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LIMIT DISTRIBUTIONS OF DIFFERENCES AND QUOTIENTS OF NON-ADJACENT k-TH RECORD VALUES

BY

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Abstract. Let $\{Y_n^{(k)}, n \geqslant 1\}$ denote the sequence of k-th record values of the sequence $\{X_n, n \geqslant 1\}$ of i.i.d. random variables with an absolutely continuous distribution function F. Fix $r \in N$. We show that, for some very broad class of distributions F, the limit distribution of the sequence

$$k(Y_{n+r}^{(k)} - Y_n^{(k)}), \quad k \geqslant 1,$$

is the gamma distribution with pdf

$$f_{r,\lambda}(x) = \frac{\lambda^r}{(r-1)!} x^{r-1} \exp(-\lambda x), \quad x \geqslant 0,$$

where $\lambda > 0$ is a parameter which depends on F. We prove the similar result for k-th lower record values $Z_n^{(k)}$. Moreover, we discuss the asymptotic behaviour of quotients of these quantities.

1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed random variables with a common distribution function (cdf) F and probability density function (pdf) f. Moreover, let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics of a sample $X_1, \ldots, X_n$.

For a fixed $k \geqslant 1$ we define the *k-th* (*upper*) *record times* $U_k(n)$, $n \geqslant 1$, of the sequence $\{X_n, n \geqslant 1\}$ as $U_k(1) = 1$,

$$U_k(n+1) = \min\{j > U_k(n): X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \quad n \geqslant 1,$$

and the *k-th upper record values* as $Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}$ for $n \geqslant 1$ (cf. [3]).

Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_1(n):U_1(n)} := R_n$ – (upper) record values of the sequence $\{X_n, n \geqslant 1\}$ and that $Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$.

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Similarly, for a fixed $k \geqslant 1$ we define the *k-th lower record times* $L_k(n)$, $n \geqslant 1$, of the sequence $\{X_n,\ n \geqslant 1\}$ as $L_k(1) = 1$,

$$L_k(n+1) = \min\{j > L_k(n)\colon X_{k:j+k-1} < X_{k:L_k(n)+k-1}\}, \quad n \geqslant 1,$$

and the *k-th lower record values* as $Z_n^{(k)} = X_{k:L_k(n)+k-1}$ for $n \geqslant 1$.

Note that for $k = 1$ we have $Z_n^{(1)} = X_{1:L_1(n)} := R_n'$ — (lower) record values of the sequence $\{X_n,\ n \geqslant 1\}$ and that $Z_1^{(k)} = X_{k:k} = \max(X_1, \ldots, X_k)$.

In [4] it has been shown that if F is an absolutely continuous distribution function with probability density function f, concentrated on the interval $S \subset \boldsymbol{R}$, and if $h(x) = f(x)/(1-F(x))$ is a differentiable function with a bounded first derivative, then

$$k(Y_{n+1}^{(k)} - Y_n^{(k)}) \xrightarrow{D} W_n, \quad k \to \infty,$$

(D — in distribution), where W_n is exponentially distributed for all n with the df

$$F_\lambda^*(x) = 1 - \exp(-\lambda x), \quad x \geqslant 0,$$

and $\lambda = h(x_0^+)$ (the right limit of $h(x)$ at the point x_0), $x_0 = \inf S$, and F_0^*, F_∞^* denote the distribution concentrated at zero and the improper distribution concentrated at infinity, respectively.

Moreover, it is shown in [2] that, under suitable assumptions, the limit distributions of sequences

$$k(Z_n^{(k)} - Z_{n+1}^{(k)}),\ k \geqslant 1,$$

and

$$n(Y_{n+1}^{(k)}/Y_n^{(k)} - 1),\ n \geqslant 1, \quad n(Z_n^{(k)}/Z_{n+1}^{(k)} - 1),\ n \geqslant 1,$$

are exponential distributions with appropriate parameters depending on F.

