z).$$

By (13) and (16), we can write

$$(24)\qquad |B_{2^{-\kappa}}(2^n, \zeta)-1|^2 = |B_{2^{-\kappa}}(2^n, \zeta-1)+B_{2^{-\kappa}}(2^n, 1)-1|^2$$
$$\leqslant 2\,[|B_{2^{-\kappa}}(2^n, \zeta-1)|^2+|B_{2^{-\kappa}}(2^n, 1)-1|^2]$$
$$\leqslant C\,[4^n|1-z|^2+e^{-\gamma 2^n}+e^{-2^n}] \leqslant C\,[4^n|1-z|^2+e^{-\gamma 2^n}]$$

for $z\in\{|z|\leqslant 1\}\cap\{|1-z|<\sigma\}$ with σ small enough.

To avoid trivial complications in writing the sums ($\sum_{n_0}^{\infty}$ instead of $\sum_1^{\infty}$ for a suitable n_0) we shall neglect the fact that the estimation (16) and, consequently, (24) is valid for σ small enough, i.e. for $|1-z|<2^{-n}$ with n large enough.

Let us note that, by (19), (21), (22), putting

$$(25)\qquad \mathcal{S} = \Delta\cap\{z;\ |1-z|>2^{-n}\}\cap G_2,$$

we have

$$\sum_{n=1}^{\infty}\|\delta_n\|^2 \leqslant C\sum_{n=1}^{\infty}\int_{\Delta\cap\{z;0<|1-z|<2^{-n}\}} 4^n|1-z|^2\, dF_\xi(z)$$

$$+C\sum_{n=1}^{\infty}\int_{\mathcal{S}}\exp(-2^n|1-z|^2)\,dF_\xi(z)+C.$$

Let $m(z)$ be a positive integer such that

$$|1-z|\,2^{m(z)}\leqslant 1 \quad\text{and}\quad |1-z|\,2^{m(z)+1}>1.$$

Put

$$g(z) = |1-z|^2\sum_{n=1}^{m(z)}4^n+\sum_{n=m(z)+1}^{\infty}\exp(-2^n|1-z|^2) = g_1(z)+g_2(z).$$

To get $\sum\|\delta_n\|^2<\infty$ it is enough to show that the function g is integrable with respect to $F_\xi(dz)$. Obviously, the function g_1 is bounded. The function g_2 is integrable with respect to $F_\xi(dz)$. Indeed, since $e^\alpha\geqslant\alpha$ for $\alpha>0$, we have

$$\exp(-2^n|1-z|^2)\leqslant\frac{2^{-n}}{|1-z|^2}.$$

Consequently, putting $h(z)=\sum_{n=m(z)+1}^{\infty}2^{-n}|1-z|^{-1}$, we get

$$\int_{\Delta\cap\{|z|<1\}} g_2(z)F_\xi(dz)\leqslant\int_{\Delta\cap\{|z|<1\}} h(z)\frac{F_\xi(dz)}{|1-z|}.$$

But $h(z) = |1-z|^{-1} 2^{-(m(z)+1)} \sum_{\nu=1}^{\infty} 2^{-\nu} < 1$ by the choice of $m(z)$, so the integrability of g_2 follows from the condition (8). This completes the proof. ■

PROPOSITION 2. *Let us fix $\alpha = 2^{-k}$ and $0 < d < 1$. Let Δ, $D_{\alpha,d}$, u, E, ξ be such as in Proposition 1. Let us write briefly*

$$C_n = B_{2^{-k}}(n, \zeta) \quad \textit{with } \zeta = (z^\nu)_{\nu=0}^{\infty},$$

$$\tilde{C}_n = B_{2^{-k}}(n, x) \quad \textit{with } x = (u^\nu \xi)_{\nu=0}^{\infty}.$$

Then

$$\sigma_n = \max_{1 \leqslant m \leqslant 2^n} |\tilde{C}_{2^n+m} - \tilde{C}_{2^n}| \to 0 \ \mu\text{-a.s.}$$

