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MULTIPARAMETER SUPERADDITIVE ERGODIC THEOREMS FOR MEAN ERGODIC L_1-CONTRACTIONS

BY

DOĞAN ÇÖMEZ* (FARGO, NORTH DAKOTA) AND MICHAEL LIN** (BEER SHEVA)

Abstract. Let T and S be commuting Markovian operators on $L_1(X)$. We prove that when the operators are mean ergodic and $\{F_{(m,n)}\}$ is a directionally (T, S)-superadditive dominated process, then the "averages" $n^{-2}F_{(n,n)}$ converge in L_1-norm. If, further, the process is strongly superadditive, then the same averages converge a.e. as well.

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1. INTRODUCTION

Let (X, Σ, μ) be a σ-finite measure space and T a linear contraction of $L_1(X)$. We will call T a *Markovian operator* if T is positive and $\int Tf\,d\mu = \int f\,d\mu$ for all $f \in L_1(X)$ (i.e. $T^*1 = 1$). We denote by $A_n(T)$ the average $n^{-1}\sum_{i=0}^{n-1} T^i$. If T and S are linear L_1-contractions, $A_{(m,n)}(T, S)$ will denote $(mn)^{-1}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} T^i S^j$. When T and S commute, $A_m(T)A_n(S) = A_{(m,n)}(T, S)$.

An L_1-contraction T is called *mean ergodic* if $A_n(T)f$ converges in L_1-norm for all $f \in L_1$ (cf. [13]).

Let T and S be positive linear contractions of $L_1(X, \mu)$. A family of integrable functions $F = \{F_{(m,n)}\}_{(m,n)\in N^2}$ with $F_{(0,0)} = F_{(n,0)} = F_{(0,n)} = 0$ for all $n \geqslant 1$ will be called a *directionally* (T, S)*-superadditive process* (or a *directionally superadditive process with respect to* T *and* S) if for all $k, l, m, n \geqslant 0$ we have

$$(1.1)\qquad F_{(m+k,n)} \geqslant F_{(m,n)} + T^m F_{(k,n)} \quad \text{and} \quad F_{(m,n+l)} \geqslant F_{(m,n)} + S^n F_{(m,l)}.$$

F is called a *strongly* (T, S)*-superadditive process* (see [4], [9] and [14]) if for all $k, l, m, n \geqslant 0$ we have

$$(1.2)\qquad F_{(m+k,n+l)} \geqslant F_{(m,n)} + T^m F_{(k,n+l)} + S^n F_{(m+k,l)} - T^m S^n F_{(k,l)}.$$

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If $\{-F_{(m,n)}\}$ is directionally (strongly) (T, S)-superadditive, then F is called *directionally* (*strongly*) (T, S)-*subadditive*; if both $\{F_{(m,n)}\}$ and $\{-F_{(m,n)}\}$ are directionally (T, S)-superadditive, then $F_{(m,n)} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} T^i S^j F_{(1,1)}$, and $\{F_{(m,n)}\}$ is called a (T, S)-*additive process*.

Remark. Directional superadditivity (called superadditivity in [7] and [14]) is called 2-*superadditivity* in [13], p. 202. The notion of superadditivity defined in [1] (which can be similarly defined in [7] also for Markov operators on L_1, but will not be used in this paper) implies directional superadditivity, but is weaker than strong superadditivity.

A superadditive process F is called *bounded* (with *time constant* γ_F) if

$$\gamma_F := \sup_{(m,n)>(0,0)} \frac{1}{mn} \|F_{(m,n)}\| < \infty,$$

and F is called *dominated* if there exists $g \in L_1$ (called a *dominant*) such that

$$\frac{1}{mn} F_{(m,n)} \leqslant A_{(m,n)}(T, S) g \qquad \text{for all } m, n > 0.$$

A dominated process is necessarily bounded, with $\int |g| \, d\mu \geqslant \gamma_F$ for every dominant g. A dominant g is called an *exact dominant* if $\int |g| \, d\mu = \gamma_F$.

Remarks. 1. A directionally superadditive process is positive ($F_{(m,n)} \geqslant 0$ for all $m, n > 0$) whenever $F_{(1,1)} \geqslant 0$.

2. If F is a directionally (strongly) superadditive process, then it is the sum of an additive process and the positive directionally (strongly) superadditive process defined by

$$F'_{(m,n)} = F_{(m,n)} - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} T^i S^j F_{(1,1)} \qquad \text{for all } m, n > 0.$$

Furthermore, $F' = \{F'_{(m,n)}\}$ is bounded (dominated) if F is bounded (dominated).

In the sequel we will use the notation $\boldsymbol{n} = (n, n)$ for any integer $n \geqslant 0$. For positive integers m, n, u, and v, $(u, v) \leqslant (m, n)$ means $u \leqslant m$ and $v \leqslant n$, and $(u, v) < (m, n)$ if $(u, v) \leqslant (m, n)$ and $(u, v) \neq (m, n)$. Unless stated otherwise, T and S will denote positive linear operators.