In this paper we extend those results and we show that for a large class of distributions F for any fixed n, $r \in N$, the sequences

$$k(Y_{n+r}^{(k)} - Y_n^{(k)}),\ k \geqslant 1, \quad \text{and} \quad k(Z_n^{(k)} - Z_{n+r}^{(k)}),\ k \geqslant 1,$$

converge in distribution as $k \to \infty$ to some gamma distributed random variables, respectively. Moreover, we show that for any fixed k, $r \in N$, the sequences

$$n(Y_{n+r}^{(k)}/Y_n^{(k)} - 1),\ n \geqslant 1, \quad \text{and} \quad n(Z_n^{(k)}/Z_{n+r}^{(k)} - 1),\ n \geqslant 1,$$

converge weakly as $n \to \infty$ to gamma or negative gamma distribution. We illustrate our results with examples of limit behaviour of differences and quotients of k-th records. In the last section we discuss alternative proofs of the main results of the paper.

NOTATION. Throughout the paper $\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$, denotes a gamma distributed random variable with the pdf

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \quad x \geqslant 0.$$

If $\beta = 0$, then $\Gamma(\alpha, \beta)$ is improper distribution concentrated at ∞, and if $\beta = \infty$, then $\Gamma(\alpha, \beta)$ is the distribution concentrated at zero. Similarly, $N\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$, denotes a negative gamma distribution with the pdf

$$\bar{f}_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} |x|^{\alpha-1} \exp \beta x, \quad x \leqslant 0.$$

Moreover, let $\bar{F}(x) = 1 - F(x)$ and

$$H(x) = -\log(\bar{F}(x)), \quad h(x) = f(x)/\bar{F}(x) = H'(x)$$

denote the hazard function and the hazard rate of F, respectively. Similarly, let

$$\bar{H}(x) = -\log \bar{F}(x), \quad \bar{h}(x) = f(x)/F(x) = -\bar{H}'(x).$$

We also define

$$q(x) = -\frac{f(x)}{\bar{F}(x) \log \bar{F}(x)} \quad \text{and} \quad \bar{q}(x) = -\frac{f(x)}{F(x) \log F(x)}.$$

2. PROBABILITY DISTRIBUTIONS OF $Y_{n+r}^{(k)} - Y_n^{(k)}$ AND $Z_n^{(k)} - Z_{n+r}^{(k)}$

It is known that the pdf of $Y_n^{(k)}$ is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!} (H(x))^{n-1} (\bar{F}(x))^{k-1} f(x), \quad x \in \mathbf{R}, \tag{2.1}$$

and the joint pdf of $(Y_m^{(k)}, Y_n^{(k)})$, $m < n$, is

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!} (H(x))^{m-1} h(x) \times \left(-\log \frac{\bar{F}(y)}{\bar{F}(x)}\right)^{n-m-1} (\bar{F}(y))^{k-1} f(y) \tag{2.2}$$

for $x < y$ and it is 0 for $x \geqslant y$ (cf. [5]).

Moreover, the pdf of $Z_n^{(k)}$ is

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!} (\bar{H}(x))^{n-1} (F(x))^{k-1} f(x), \quad x \in \mathbf{R}, \tag{2.3}$$

and the joint pdf of $(Z_m^{(k)}, Z_n^{(k)})$, $m < n$, is

$$f_{Z_m^{(k)},Z_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!\,(n-m-1)!}(\bar{H}(x))^{m-1}\bar{h}(x) \times\left(-\log\frac{F(y)}{F(x)}\right)^{n-m-1}(F(y))^{k-1}f(y) \tag{2.4}$$

for $x \geqslant y$ and it is 0 otherwise.