Proof. Put

$$\sigma_{n,m} = \tilde{C}_{2^n+m} - \tilde{C}_{2^n}.$$

Let us write m in the form

$$m = \sum_{r=1}^{n} \varepsilon_r 2^{n-r}, \qquad \varepsilon_r = 0, 1.$$

The standard dyadic expansion method leads to the formula

$$\sigma_{n,m} = \sum_{r=1}^{n} \varepsilon_r H_{r,n}^{(j_r)}$$

for suitable j_r's, where

$$H_{r,n}^{(j)} = \int_{\Delta} K_{r,n}^{(j)}(z) E(dz) \xi$$

with

$$K_{r,n}^{(j)}(z) = C_{2^n + j2^{n-r}}(z) - C_{2^n+(j-1)2^{n-r}}(z) \quad \text{for } r = 1, \ldots, n;\ j = 1, \ldots, 2^r.$$

Thus, we can write

$$\sigma_n = \max_{1 \leqslant m \leqslant 2^n} |\sigma_{n,m}| \leqslant \max_{(j_1,\ldots,j_n)} \sum_{r=1}^{n} |H_{r,n}^{(j_r)}|,$$

where the maximum is taken over all vectors $(j_1, \ldots, j_n)$ with different entries in $(1, \ldots, 2^r)$.

Consequently, we get

$$|\sigma_n|^2 \leqslant \max_{(j_1,\ldots,j_n)} \left(\sum_{r=1}^{n} |H_{r,n}^{(j_r)}| \frac{r}{r} \right)^2$$

$$\leqslant \max_{(j_1,\ldots,j_n)} \sum_{r=1}^{n} |H_{r,n}^{(j_r)}|^2 r^2 \sum_{r=1}^{n} r^{-2} \leqslant 2 \sum_{r=1}^{n} r^2 \sum_{j=1}^{2^r} |H_{r,n}^{(j)}|^2$$

and, finally,

$$\sum_{n=1}^{\infty} \|\sigma_n\|^2 \leqslant 2 \sum_{n=1}^{\infty} \sum_{r=1}^{n} r^2 \sum_{j=1}^{2^r} \|H_{r,n}^{(j)}\|^2.$$

We shall show that

$$\sum_n \|\sigma_n\|^2 < \infty. \tag{26}$$

Putting

$$m_1 = 2^n+(j-1)2^{n-r},$$

$$m_2 = 2^n+j2^{n-r}, \quad r = 1, \ldots, n, \; j = 1, \ldots, 2^r,$$

we can write

$$K_{r,n}^{(j)} = C_{m_2}-C_{m_1}.$$

We estimate $C_{m_2}-C_{m_1}$ on several parts of Δ getting in this way the estimation of

$$\|H_{r,n}^{(j)}\|^2 = \int_{\Delta} |C_{m_2}(z)-C_{m_1}(z)|^2 \, dF_\xi(z).$$

From (19) and (22) it follows immediately that

$$A = \sum_{n=1}^{\infty} \sum_{r=1}^{n} r^2 \sum_{j=1}^{2^r} \int_T |K_{r,n}^{(j)}|^2 \, dF_\xi < \infty,$$

where $T = \{\operatorname{Re} z \leqslant 0\} \cup \{\operatorname{Re} z^{2^k} \leqslant 1-d\}$.

Indeed, for $z \in T$ we have, by (19) and (22),

$$|K_{r,n}^{(j)}(z)|^2 = |C_{m_2}(z)-C_{m_1}(z)|^2 \leqslant C\exp(-d2^n).$$

Consequently,

$$A \leqslant C \sum_{n=1}^{\infty} \exp(-d2^n)\, n^3\, 2^n < \infty.$$

Thus, to prove (26) it is enough to show that

$$\sum_{n=1}^{\infty} \sum_{r=1}^{n} r^2 \sum_{j=1}^{2^r} \int_S |H_{r,n}^{(j)}|^2 \, dF_\xi < \infty,$$

where $S = \{|z| \leqslant 1\} \cap \{\operatorname{Re} z^{2^k} > 1-d\} \cap \{\operatorname{Re} z > 0\}$.

In this case the estimation should be such more delicate. Let us introduce the following notation:

$$C_t = B_{2^{-k}}(t, \zeta), \qquad \phi_t = \exp\left(-t(1-z^{2^k})\right),$$
$$\beta_t = 1+ \sum_{\nu=1}^{2^k-1} \frac{\alpha_\nu^{(k)}(z)}{\Gamma(\nu 2^{-k})} \int_{[0,z^{2^k}t]} e^{-u} u^{\nu 2^{-k}-1} \, du. \tag{27}$$

Here the function $\alpha_\nu^{(k)}(z)$ is the same as in the formula (12).