Multiparameter additive and superadditive processes with respect to L_1–L_∞ contractions (i.e., Dunford–Schwartz operators) have been a subject of intensive study by various authors. It is known that if T and S are Dunford–Schwartz operators, then the (ordinary) averages $n^{-2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T^i S^j f$ or the unrestricted averages $(mn)^{-1} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} T^i S^j f$ converge a.e. (see [3], [8], [10], and [12]) when $f \in L_1$ or $f \in L \log^+ L$, respectively. Also, if T and S are Markovian Dunford–Schwartz operators and $F = \{F_{(m,n)}\}$ is a bounded strongly (T, S)-superadditive process, then $n^{-2} F_n$ converge a.e. (see [4] and [9]). Almost everywhere convergence of unrestricted "averages" of strongly superadditive processes with respect to operators induced by measure-preserving transformations is established in [14], and the same convergence is obtained in [1] assuming only superadditivity of the process. Moving averages were considered in [11] for the superadditive processes of [1], and for strongly superadditive processes norm convergence was obtained in [5] without restricting the moving averages. On the other hand, it is known (e.g., [13], p. 151) that the pointwise ergodic theorem need not hold for a general Markovian L_1-contraction, so the above results are not valid if T and S are only L_1- (or only L_∞-) contractions. In [6], we obtained ergodic theorems for one-parameter superadditive processes with respect to a mean ergodic Markovian operator, which need not be an L_∞-contraction. In this paper we study the norm and a.e. convergence of the "averages" $n^{-2} F_n$ of superadditive processes with respect to commuting *mean ergodic* L_1-contractions. The results obtained extend some ergodic theorems, proved previously by various authors (see [14], [7], [4], and [9]), to the setting of multiparameter superadditive processes relative to mean ergodic operators. It should be noted that the result of [1] has not yet been extended to Markovian operators.

2. NORM CONVERGENCE

First we note that the time constant γ_F is attained as a limit:

PROPOSITION 2.1. *If T and S are commuting Markovian operators on L_1, and if F is a bounded directionally (T, S)-superadditive process, then*

$$\gamma_F = \lim_{m,n\to\infty} \frac{1}{mn} \int |F_{(m,n)}|\, d\mu.$$

Proof. See [4], pp. 615–616, or [9]. The proof there uses only the directional superadditivity of the process F and the fact that T and S are Markovian. ∎

In investigating the norm and a.e. convergence of multiparameter processes, we will often use the Brunel operator associated with the contractions under discussion (see [3] and [13], p. 213, for the definition of the Brunel operator). Moreover, we will often make use of the following theorem, which was proved in [6], that is why we will state it here (*two*-parameter version) for easy reference.

THEOREM A ([6], Theorem 2.2). *Let T and S be commuting contractions of L_1, and let U be the corresponding Brunel operator.*

(i) *If U is mean ergodic, then $A_n(T^m, S^m) f$ converges in L_1-norm for m fixed and every $f \in L_1$.*

(ii) *If the moduli τ and σ of T and S, respectively, commute, then U is mean ergodic if and only if $A_n(T, S) f$ converges in L_1-norm for every $f \in L_1$.*

If T and S are commuting mean ergodic contractions on a Banach space B with $E_1 = \lim_{n\to\infty} A_n(T)$ and $E_2 = \lim_{n\to\infty} A_n(S)$, then, for every $x \in B$, $\lim_{n\to\infty} \|A_n(T, S)x - E_1 E_2 x\| = 0$ ([6], Lemma 2.3). Hence, it follows from Theorem A (ii) that if T and S are (commuting) mean ergodic positive L_1-contrations, so is their Brunel operator.

THEOREM 2.2. *Let T and S be commuting Markovian operators on L_1 with mean ergodic Brunel operator U. If F is a dominated directionally (T, S)-superadditive process, then $n^{-2} F_n$ converges in L_1-norm. The limit function is both T- and S-invariant.*

Proof. By Theorem A (i), $n^{-2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T^i S^j F_{(1,1)}$ converges in L_1--norm, so we can assume that F is a positive process. Since F is dominated,

$$n^{-2} F_n \leqslant n^{-2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T^i S^j g \quad \text{for some } g \in L_1^+.$$

Again by Theorem A (i), $A_n(T, S)g$ converges in L_1-norm. By the "splitting theorem" ([13], p. 77), the limit g^* is T- and S-invariant. Since U is a convex combination of T^i and S^j's, also $Ug^* = g^*$. If $C = \{g^* > 0\}$ and $D = X - C$, then

$$\|1_D n^{-2} F_n\|_1 \to 0.$$

Therefore, it is enough to prove the norm convergence on C. Since $Tg^* = Sg^* = g^*$, $L_1(C)$ is both T- and S-invariant. Let $\hat{T} = T|_{L_1(C)}$ and $\hat{S} = S|_{L_1(C)}$, and define $G_{(m,n)} = 1_C F_{(m,n)}$. Then $G = \{G_{(m,n)}\}$ is bounded, and

$$\begin{aligned} G_{(m+k,n)} = 1_C F_{(m+k,n)} &\geqslant 1_C F_{(m,n)} + 1_C T^m F_{(k,n)} \\ &\geqslant 1_C F_{(m,n)} + 1_C T^m 1_C F_{(k,n)} = G_{(m,n)} + \hat{T}^m G_{(k,n)}. \end{aligned}$$

Similarly, $G_{(m,n+l)} \geqslant G_{(m,n)} + \hat{S}^n G_{(m,l)}$. Thus G is a bounded $(\hat{T}, \hat{S})$-superadditive process. Then $n^{-2} G_n$ converges in norm by Theorem 5.2 in [7], since the equivalent finite measure $g^* d\mu$ is invariant for $\hat{T}$ and $\hat{S}$. ∎

Remarks. 1. Strong superadditivity is not needed for the norm convergence.

2. If T and S have a common invariant equivalent probability, the norm convergence is obtained in [7] for bounded processes (see [14] for T and S induced by measure-preserving transformations). However, our proof of Theorem 2.2 requires that F be dominated (not just bounded).