Lemma 1. *The distribution function of the random variable* $\Delta_{n,r}^{(k)} = Y_{n+r}^{(k)} - Y_n^{(k)}$ *is of the form*

$$F_{\Delta_{n,r}^{(k)}}(x) = 1 - \sum_{i=0}^{r-1}\frac{k^i}{i!}\int_S\left(-\log\frac{\bar{F}(u+x)}{\bar{F}(u)}\right)^i\left(\frac{\bar{F}(u+x)}{\bar{F}(u)}\right)^k dF_{Y_n^{(k)}}(u)$$

for $x \geqslant 0$ *and it is* 0 *otherwise.*

Proof. Using (2.2) we see that the pdf of $\Delta_{n,r}^{(k)}$ is

$$f_{\Delta_{n,r}^{(k)}}(w) = \frac{k^{n+r}}{(n-1)!\,(r-1)!}\int_R\left(-\log(\bar{F}(u))\right)^{n-1}\frac{f(u)}{\bar{F}(u)} \times\left(-\log\frac{\bar{F}(w+u)}{\bar{F}(u)}\right)^{r-1}(\bar{F}(w+u))^{k-1}f(w+u)\,du \tag{2.5}$$

for $w \geqslant 0$. Therefore, the df of $\Delta_{n,r}^{(k)}$ is

$$F_{\Delta_{n,r}^{(k)}}(x) = \frac{k^{n+r}}{(n-1)!\,(r-1)!}\int_R\left(-\log(\bar{F}(u))\right)^{n-1}(\bar{F}(u))^{k-1}f(u) \times\left\{\int_\alpha^1(-\log t)^{r-1}t^{k-1}\,dt\right\}du,$$

where

$$\alpha := \alpha(u, x) = \bar{F}(u+x)/\bar{F}(u).$$

Since

$$\int(-\log t)^{r-1}t^{k-1}\,dt = t^k\left\{\sum_{i=0}^{r-1}\frac{(r-1)!}{i!\,k^{r-i}}(-\log t)^i\right\} + C,$$

we have

$$\int_\alpha^1(-\log t)^{r-1}t^{k-1}\,dt = \frac{(r-1)!}{k^r} - \alpha^k\left\{\sum_{i=0}^{r-1}\frac{(r-1)!}{i!\,k^{r-i}}(-\log\alpha)^i\right\} \tag{2.6}$$

and

$$F_{\Delta_{n,r}^{(k)}}(x) = \frac{k^n}{(n-1)!}\int_R\left(-\log(\bar{F}(u))\right)^{n-1}(\bar{F}(u))^{k-1}f(u)$$

$$-\frac{k^n}{(n-1)!}\int_{\mathbf{R}}\left(-\log(\bar{F}(u))\right)^{n-1}(\bar{F}(u))^{k-1}f(u)$$

$$\times\left(\frac{\bar{F}(u+x)}{\bar{F}(u)}\right)^k\left\{\sum_{i=0}^{r-1}\frac{k^i}{i!}\left(-\log\frac{\bar{F}(u+x)}{\bar{F}(u)}\right)^i\right\}du$$

$$=1-\int_S\left(\frac{\bar{F}(u+x)}{\bar{F}(u)}\right)^k\left\{\sum_{i=0}^{r-1}\frac{k^i}{i!}\left(-\log\frac{\bar{F}(u+x)}{\bar{F}(u)}\right)^i\right\}dF_{Y_n^{(k)}}(u),$$

which completes the proof of Lemma 1. ∎

Using (2.3) and (2.4) instead of (2.1) and (2.2) we prove the similar result for k-th lower records.

LEMMA 2. *The distribution function of the random variable* $D_{n,r}^{(k)}=Z_n^{(k)}-Z_{n+r}^{(k)}$ *is of the form*

$$F_{D_{n,r}^{(k)}}(x)=1-\sum_{i=0}^{r-1}\frac{k^i}{i!}\int_S\left(-\log\frac{F(u-x)}{F(u)}\right)^i\left(\frac{F(u-x)}{F(u)}\right)^k dF_{Z_n^{(k)}}(u)$$

for $x\geqslant 0$ *and it is* 0 *otherwise.*

Proof. Using (2.4) we see that the pdf of $D_{n,r}^{(k)}$ is (after similar evaluations as in Lemma 1)

$$f_{D_{n,r}^{(k)}}(w)=\frac{k^{n+r}}{(n-1)!(r-1)!}\int_{\mathbf{R}}\left(-\log F(u)\right)^{n-1}h(u) \tag{2.7}$$