Then, in particular, we have $2^k C_t = \phi_t \beta_t$ and, consequently,

(28) $$2^k |C_{t_1} - C_{t_2}| \leqslant |\phi_{t_2} - \phi_{t_1}|\,|\beta_{t_2}| + |\phi_{t_1}|\,|\beta_{t_2} - \beta_{t_1}|.$$

Let us notice that, for $t_1 < t_2$ and $z \in S$, considering the integral $\int_{t_1}^{t_2} \exp(-s(1-z^{2^k}))\,ds$, we get easily

(29) $$|\phi_{t_2} - \phi_{t_1}| \leqslant |1 - z^{2^k}| \int_{t_1}^{t_2} \exp(-s(1 - \operatorname{Re} z^{2^k}))\,ds$$
$$\leqslant C|1-z|(t_2 - t_1)\exp(-t_1(1 - \operatorname{Re} z^{2^k})).$$

In particular, for $z \in S$ we have

(30) $$|\phi_{t_2} - \phi_{t_1}| \leqslant \frac{C}{|1-z|}\frac{t_2 - t_1}{t_1}.$$

Indeed,

$$\exp(-t_1(1 - \operatorname{Re} z^{2^k})) \leqslant \frac{1}{t_1(1 - \operatorname{Re} z^{2^k})}.$$

But, for $z = re^{i\theta} \in S$ we have $|\theta| < \pi/2^{k+1}$, so $\operatorname{Re} z^{2^k} = r^{2^k}\cos 2^k\theta \leqslant r\cos\theta = \operatorname{Re} z$. Since $|z| \leqslant 1$, we also have $1 - \operatorname{Re} z \geqslant \frac{1}{2}|1-z|^2$. Thus, for $z \in S$, $1 - \operatorname{Re} z^{2^k} \geqslant \geqslant 1 - \operatorname{Re} z \geqslant \frac{1}{2}|1-z|^2$ and, consequently, (29) implies

$$|\phi_{t_2} - \phi_{t_1}| \leqslant C|1-z|\frac{t_2 - t_1}{t_1}\frac{1}{|1-z|^2}.$$

We are going to estimate $\beta_{t_2}(z) - \beta_{t_1}(z)$ for $z \in S$. In particular, we have $\frac{1}{2} < |z| \leqslant 1$ and $\operatorname{Arg} z \in (0, \pi/2)$.

Let us put

$$\varphi_k^{(\nu)}(u) = \frac{1}{\Gamma(\nu 2^{-k})} e^{-u} u^{\nu 2^{-k} - 1}.$$

We shall often omit the indices writing φ instead of $\varphi_k^{(\nu)}$, when it is clear. We can write

$$\beta_{t_2}(z) - \beta_{t_1}(z) = \sum_{\nu=1}^{2^k - 1} \alpha_\nu^{(k)}(z)\Big[\int_{[0, z^{2^k} t_2]} \varphi_k^{(\nu)}\,du - \int_{[0, z^{2^k} t_1]} \varphi_k^{(\nu)}\,du\Big].$$

By (15), for $t > 2$,

$$\Big|\int_{[0, z^{2^k} t]} \varphi du - \int_{[0,t]} \varphi du\Big| \leqslant \Big|\int_{[0, z^{2^k} t]} \varphi du - \int_{[0, |z^{2^k}| t]} \varphi du\Big| + \Big|\int_{[0, |z^{2^k}| t]} \varphi du - \int_{[0,t]} \varphi du\Big| \leqslant Ce^{-\gamma t}$$

with some $C > 0$ and $0 < \gamma < \frac{1}{2}$.

Thus,

$$\Big|\int_{[0,z^{2^k}t_2]}\varphi du-\int_{[0,z^{2^k}t_1]}\varphi du\Big|$$

$$\leqslant\Big|\int_{[0,z^{2^k}t_2]}\varphi du-\int_{[0,t_2]}\varphi du\Big|+\Big|\int_{[0,t_2]}\varphi du-\int_{[0,t_1]}\varphi du\Big|+\Big|\int_{[0,t_1]}\varphi du-\int_{[0,z^{2^k}t_1]}\varphi du\Big|$$

$$\leqslant C\Big[\exp(-\gamma t_2)+\exp(-\gamma t_1)+\int_{t_1}^{t_2}\varphi du\Big]\leqslant C\Big[\exp(-\gamma t_1)+\int_{t_1}^{\infty}\varphi_k^{(\nu)}(u)\,du\Big]$$

$$\leqslant C\left[\exp(-\gamma t_1)+\exp\left(-\frac{t_1}{2}\right)\right]\leqslant C\exp(-\gamma t_1).$$