3. If $T_1, T_2, \ldots, T_d$ are commuting Markovian operators on L_1, a family of L_1-functions $F = \{F_{\vec{n}}\}$ is called *directionally* $(T_1, \ldots, T_d)$-*superadditive* if

$$F_{\vec{n}+k\vec{e}_i} \geqslant F_{\vec{n}} + T_i^{n_i} F_{\vec{n}+(k-n_i)\vec{e}_i} \quad \text{for all } 1 \leqslant i \leqslant d,$$

where $\vec{n} = (n_1, n_2, \ldots, n_d)$, and $\vec{e}_i$ is the i-th coordinate unit vector, $1 \leqslant i \leqslant d$. In this case, if the associated Brunel operator U is mean ergodic, then the same arguments yield the norm convergence of the averages $n^{-d}F_{\mathbf{n}}$, where $\mathbf{n}=(n, n, \dots, n)$.

4. The superadditivity of [1], defined for d commuting measure-preserving transformations, can also be defined for d Markovian operators on L_1; strong $(T_1, T_2, \dots, T_d)$-superadditivity can be defined by using an analogue of the formula which computes the volume of a d-dimensional box from the d-dimensional distribution. For example, for commuting Markovian operators T, S and R on L_1, the strong superadditivity will be defined by

$$F_{(m+k,n+l,j+i)} \geqslant F_{(m,n,j)} + T^m F_{(k,n+l,j+i)} + S^n F_{(m+k,l,j+i)} + R^j F_{(m+k,n+l,i)}$$
$$- T^m S^n F_{(k,l,j+i)} - T^m R^j F_{(k,n+l,i)} - S^n R^j F_{(m+k,l,i)} + T^m S^n R^j F_{(k,l,i)}.$$

3. ALMOST EVERYWHERE CONVERGENCE

First we obtain the following a.e. result for additive processes (which generalizes Theorem 2.8 of [6]).

THEOREM 3.1. *Let T and S be commuting (not necessarily positive) contractions on L_1 whose moduli also commute. If the associated Brunel operator U is mean ergodic, then $A_n(T, S)f$ converge a.e. and in L_1-norm for all $f \in L_1$.*

Proof. By definition, the Brunel operator of T and S is the same as that of their moduli $\tau = |T|$ and $\sigma = |S|$. By Theorem A (i), $A_n(T, S)f$ and $A_n(\tau, \sigma)f$ converge in L_1-norm for all $f \in L_1$. Therefore, by Corollary (a) to Theorem 1 in [12], $A_n(T, S)f$ converges a.e. for all $f \in L_1$. Replacing T and S by τ and σ, respectively, we have also the a.e. convergence of $A_n(\tau, \sigma)f$ for all $f \in L_1$. ∎

The previous result will be extended to directionally superadditive processes that have an exact dominant. For norm convergence, domination is sufficient by Theorem 2.2.

THEOREM 3.2. *Let T and S be commuting Markovian operators on L_1 whose associated Brunel operator is mean ergodic. If F is a directionally (T, S)-superadditive process with an exact dominant, then $\lim_{n\to\infty} n^{-2}F_{\mathbf{n}}$ exists a.e. and in L_1-norm.*

Proof. By Theorem 3.1, $A_n(T, S)F_{(1,1)}$ converge a.e., so we can assume that F is non-negative. Let $0 \leqslant g \in L_1$ be the exact dominant. Then

$$\frac{1}{n^2}F_{\mathbf{n}} \leqslant \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^i S^j g = A_n(T, S)g.$$

Since $A_n(T, S)g$ converges a.e. and in L_1, by Theorem 3.1 (or Theorems 2.2 and 2.8 in [6]), the a.e. limit is necessarily the L_1-limit. Hence

$$\int \left(\limsup \frac{1}{n^2}F_{\mathbf{n}}\right) d\mu \leqslant \int \left(\lim_n A_n(T, S)g\right) d\mu = \lim_n \int A_n(T, S)g\, d\mu = \gamma_F.$$

By Proposition 2.1, for a given $\varepsilon > 0$, we can find a positive integer n_0 such that

$$\frac{1}{n_0^2}\int F_{n_0}\,d\mu > \gamma_F - \varepsilon.$$

We use Theorem A: since U is mean ergodic, by part (i) the averages $A_n(T^{n_0}, S^{n_0})f$ converge in L_1-norm for all $f \in L_1$, and by part (ii) also the Brunel operator associated with T^{n_0} and S^{n_0} is mean ergodic. The process

$$H_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} T^{n_0 i} S^{n_0 j} F_{n_0}$$

is a (T^{n_0}, S^{n_0})-additive process, so $n^{-2}H_n$ converges a.e. and in norm to the same limit, by Theorem 3.1. Since T and S are Markovian, we obtain

$$\int\left(\lim_n \frac{1}{n^2} H_n\right)d\mu = \lim_n \frac{1}{n^2}\int H_n\,d\mu = \int F_{n_0}\,d\mu > n_0^2(\gamma_F - \varepsilon).$$

For any n let $k_n = [n/n_0]$ and $j_n = k_n n_0$. By directional superadditivity and positivity of F we have

$$\frac{1}{n^2}F_n \geqslant \frac{1}{n^2}F_{j_n} \geqslant \frac{1}{(k_n+1)^2 n_0^2}F_{j_n}.$$

By directional superadditivity, $F_{(nn_0, nn_0)} \geqslant H_n$ (the proof by induction uses the inequality $F_{(kn_0, n_0)} \geqslant \sum_{i=0}^{k-1} T^{in_0} F_{(n_0, n_0)}$, which is also proved by induction). Hence we get

$$\int\left(\liminf \frac{1}{n^2}F_n\right)d\mu \geqslant \int\left(\liminf \frac{1}{k_n^2 n_0^2}F_{j_n}\right)d\mu \geqslant \frac{1}{n_0^2}\int \lim_n \frac{1}{n^2}H_n\,d\mu > \gamma_F - \varepsilon,$$

which implies

$$\int\left(\limsup \frac{1}{n^2}F_n - \liminf \frac{1}{n^2}F_n\right)d\mu < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this gives $\liminf n^{-2}F_n = \limsup n^{-2}F_n$ a.e., and hence $\lim n^{-2}F_n$ exists a.e. ∎

Remarks. 1. In the one-dimensional case, for T Markovian there is an exact dominant if $\{F_n\}$ is a bounded superadditive process (cf. [2]).