$$\times\left(-\log\frac{F(u-w)}{F(u)}\right)^{r-1}(F(u-w))^{k-1}f(u-w)\,du$$

for $w\geqslant 0$. Therefore the df of $D_{n,r}^{(k)}$ is

$$F_{D_{n,r}^{(k)}}(x)=\frac{k^{n+r}}{(n-1)!(r-1)!}\int_{\mathbf{R}}\left(-\log F(u)\right)^{n-1}(F(u))^{k-1}f(u)$$

$$\times\left\{\int_\beta^1(-\log t)^{r-1}t^{k-1}dt\right\}du,$$

where

$$\beta:=\beta(u,x)=F(u-x)/F(u).$$

Using (2.6) we easily complete the proof of Lemma 2. ∎

3. PROBABILITY DISTRIBUTIONS OF $Y_{n+r}^{(k)}/Y_n^{(k)}$ AND $Z_n^{(k)}/Z_{n+r}^{(k)}$

We start this section with the following lemma.

LEMMA 3. *For all real numbers A, B, C, $0 \leqslant A < B \leqslant C$, and $n, r \in N$,*

$$\int_A^B u^{n-1}(C-u)^{r-1}\,du = \frac{\Gamma(n)\Gamma(n)}{\Gamma(n+r)}\sum_{i=0}^{r-1}\binom{n+r-1}{i}\left\{B^{n+r-1}\left(\frac{C}{B}-1\right)^i - A^{n+r-1}\left(\frac{C}{A}-1\right)^i\right\}.$$

Now let us state and prove the results on the probability distributions of quotients $Y_{n+r}^{(k)}/Y_n^{(k)}$ and $Z_n^{(k)}/Z_{n+r}^{(k)}$.

LEMMA 4. *The distribution function of the random variable $U_{n,r}^{(k)} = Y_{n+r}^{(k)}/Y_n^{(k)}$ is of the form*

$$(3.1)\quad F_{U_{n,r}^{(k)}}(z) = \begin{cases} p_n' - \sum_{j=0}^{r-1}\binom{n+r-1}{j}\int_0^\infty R^{n+r-1}(y,z)(R^{-1}(y,z)-1)^j\,dF_{Y_{n+r}^{(k)}}(y) & \text{for } z \leqslant 0, \\ p_n' + \sum_{j=0}^{r-1}\binom{n+r-1}{j}\int_{-\infty}^0 R^{n+r-1}(y,z)(R^{-1}(y,z)-1)^j\,dF_{Y_{n+r}^{(k)}}(y) & \text{for } 0 < z < 1, \\ p_n'' - \sum_{j=0}^{r-1}\binom{n+r-1}{j}\int_0^\infty R^{n+r-1}(y,z)(R^{-1}(y,z)-1)^j\,dF_{Y_{n+r}^{(k)}}(y) & \text{for } z \geqslant 1, \end{cases}$$

where

$$p_n' = P\{Y_n^{(k)} < 0,\ Y_{n+r}^{(k)} > 0\}, \qquad p_n'' = P\{Y_n^{(k)} < 0\} + P\{Y_{n+r}^{(k)} > 0\},$$

and

$$R(y,z) = H(y/z)/H(y).$$

Proof. Note that if (X, Y) is an absolutely continuous random vector with a pdf $f(x, y)$ such that $X \leqslant Y$, then the distribution function of the random variable $Z = Y/X$ is

$$(3.2)\quad F_Z(z) = \begin{cases} P\{X<0,\ Y>0\} - \int_0^\infty\int_{-\infty}^{y/z} f(x,y)\,dx\,dy & \text{for } z \leqslant 0, \\ P\{X<0,\ Y>0\} + \int_{-\infty}^0\int_{-\infty}^{y/z} f(x,y)\,dx\,dy & \text{for } 0<z<1, \\ P\{X<0\} + P\{Y>0\} - \int_0^\infty\int_{-\infty}^{y/z} f(x,y)\,dx\,dy & \text{for } z \geqslant 1. \end{cases}$$

