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A REMARK ON THE POISSON KERNELS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. In this note we give an improvement of the estimate of the Poisson kernels for second order differential operators on homogeneous manifolds of negative curvature obtained for the first time, using some probabilistic techniques, in [1] and then improved by the author in [4].

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INTRODUCTION AND THE MAIN ESTIMATE

In this note we consider a class of second order differential operators on a connected, simply connected homogeneous manifold of negative curvature. Such a manifold is a semidirect product $G = N \times_s A$, where N is a nilpotent Lie group and $A = \boldsymbol{R}^+$ normalizes N (see [3]).

On the Lie algebra level we have

$$\mathfrak{g} = \mathfrak{n} \times_s \boldsymbol{R}.$$

Then the negative curvature assumption also implies that $H = (0, 1) \in \mathfrak{g}$ may be chosen so that the eigenvalues d_j of $\mathrm{ad}(H)$ on $\mathfrak{n}$ all have positive real parts, which can be made arbitrarily large (see [1]).

Let the general element of $G = N \times_s \boldsymbol{R}^+$ be denoted by (x, a) or simply by xa. On G we consider the second order left-invariant operator

$$\mathcal{L}^\gamma = \sum_j (X_j^a)^2 + X^a + a^2 \partial_a^2 + (1-\gamma)\, a\partial_a,$$

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where $X, X_1, \ldots, X_m$ are left-invariant vector fields on N, the vector fields $X_1, \ldots, X_m$ generate $\mathfrak{n}$ as a Lie algebra, and for $Y \in \mathfrak{n}$

$$Y^a = \mathrm{Ad}_{\exp(\log a)H} Y = \exp((\log a) D), \tag{1}$$

where $D = \mathrm{ad}_H$ is a derivation of the Lie algebra $\mathfrak{n}$ of the Lie group N.

Let μ_t^γ be the semigroup of measures generated by $\mathcal{L}^\gamma$. It is known (see [2]) that if $\gamma \geqslant 0$, then there exists a unique (up to a positive multiplicative constant) positive Radon measure ν_γ with a smooth density m_γ on N such that

$$\check{\mu}_t^\gamma * \nu_\gamma = \nu_\gamma, \quad \gamma \geqslant 0.$$

ν_γ or its density m_γ is called the *Poisson kernel* for the operator $\mathcal{L}^\gamma$. For $\gamma > 0$ the measure ν_γ is bounded, while ν_0 is unbounded. These measures have been studied by many authors and in various contexts; see e.g. [1] and the literature quoted there. In particular in [1], using some probabilistic techniques, it has been proved (see Theorem 6.1 there) that for every $\gamma \geqslant 0$ there exists a constant C_γ such that for all $x \in N$ the following estimates for the Poisson kernels hold:

$$C_\gamma^{-1}(|x|+1)^{-Q-\gamma} \leqslant m_\gamma(x) \leqslant C_\gamma(|x|+1)^{-Q-\gamma}, \tag{2}$$

where $|\cdot|$ denotes the "homogeneous norm" on N, and Q means the "homogeneous dimension" of N (see e.g. [1] or [4] for precise definitions).

It turns out (see Theorem 1.2 in [4]) that we can control constants which appear in the proof of (2) in [1] and show that we may in fact choose C_γ independent of γ for, say, $0 \leqslant \gamma \leqslant 1$, i.e.,

$$C^{-1}(|x|+1)^{-Q-\gamma} \leqslant m_\gamma(x) \leqslant C(|x|+1)^{-Q-\gamma}, \quad 0 \leqslant \gamma \leqslant 1. \tag{3}$$

However, as written in (3), the constant C still depends on the particular operator or, speaking more precisely, on the derivation D in (1). What can be easily shown is that this dependence is continuous, i.e., $|C_{D_1} - C_{D_2}|$ is as small as we want provided that $\|D_1 - D_2\|_{\mathfrak{n}\to\mathfrak{n}}$ is sufficiently small. In fact, this observation follows easily from careful reading of the proof of the estimate (2) in [1]. This remark together with the result of the author from [4] allow us to state the following

THEOREM. *Let m_γ be the Poisson kernel for $\mathcal{L}^\gamma$. Then for every $c_1, c_2 > 0$ there exists a positive constant C such that, for every $x \in N$, $\|D\|_{\mathfrak{n}\to\mathfrak{n}} \leqslant c_1$ and, for every $\gamma \in [0, c_2]$,*

$$C^{-1}(|x|+1)^{-Q-\gamma} \leqslant m_\gamma(x) \leqslant C(|x|+1)^{-Q-\gamma}.$$

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