2. The one-dimensional result of [6] does not require the positive contraction T to be Markovian.

3. Our proof requires an exact dominant. In [1] there is a (two-parameter) example of a bounded (in fact, dominated) superadditive process which has no exact dominant.

4. In [1] there is also a two-parameter example of a superadditive process which is not strongly superadditive, but still has an exact dominant.

THEOREM 3.3. *Let T and S be commuting Markovian operators on L_1 whose associated Brunel operator U satisfies $U1 \leqslant 1$. If F is a directionally (T, S)-superadditive process with an exact dominant, then $n^{-2} F_n$ converges a.e.*

Proof. By Theorem 3 in [12] (the proof of Theorem 2.4 in [6] is incomplete), $A_n(T, S)f$ converges a.e. for every $f \in L_1$. Hence, by considering

$$F'_n = F_n - \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T^i S^j F_{(1,1)},$$

we may assume that F_n is positive.

If $g \in L_1^+$ is a dominant, we have

$$0 \leqslant n^{-2} F_n \leqslant A_n(T, S) g \leqslant \delta A_{m(n)}(U) g,$$

where $\delta > 0$ is a constant (from the construction of the Brunel operator), and $m(n) = [\sqrt{n}]+1$.

The measure is subinvariant for the Brunel operator U. Hence both its conservative and dissipative parts (C and D, respectively) are absorbing ([13], p. 131). The conservative part is further decomposed into two absorbing sets: C_1 — on which U has an equivalent finite invariant measure, and C_0 — on which there is no absolutely continuous finite invariant measure. By the Dunford–Schwartz theorem, $A_m(U)f$ converges a.e. for all $f \in L_1$, and the limit is 0 a.e. on $D_1 = D \cup C_0$. Hence $n^{-2} F_n \to 0$ a.e. on D_1.

It remains to prove convergence a.e. on C_1. Since U is Markovian, $U^* 1_{C_1} = 1_{C_1}$, so also $T^* 1_{C_1} = 1_{C_1} = S^* 1_{C_1}$, by the Brunel–Falkowitz lemma. Hence $L_1(C_1)$ is invariant under both T and S, and their respective restrictions are denoted by $\hat{T}$ and $\hat{S}$. As in Theorem 2.2, if we define $G_{(m,n)} = 1_{C_1} F_{(m,n)}$, we obtain a $(\hat{T}, \hat{S})$-superadditive process. Similarly, $H = 1_{D_1} F$ is a superadditive process with respect to $\bar{T}$ and $\bar{S}$, the restrictions of T and S to $L_1(D_1)$ (which is also invariant). If g is a dominant for F, then $\hat{g} = 1_{C_1} g$ is a dominant for G and $\bar{g} = 1_{D_1} g$ is a dominant for H. Since all the restrictions are also Markovian, Proposition 2.1 shows that $\gamma_F = \gamma_G + \gamma_H$. Hence, if g is an exact dominant for F, $\hat{g}$ is an exact dominant for G, and since $\hat{T}$ and $\hat{S}$ are mean ergodic, the previous theorem yields a.e. convergence on C_1 of $n^{-2} G_n = n^{-2} 1_{C_1} F_n$. ∎

Next, we show that for strongly superadditive processes, boundedness suffices for a.e. convergence. The two different conditions imposed in the previous theorems on the Brunel operator U, that it be mean ergodic or that it be an L_∞-contraction, are special cases of the condition that U is *pointwise ergodic*, i.e., $A_n(U)f(x)$ converges a.e. for every $f \in L_1$ (and, by Fatou's lemma, the limit function is integrable).

THEOREM 3.4. *Let T and S be commuting Markovian operators on L_1 whose associated Brunel operator U is pointwise ergodic, and let F be a bounded strongly (T, S)-superadditive process. Then $n^{-2} F_n$ converges a.e. to an L_1-function.*

Proof. By the proof of Lemma 4.3 of [9] (which assumes the subinvariance of the measure, i.e., the fact that U is also an L_∞-contraction, only for deducing that U is pointwise ergodic), pointwise ergodicity of U implies that for F positive we have

$$\mu\left(\left\{x\colon \limsup_n \frac{1}{n^2} F_n(x) > \alpha\right\}\right) \leqslant \frac{\delta\gamma_F}{\alpha} \quad \text{for all } \alpha > 0. \tag{3.1}$$

Since U is pointwise ergodic, the a.e. convergence of additive processes follows from Theorem 1 (b) in [12], and Fatou's lemma yields the integrability of the limit function. Now the proof of Theorem 3.1 in [9] proves our theorem (the integrability of the limit for F positive is also there). ∎

Remark. The result of [1], for T and S induced by transformations preserving the same measure, requires only the superadditivity introduced there, which is weaker than strong superadditivity. It is not clear if Theorem 3.4 can be proved for operators in this setting.

When the Brunel operator satisfies $U1 \leqslant 1$, (3.1) can be improved to give a "maximal inequality". This can be obtained from one of the inequalities in the proof of Lemma 4.3 in [9] (using the maximal inequality for U). We present here a simple proof, which does not depend on the intricate arguments of [2].