Put $X = Y_n^{(k)}$ and $Y = Y_{n+r}^{(k)}$; then $Z = U_{n,r}^{(k)}$ and we obtain (3.1) combining (3.2) with (2.2) and Lemma 3. For instance, for $z < 0$

$$F_{U_{n,r}^{(k)}}(z) = p_n' - \frac{k^{n+r}}{\Gamma(n)\Gamma(r)} \int_0^\infty (\bar{F}(y))^{k-1} f(y) \{ \int_{-\infty}^{y/z} (H(x))^{n-1} (H(y)-H(x))^{r-1} h(x)\, dx \}\, dy$$

$$= p_n' - \frac{k^{n+r}}{\Gamma(n)\Gamma(r)} \int_0^\infty (\bar{F}(y))^{k-1} f(y) \{ \int_0^{H(y/z)} u^{n-1} (H(y)-u)^{r-1}\, du \}\, dy.$$

Using Lemma 3 with $A = 0$, $B = H(y/z)$ and $C = H(y)$ we obtain

$$F_{U_{n,r}^{(k)}}(z) = p_n' - \int_0^\infty (\bar{F}(y))^{k-1} f(y) \left\{ \sum_{i=0}^{r-1} \binom{n+r-1}{i} (H(y/z))^{n+r-1} \left(\frac{H(y)}{H(y/z)} - 1 \right)^i \right\} dy$$

$$= p_n' - \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_0^\infty \left(\frac{H(y/z)}{H(y)} \right)^{n+r-1} \left(\frac{H(y)}{H(y/z)} - 1 \right)^j dF_{Y_{n+r}^{(k)}}(y).$$

The remaining cases $0 < z < 1$ and $z \geqslant 1$ may be treated similarly. ∎

In the same way we prove the following result.

Lemma 5. *The distribution function of the random variable* $T_{n,r}^{(k)} = Z_n^{(k)}/Z_{n+r}^{(k)}$ *is of the form*

$$F_{T_{n,r}^{(k)}}(z)$$

$$= \begin{cases} \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_{-\infty}^{0} \bar{R}^{n+r-1}(y, z) (\bar{R}^{-1}(y, z) - 1)^j\, dF_{Z_{n+r}^{(k)}}(y) & \text{for } z < 1, \\ 1 - \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_0^\infty \bar{R}^{n+r-1}(y, z) (\bar{R}^{-1}(y, z) - 1)^j\, dF_{Z_{n+r}^{(k)}}(y) & \text{for } z \geqslant 1, \end{cases}$$

where $\bar{R}(y, z) = \bar{H}(yz)/\bar{H}(y)$.

4. LIMIT DISTRIBUTIONS OF DIFFERENCES OF k-TH RECORD VALUES

Theorem 1. *Let F be an absolutely continuous distribution function with density f and the interval $S \subset \mathbf{R}$ as the support, such that $h(x) = f(x)/(1-F(x))$ is a differentiable function with bounded first derivative*

$$(4.1) \qquad |h'(x)| \leqslant M, \quad x \in S.$$

Let us fix $r \in N$ and assume that $\{F_k, k \geqslant 1\}$ is a sequence of distribution functions of the form

$$F_k(x) = 1 - \sum_{i=0}^{r-1} \frac{k^i}{i!} \int_S \left(-\log \frac{1-F(u+x/k)}{1-F(u)} \right)^i \left(\frac{1-F(u+x/k)}{1-F(u)} \right)^k dG_k(u)$$

for $x \geqslant 0$ *and it equals* 0 *otherwise, where* $\{G_k,\ k \geqslant 1\}$ *is a sequence of distribution functions such that*

$$G_k \to G, \quad k \to \infty, \tag{4.2}$$

and G *is a distribution concentrated at a point* $x_0 \in \partial S$. *Then*

$$F_k \to F^*_{r,\lambda}, \quad k \to \infty,$$

where $F^*_{r,\lambda}$ *is the df of* $\Gamma(r, \lambda)$ *random variable and*

$$\lambda = \begin{cases} \lim_{x \to x_0^+} h(x) & \text{if } x_0 = \inf S, \\ \lim_{x \to x_0^-} h(x) & \text{if } x_0 = \sup S. \end{cases}$$