Finally, since $|\alpha_\nu^{(k)}(z)|\equiv 1$, we get easily the estimation

(31) $$|\beta_{t_2}(z)-\beta_{t_1}(z)|\leqslant C\exp(-\gamma t_1)\quad \text{for } z\in G_2.$$

Since $2^k C_t=\varphi_t\beta_t$ and, by (15′), β_t is bounded, we get one more estimation

(32) $$|C_{t_2}-C_{t_1}|\leqslant|C_{t_2}|+|C_{t_1}|\leqslant C\exp\big(-t_1(1-\operatorname{Re} z2^k)\big)\leqslant\frac{C}{t_1|1-z|^2}.$$

Let us write

(33) $$\int_S|C_{m_2}-C_{m_1}|^2\,dF_\xi=\int_{S\cap\{|1-z|\leqslant 2^{-n}\}}|C_{m_2}-C_{m_1}|^2\,dF_\xi$$

$$+\int_{S\cap\{2^{-n}<|1-z|\leqslant 2^{-(n-r)}\}}|C_{m_2}-C_{m_1}|^2\,dF_\xi+\int_{S\cap\{2^{-(n-r)}<|1-z|\leqslant 1/2\}}|C_{m_2}-C_{m_1}|^2\,dF_\xi.$$

Let us put

$$Z_k=\{z;\ 2^{-(k+1)}<|1-z|\leqslant 2^{-k}\},$$

and note that by (8) the measure $\tilde{F}_\xi(dz)=|1-z|^{-2}F_\xi(dz)$ is finite on S.

By (28) and (32), we have

(34) $$\int_{S\cap\{|1-z|\leqslant 2^{-n}\}}|C_{m_2}-C_{m_1}|^2\,dF_\xi\leqslant C2^{2n-2r}\sum_{k=n}^{\infty}2^{-2k}F_\xi(Z_k)+C\exp(-\gamma 2^n).$$

By (30) and (31), we obtain

(35) $$\int_{S\cap\{2^{-n}<|1-z|\leqslant 2^{-(n-r)}\}}|C_{m_2}-C_{m_1}|^2\,dF_\xi$$

$$\leqslant C2^{-2r}\sum_{k=n-r}^{n}\tilde{F}_\xi(Z_k)+C\exp(-\gamma 2^n).$$

By (31) and (32), we get

$$\int_{S\cap\{2^{-(n-r)}<|1-z|\leqslant 1/2\}} |C_{m_2}-C_{m_1}|^2\,dF_\xi \leqslant C2^{-2n}\sum_{k=1}^{n-r} 2^{2k}\tilde{F}_\xi(\mathbf{Z}_k)+C\exp(-\gamma 2^n). \tag{36}$$

Now it is rather easy to check that

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n} r^2 2^r \int_S |C_{m_2}-C_{m_1}|^2\,dF_\xi < \infty. \tag{37}$$

Since

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n} r^2 2^r \exp(-\gamma 2^n) < \infty,$$

it is enough to estimate the sum

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n} r^2 2^r 2^{2n-2r}\sum_{k=n}^{\infty} 2^{-2k}F_\xi(\mathbf{Z}_k)+\sum_{n=1}^{\infty}\sum_{r=1}^{n} r^2 2^r 2^{-2r}\sum_{k=n-r}^{n}\tilde{F}_\xi(\mathbf{Z}_k)$$
$$+\sum_{n=1}^{\infty}\sum_{r=1}^{n} r^2 2^r 2^{-2n}\sum_{k=1}^{n-1} 2^{2k}\tilde{F}_\xi(\mathbf{Z}_k)=\sigma_1+\sigma_2+\sigma_3.$$

We have

$$\sigma_1 \leqslant C\sum_{r=1}^{\infty} r^2 2^{-r}\sum_{n=1}^{\infty} 2^{2n}\sum_{k=n}^{\infty} 2^{-2k}F_\xi(\mathbf{Z}_k)$$
$$\leqslant C\sum_{k=1}^{\infty} 2^{-2k}\tilde{F}_\xi(\mathbf{Z}_k)\sum_{n=1}^{k} 2^{2n}\leqslant C\sum_{k=1}^{\infty} F_\xi(\mathbf{Z}_k)<\infty.$$