PROPOSITION 3.5. *Let T and S be commuting Markovian operators on L_1 whose associated Brunel operator U satisfies $U1 \leqslant 1$, and let F be a bounded positive strongly (T, S)-superadditive process. Then, for any $\alpha > 0$,*

$$\mu\left(\left\{x\colon \sup_n \frac{1}{n^2} F_n(x) > \alpha\right\}\right) \leqslant \frac{\delta\gamma_F}{\alpha}.$$

Proof. Strong superadditivity of the positive process implies that (see [9] and [4]), for each $k > 1$, the functions

$$g_k = \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} [F_{(i,j)} - TF_{(i-1,j)} - SF_{(i,j-1)} + TSF_{(i-1,j-1)}]$$

are in L_1^+, and satisfy $\int g_k\, d\mu = k^{-2} \int F_k\, d\mu \leqslant \gamma_F$, and

$$G_n^k := \sum_{i,j=0}^{n-1} T^i S^j g_k \geqslant \left(1 - \frac{n}{k}\right)^2 F_n \quad \text{for all } 1 \leqslant n \leqslant k.$$

Brunel's estimate ([13], Theorem 3.4, p. 213) yields

$$\left(1 - \frac{n}{k}\right)^2 \frac{1}{n^2} F_n \leqslant \frac{1}{n^2} G_n^k \leqslant \delta A_{m(n)}(U) g_k \quad \text{for all } 1 \leqslant n \leqslant k. \tag{3.2}$$

Fix integers N and r, and put $k = rN$. Since $1 - n/(rN) \geqslant (r-1)/r$ for $n \leqslant N$, (3.2) yields

$$\max_{1 \leqslant n \leqslant N} \frac{(r-1)^2}{r^2 n^2} F_n \leqslant \max_{1 \leqslant n \leqslant N} \left(1 - \frac{n}{rN}\right)^2 \frac{1}{n^2} F_n \leqslant \max_{1 \leqslant n \leqslant N} \delta A_{m(n)}(U) g_k. \tag{3.3}$$

Since $\|g_k\| \leqslant \gamma_F$, the maximal ergodic inequality for U ([13], p. 51) yields for $\alpha > 0$

$$\mu\left\{x:\ \max_{1\leqslant n\leqslant N}\frac{1}{n^2}F_n(x) > \alpha\right\} \leqslant \mu\left\{x:\ \max_{1\leqslant n\leqslant N}\frac{r^2}{(r-1)^2}\,\delta A_{m(n)}\,g_k(x) > \alpha\right\}$$

$$\leqslant \frac{r^2\,\delta\,\|g_k\|}{(r-1)^2\,\alpha} \leqslant \frac{r^2\,\delta\gamma_F}{(r-1)^2\,\alpha}.$$

By letting $r \to \infty$, and then $N \to \infty$, we obtain the desired inequality. ∎

4. ON CONDITIONS RELATED TO UNRESTRICTED ALMOST EVERYWHERE CONVERGENCE

Smythe [14] proved unrestricted a.e. ergodic convergence for bounded strongly (T, S)-superadditive process $F \subset L\log^+ L$ with some additional condition (which will be called $\mathbf{C}$), which, as remarked there, is not easy to verify (although it is, in a sense, necessary). Some stronger but easier-to-check conditions were proposed in [14]. Let

$$\Delta_{(k,l)} = F_{(k,l)} - TF_{(k-1,l)} - SF_{(k,l-1)} + TSF_{(k-1,l-1)}$$

and define

$$\varphi_{(m,n)} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta_{(i,j)}.$$

It was shown in [14] that $\mathbf{C}$ is implied by

$$\sup_{(m,n)}\|\varphi_{(m,n)}\|_{L\log^+ L} < \infty. \tag{4.1}$$

Also, if

$$\sup_{(m_i,n_i)}\left\|\frac{1}{m_i n_i}\sum_{(k,l)\leqslant(m_i,n_i)}(F_{(1,1)} - \Delta_{(k,l)})\right\|_{L\log^+ L} < \infty, \tag{4.2}$$

then (4.2) and the condition $F \subset L\log^+ L$ imply (4.1) and the unrestricted a.e. ergodic convergence [14]. However, neither of the conditions (4.1) and (4.2) is much easier to verify, and it was conjectured in [14] that the conditions $F \subset L\log^+ L$ and (4.2) can be replaced by the single condition

$$\sup_{(m,n)}\left\|\frac{1}{mn}F_{(m,n)}\right\|_{L\log^+ L} < \infty. \tag{4.3}$$

In other words, the question is if Theorem 2.2 in [14] is valid when its assumptions are replaced by (4.3). If condition (4.3) implied (4.1) (it certainly implies $F \subset L\log^+ L$), then Theorem 2.2 of [14] could be applied to obtain the a.e. unrestricted convergence. It was shown in [14] that in many interesting cases condition (4.3) implies (4.2), and hence (4.1). Unfortunately, these implications are not true in general. We will construct a dominated superadditive process $F = \{F_{(m,n)}\} \subset L\log^+ L$ for which (4.3) holds and (4.1) does not.

LEMMA 4.1. *Let S be a Markovian operator and $G = \{G_j\}$ an S-superadditive process. Then $F_{(m,n)} := mG_n$ defines a strongly (I, S)-superadditive process (I is the identity transformation). If G is bounded, then F has an exact dominant.*

The proof of strong superadditivity is a simple computation. If G is bounded, then by [2] it has an exact dominant, which is then an exact dominant for F.

Let (X, Σ, μ) be the unit interval with Lebesgue measure.

CLAIM 0. *There exists a sequence $\{p_k\}_{k\geq 0}$ of non-negative measurable functions on X with:*

(i) $$\|p_n\|_\infty < \infty \quad \textit{for every } n.$$

(ii) $$\sum_{n=0}^{\infty} \|p_n\|_1 < \infty.$$

(iii) $$\|2p_n\|_{L\log^+ L} \to 0 \quad \textit{as } n \to \infty.$$

(iv) $$\Big\|\sum_{n=0}^{\infty} p_n\Big\|_{L\log^+ L} = \infty.$$

Proof. Let $\alpha_n = n+1$ for $n \geq 0$. Put $t_0 = 1$, and for $n \geq 1$ put

$$t_n = \frac{1}{[(n+1)\log(n+1)]^2}.$$

We now define $p_n = \alpha_n \chi_{[0,t_n]}$. Then $p_n \geq 0$, and we have:

(i) $$\|p_n\|_\infty = \alpha_n < \infty \quad \text{for every } n \geq 0.$$

(ii) $$\sum_{n=1}^{\infty} \|p_n\|_1 = \sum_n t_n \alpha_n = \sum_n \frac{n+1}{[(n+1)\log(n+1)]^2} = \sum_n \frac{1}{(n+1)[\log(n+1)]^2} < \infty.$$

(iii) $$\|2p_n\|_{L\log^+ L} = t_n 2\alpha_n \log(2\alpha_n) = \frac{2}{(n+1)\log(n+1)} + \frac{\log 4}{(n+1)[\log(n+1)]^2} \to 0 \quad \text{as } n \to \infty.$$

(iv) $$\Big\|\tfrac{1}{2}\sum_{n=1}^{\infty} p_n\Big\|_{L\log^+ L} \geq c \sum_{n\geq 1} (t_{n-1} - t_n)\, n \log n \sim \sum_n \frac{1}{(n+1)\log^+(n+1)} = \infty. \ \blacksquare$$

Let S be a measure-preserving transformation on (X, Σ, μ). The operator induced by S will also be denoted by S.

For any sequence $\{p_k\}_{k\geq 0}$ of positive functions with properties (i)–(iv), we can construct a strongly (I, S)-superadditive positive process as follows. Let $\{N_k\colon k\geq -1\}$ be a strictly increasing sequence of integers with $N_{-1}=0$; let $q_j=0$ for $j\notin\{N_k\colon k\geq -1\}$, $q_{N_k}=p_{k+1}$. Define inductively $f_0=0$, and $f_{j+1}=Sf_j+q_j$. Thus, $f_1=p_0$, for $n\leq N_0$ we have $f_n=S^{n-1}p_0$, and $f_{N_0+1}=S^{N_0}p_0+p_1$. We now put $F_{(m,n)}=m\sum_{j=0}^{n}f_j$. Clearly, $F_{(0,n)}=F_{(m,0)}=0$. The definitions yield that

$$F_{(m,n)}=m\sum_{r=0}^{k}\Big[\sum_{j=0}^{n-N_{r-1}-1}S^jp_r\Big] \quad \text{for } k \text{ defined by } N_{k-1}<n\leq N_k. \tag{4.4}$$

Let $G_0=0$, and for $n>0$ define $G_n=\sum_{r=0}^{k}\big[\sum_{j=0}^{n-N_{r-1}-1}S^jp_r\big]$ with $k=k(n)$ defined by $N_{k-1}<n\leq N_k$.

Claim 1. *The process $\{G_n\}$ is S-superadditive.*

Proof. We have to prove $G_{n+l}\geq G_n+S^nG_l$, which is obvious if n or l is zero. Let $n>0$ and $l>0$, and take integers k, u, v such that $N_{k-1}<n\leq N_k$, $N_{u-1}<l\leq N_u$, and $N_{v-1}<n+l\leq N_v$. Then

$$G_{n+l}-S^nG_l=\sum_{r=0}^{v}\Big[\sum_{j=0}^{n+l-N_{r-1}-1}S^jp_r\Big]-S^n\sum_{r=0}^{u}\Big[\sum_{j=0}^{l-N_{r-1}-1}S^jp_r\Big].$$

When $u>k$, this yields, since always $v\geq u$, that

$$G_{n+l}-S^nG_l\geq\sum_{r=0}^{u}\Big[\sum_{j=0}^{n-1}S^jp_r\Big]\geq\sum_{r=0}^{k}\Big[\sum_{j=0}^{n-1}S^jp_r\Big]\geq G_n.$$

When $u\leq k$, we subtract more and add less (since $k\leq v$) to obtain

$$G_{n+l}-S^nG_l\geq\sum_{r=0}^{k}\Big[\sum_{j=0}^{n+l-N_{r-1}-1}S^jp_r\Big]-S^n\sum_{r=0}^{k}\Big[\sum_{j=0}^{l-N_{r-1}-1}S^jp_r\Big]\geq G_n,$$

proving the claim. ∎

Claim 2. *F is a bounded (I, S)-superadditive process. Furthermore, $F\subset L\log^+L$.*

Proof. By Claim 1 and Lemma 4.1, F is strongly (I, S)-superadditive. Let $\phi=\sum_{r=0}^{\infty}p_r$, which is in L_1 by property (ii) of $\{p_r\}$. Then ϕ is a dominant for the S-superadditive process G, since

$$G_n=\sum_{r=0}^{k}\Big[\sum_{j=0}^{n-N_{r-1}-1}S^jp_r\Big]\leq\sum_{r=0}^{k}\Big[\sum_{j=0}^{n-1}S^jp_r\Big]\leq\sum_{j=0}^{n-1}S^j\Big(\sum_{r=0}^{k}p_r\Big)\leq\sum_{j=0}^{n-1}S^j\phi.$$

Hence F has an exact dominant, by Lemma 4.1. Property (iv) of $\{p_r\}$ shows that $\phi\notin L\log^+L$. Equality (4.4) and property (i) of $\{p_r\}$ yield that each $F_{(m,n)}$ is bounded, so $F\subset L\log^+L$. ∎

We will now assume that S is ergodic, and choose $\{N_k\}_{k\geqslant 0}$ so that our process satisfies (4.3) but not (4.1). First we will make a careful selection of the sequence $\{N_k\}_{k\geqslant 0}$ as follows.