Moreover, we have $\sigma_2 \leqslant C\sum_{n=1}^{\infty}\sum_{k=0}^{n}\tilde{F}_\xi(\mathbf{Z}_k)\sum_{r=n-k}^{n} r^2 2^{-r}$. Since $r2^{-r}\leqslant 2^{-r/2}$, for r large enough we can write

$$\sigma_2 \leqslant C\sum_{n=1}^{\infty}\sum_{k=0}^{n}\tilde{F}_\xi(\mathbf{Z}_k)\left(\frac{1}{\sqrt{2}}\right)^r \leqslant C\sum_{n=1}^{\infty}\sum_{k=0}^{n}\tilde{F}_\xi(\mathbf{Z}_k)\left(\frac{1}{\sqrt{2}}\right)^{n-k}$$
$$\leqslant \sum_{k=0}^{\infty}\tilde{F}_\xi(\mathbf{Z}_k)\sum_{n=k}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{n-k}\leqslant C\sum_{k=0}^{\infty}\tilde{F}_\xi(\mathbf{Z}_k)<\infty,$$

$$\sigma_3 \leqslant C \sum_{n=1}^{\infty} 2^{-2n} \sum_{r=1}^{n} r^2 2^r \sum_{k=1}^{n-r} 2^{2k} \tilde{F}_\xi(Z_k) \leqslant C \sum_{n=1}^{\infty} 2^{-2n} \sum_{k=1}^{n-1} 2^{2k} \tilde{F}_\xi(Z_k) \sum_{r=1}^{n-k} r^2 2^r$$

$$\leqslant C \sum_{n=1}^{\infty} 2^{-2n} \sum_{k=1}^{n-1} 2^{2k} \tilde{F}_\xi(Z_k)(n-k)^3 2^{n-k} \leqslant C \sum_{k=0}^{\infty} 2^k F_\xi(Z_k) \sum_{n=k}^{\infty} (n-k)^3 2^{-n}$$

$$\leqslant C \sum_{k=0}^{\infty} 2^k F_\xi(Z_k) \sum_{s=1}^{\infty} s^3 2^{-s} 2^{-k} \leqslant C \sum_{k=0}^{\infty} F_\xi(Z_k) < \infty,$$

which completes the proof of Proposition 2. ∎

Proof of the Theorem. The Theorem follows immediately from Propositions 1 and 2.

5. FINAL REMARKS

From the proof of the Theorem it is clear that the condition (8) can be dropped in the case when $|1-z| \leqslant C|1-\operatorname{Re} z|$ for $z \in \Delta$ near 1. It has a simple geometric interpretation: near 1 the set Δ is situated in the angle $\{1+re^{i\theta};\ r \geqslant 0,\ |\theta-\pi| \leqslant c < \pi/2\}$. In particular, this is the case when u is self-adjoint,

$$u = \int_a^1 \lambda E(d\lambda) \quad \text{with } -\infty < a < 1.$$

We did not manage to show the necessity of the condition (8). It seems to be probable that (8) can be replaced by the weaker assumption

$$\int_{|z|<1} \frac{dF_\xi(z)}{|1-z|^\beta} < \infty \quad \text{with some } 0 < \beta < 2.$$

Concluding the paper let us remark that the Borel methods of summability are also very efficient in the case of the uniform convergence (in operator norm) for not necessarily normal operators. As an example let us formulate the following simple result.

Let u be a bounded linear operator acting in a Hilbert space and let $u = a+ib$ be its decomposition into real and imaginary parts. Assume that $a \leqslant I-\delta$ for some $\delta > 0$ (b is arbitrary). Then, for $x = (u^n)$, we have

$$\|B_1(t, x)\| \to 0 \quad \text{as } t \to \infty.$$

Indeed,

$$B_1(t, x) = \exp\big(t[(a-1)+ib]\big).$$

By the Trotter–Lie product formula,

$$\exp\big(t[(a-I)+ib]\big) = \lim_{n\to\infty} \left[\exp\left(\frac{t}{n}(a-I)\right)\exp\left(i\frac{t}{n}b\right)\right]^n.$$

Consequently, we get

$$\|B_1(t, x)\| \leqslant \overline{\lim_n} \left\| \exp\left(\frac{t}{n}(a-I)\right) \right\|^n = e^{-\delta t} \to 0 \quad \text{as } t \to \infty.$$

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