Define $\gamma_k = \|\sum_{i=0}^{k} p_k\|_1 = \sum_{i=0}^{k} \|p_k\|_1$. By the ergodic theorem,

$$\lim_{n\to\infty} A_n(S)\,p_0 = \|p_0\|_1 \text{ a.e.}$$

By Egorov's theorem, for any $\varepsilon_0 > 0$ there exist a positive integer N_0 and a set $A_0 \in \Sigma$ with $\mu(A_0) \geqslant 1-\varepsilon_0$, such that

$$\forall n \geqslant N_0\colon\ |A_n(S)\,p_0(x)-\gamma_0| < \varepsilon_0 \text{ on } A_0.$$

On the remaining part of X, $A_n(S)\,p_0$ is obviously bounded by α_0. Similarly, $A_n(S)\,p_1 \to \|p_1\|_1$ a.e., and hence

$$\lim_{n\to\infty} \frac{1}{n+N_0} \sum_{j=0}^{n-1} p_1(S^j x) \to \|p_1\| \text{ a.e.} \qquad (\text{since } n/(n+N_0) \to 1).$$

Thus

$$A_n(S)\,p_0(x) + \frac{1}{n+N_0} \sum_{j=0}^{n-1} p_1(S^j x) \to \gamma_1 \text{ a.e.}$$

Therefore, for any $\varepsilon_1 > 0$ there exists a positive integer $N_1 > N_0$ and $A_1 \in \Sigma$ with $\mu(A_1) > 1-\varepsilon_1$ such that

$$\forall n \geqslant N_1\colon\ \left| A_n(S)\,p_0(x) + \frac{1}{n} \sum_{j=0}^{n-N_0-1} p_1(S^j x) - \gamma_1 \right| < \varepsilon_1 \text{ on } A_1.$$

On the remaining part of X, $A_n(S)\,p_0(x) + n^{-1}\sum_{j=0}^{n-N_0-1} p_1(S^j x)$ is bounded by $\alpha_1+\alpha_0$. We choose a sequence $\varepsilon_k \downarrow 0$ satisfying

$$\text{(4.5)} \qquad \varepsilon_k \Big(2\sum_{i=0}^{k} \alpha_i\Big) \log^+ \Big(2\sum_{i=0}^{k} \alpha_i\Big) < 1 \qquad \text{for all } k \geqslant 0,$$

and continuing the above process inductively, we obtain a strictly increasing sequence of positive integers N_k and a sequence of sets $A_k \in \Sigma$ with $\mu(A_k) > 1-\varepsilon_k$, such that

$$\text{(4.6)} \qquad \forall n \geqslant N_k\colon\ \left| \frac{1}{n} \sum_{r=0}^{k} \sum_{j=0}^{n-N_{r-1}-1} p_r(S^j x) - \gamma_k \right| < \varepsilon_k \text{ on } A_k.$$

On all of X, so in particular on the remaining part $X-A_k$, the function

$$\frac{1}{n}\sum_{j=0}^{n-1} p_0(S^j x) + \frac{1}{n}\sum_{r=1}^{k} \sum_{j=0}^{n-N_{r-1}-1} p_r(S^j x) = \frac{1}{n}\sum_{r=0}^{k} \sum_{j=0}^{n-N_{r-1}-1} p_r(S^j x)$$

is bounded by $\sum_{i=0}^{k} \alpha_i$.

CLAIM 3. *F satisfies* (4.3).

Proof. Recall that for $n > 0$ we defined $G_n = \sum_{r=0}^{k} \left[\sum_{j=0}^{n-N_{r-1}-1} S^j p_r\right]$ with $k = k(n)$ defined by $N_{k-1} < n \leqslant N_k$, and $(mn)^{-1} F_{(m,n)} = n^{-1} G_n$. For $n > N_0$ we have $k(n) \geqslant 1$, and we can define

$$H_n(x) = \frac{1}{n} \sum_{r=0}^{k-1} \sum_{i=0}^{n-N_{r-1}-1} S^i p_r(x) \quad \text{and} \quad J_n(x) = \frac{1}{n} \sum_{i=0}^{n-N_{k-1}-1} S^i p_k(x).$$

We then have $n^{-1} G_n = H_n + J_n$. Since the function $\omega(t) = |t| \log^+ |t|$ is convex, we obtain

$$\text{(4.7)} \quad \|H_n + J_n\|_{L\log^+ L} = \left\| \frac{2H_n + 2J_n}{2} \right\|_{L\log^+ L} \leqslant \frac{1}{2} [\|2H_n\|_{L\log^+ L} + \|2J_n\|_{L\log^+ L}].$$

Since $n > N_{k-1}$, we have, by (4.6), $\gamma_{k-1} - \varepsilon_{k-1} \leqslant H_n(x) \leqslant \gamma_{k-1} + \varepsilon_{k-1}$ on A_{k-1}, and on the rest of the space $H_n(x) \leqslant \sum_{i=0}^{k-1} \alpha_i$. Since $\omega(t)$ is increasing on $[0, \infty)$, the condition (4.5) yields

$$\begin{aligned} \|2H_n\|_{L\log^+ L} &= \int_{A_{k-1}} \omega(2H_n)\, d\mu + \int_{A_{k-1}^c} \omega(2H_n)\, d\mu \\ &\leqslant \omega(a_k)\, \mu(A_{k-1}) + \varepsilon_{k-1}\, \omega\Big(2 \sum_{i=0}^{k-1} \alpha_i\Big) \leqslant \omega(a_k) + 1, \end{aligned}$$

where $a_k = 2(\gamma_{k-1} + \varepsilon_{k-1})$. But for every k we have $\gamma_k \leqslant \sum_{i=0}^{\infty} \|p_i\|_1 = \|\phi\|_1$, and $\varepsilon_k \leqslant \varepsilon_0$, so we obtain $\|H_n\|_{L\log^+ L} \leqslant \omega(2\|\phi\|_1 + 2\varepsilon_0) + 1$ for every $n > 0$.

Now, since ω is nondecreasing and convex, we have

$$\begin{aligned} \|2J_n\|_{L\log^+ L} &= \left\| \omega\Big(\frac{2}{n} \sum_{i=0}^{n-N_{k-1}-1} S^i p_k\Big) \right\|_1 \leqslant \left\| \omega\Big(\frac{2}{n - N_{k-1}} \sum_{i=0}^{n-N_{k-1}-1} S^i p_k\Big) \right\|_1 \\ &\leqslant \left\| \frac{1}{n - N_{k-1}} \sum_{i=0}^{n-N_{k-1}-1} \omega(2S^i p_k) \right\|_1 \leqslant \frac{1}{n - N_r} \sum_{i=0}^{n-N_{k-1}-1} \|\omega(2S^i p_k)\|_1. \end{aligned}$$

By the measure-preserving property of S,

$$\|\omega(2S^i p_k)\|_1 = \|\omega(2p_k)\|_1 = \|2p_k\|_{L\log^+ L}.$$

Therefore, we have

$$\left\| \frac{1}{mn} F_{(m,n)} \right\|_{L\log^+ L} \leqslant \omega(2\|\phi\|_1 + 2\varepsilon_0) + 1 + \sup_k \|2p_k\|_{L\log^+ L}$$

for $n > N_0$ and any $m > 0$, which, by property (iii) of $\{p_k\}$, yields (4.3). ∎

CLAIM 4. *F fails to satisfy* (4.1).

Proof. Observe that, for any $i, j \geqslant 1$, the definitions yield

$$\Delta_{(i,j)} = F_{(i,j)} - SF_{(i,j-1)} - TF_{(i-1,j)} + TSF_{(i-1,j-1)} = G_j - SG_{j-1} \stackrel{\text{def}}{=} \Delta_j.$$

Hence

$$\varphi_{(m,n)} = \frac{1}{n} \sum_{j=1}^{n} (G_j - SG_{j-1}) \stackrel{\text{def}}{=} \varphi_n.$$

Let $k = k(j)$ be defined by $N_{k-1} < j \leqslant N_k$. When $N_{k-1} < j-1$, we obtain $\Delta_j = \sum_{r=0}^{k} p_r$. When $N_{k-1} = j-1$, we obtain

$$\Delta_j = \sum_{r=0}^{k-1} p_r + \sum_{l=0}^{j-N_{k-1}-1} S^l p_k.$$

Thus, we always have $\Delta_j \geqslant \sum_{r=0}^{k} p_r$, so

$$\varphi_n \geqslant \frac{1}{n} \sum_{j=1}^{n} (p_0 + p_1 + \ldots + p_{k(j)}).$$

The set $\{j\colon 1 \leqslant j \leqslant n, k(j) > r\}$ contains $n - N_{r-1}$ numbers, and p_r is in the sum for each of these j. Hence

$$\varphi_n \geqslant \frac{1}{n}\Big[np_0 + \sum_{r=1}^{k(n)} (n - N_{r-1}) p_r\Big] = \frac{1}{n} \sum_{r=0}^{k(n)} (n - N_{r-1}) p_r.$$

Given $M > 0$, by (iv), find k such that $\|\frac{1}{2}\sum_{l=0}^{k} p_l\|_{L\log^+ L} > M$. Then choose $N > N_k$ such that, for all $n \geqslant N$, $(n - N_k)/n > \frac{1}{2}$. Hence, for $n \geqslant N$, we have $k(n) > k$, and $(n - N_{r-1})/n \geqslant \frac{1}{2}$ for $r \leqslant k$, so

$$\varphi_{(m,n)} = \varphi_n \geqslant \frac{1}{n} \sum_{r=0}^{k} (n - N_{r-1}) p_r \geqslant \frac{1}{2} \sum_{r=0}^{k} p_r.$$

Consequently, $\|\varphi_{(m,n)}\|_{L\log^+ L} \geqslant \|\frac{1}{2}\sum_{r=0}^{k} p_r\|_{L\log^+ L} > M$, so (4.1) does not hold. ∎

Remarks. 1. Notice that, since all the terms are positive, $F_{(m,n)} \geqslant m\sum_{j=0}^{k(n)} p_j$ by (4.4), so $\sup_{m,n} \|F_{(m,n)}\|_{L\log^+ L} = \infty$ by (iv).

2. In our example, unrestricted a.e. ergodic convergence holds, by the one-dimensional "subadditive ergodic theorem" applied to $\{G_m\}$. Thus, the conjecture (see [14]) that condition (4.3) implies the unrestricted a.e. ergodic convergence is still open.

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Doğan Çömez
Department of Mathematics
North Dakota State University
Fargo, ND 58105, USA
E-mail: dogan.comez@ndsu.nodak.edu

Michael Lin
Department of Mathematics
Ben-Gurion University of the Negev
Beer Sheva, Israel
E-mail: lin@math.bgu.ac.il

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