Proof. Applying Taylor's formula to the function $s(z) = -\log(1-F(z))$ we obtain

$$-\log \frac{1-F(u+x/k)}{1-F(u)} = s\left(u+\frac{x}{k}\right) - s(u) = h(u)\frac{x}{k} + h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k^2},$$

where $0 < \theta < 1$. Therefore

$$1-F_k(x) = \sum_{i=0}^{r-1} \frac{1}{i!} \int_S \left(xh(u) + h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right)^i$$
$$\times \exp(-xh(u)) \exp\left(-h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right) dG_k(u).$$

By (4.1) we have

$$H_{k,r}(x) \exp\left(-\frac{Mx^2}{k}\right) \leqslant 1-F_k(x) \leqslant H_{k,r}(x) \exp\left(\frac{Mx^2}{k}\right),$$

where

$$H_{k,r}(x) = \sum_{i=0}^{r-1} \frac{1}{i!} \int_S \left(xh(u) + h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right)^i \exp(-xh(u))\, dG_k(u).$$

Let us fix $i \in \{0, 1, \ldots, r-1\}$. Using the binomial formula we obtain

$$\int_S \left(xh(u) + h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right)^i \exp(-xh(u))\, dG_k(u)$$
$$= \int_S (xh(u))^i \exp(-xh(u))\, dG_k(u)$$
$$+ \sum_{j=0}^{i-2} \binom{i-1}{j} \int_S (xh(u))^j \left(h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right)^{i-1-j} \exp(-xh(u))\, dG_k(u)$$
$$:= I_1 + I_2,$$

say. Now, by (4.2) we have

$$I_1 \to (\lambda x)^i \exp(-\lambda x), \quad k \to \infty,$$

and by (4.1) again

$$|I_2| \leqslant \sum_{j=0}^{i-2} \binom{i-1}{j} \left(\frac{Mx^2}{k}\right)^{i-1-j} \int_S (xh(u))^j \exp(-xh(u))\, dG_k(u) \to 0, \quad k \to \infty,$$

where λ is given by (4.3) below. This proves that for $x \geqslant 0$

$$\lim_{k\to\infty} H_{k,r}(x) = \sum_{i=0}^{r-1} \frac{1}{i!} (\lambda x)^i \exp(-\lambda x),$$

which is the tail of $\Gamma(r, \lambda)$ distribution function. ∎

Using Lemma 1 one can see (cf. Example 1 in Section 6) that if $f(x) = \lambda \exp(-\lambda x)$, $x \geqslant 0$, then for all $n, r \in N$ the random variable $k(Y_{n+r}^{(k)} - Y_n^{(k)})$ has the gamma $\Gamma(r, \lambda)$ distribution. The following theorem states that for a broad class of distributions F the asymptotic distribution of $k(Y_{n+r}^{(k)} - Y_n^{(k)})$ is also $\Gamma(r, \lambda)$ with λ depending on F.

THEOREM 2. *Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. random variables with df F and pdf f, with the interval $S \subset \boldsymbol{R}$ as the support, and that $h(x) = f(x)/\bar{F}(x)$ is a differentiable function with bounded first derivative. Then for any fixed $n, r \in N$*

$$k(Y_{n+r}^{(k)} - Y_n^{(k)}) \xrightarrow{D} \Gamma(r, \lambda), \quad k \to \infty,$$

where

$$\lambda = \lim_{x \to x_0^+} h(x) \tag{4.3}$$

and $x_0 = \inf S$.

Proof. By Lemma 1 the df of $k\Delta_{n,r}^{(k)}$ is

$$F_{k\Delta_{n,r}^{(k)}}(x) = 1 - \sum_{i=0}^{r-1} \frac{k^i}{i!} \int_S \left(-\log \frac{1-F(u+x/k)}{1-F(u)}\right)^i \left(\frac{1-F(u+x/k)}{1-F(u)}\right)^k dG_k(u),$$

where G_k is the distribution function of $Y_n^{(k)}$. Since $Y_n^{(k)} \xrightarrow{D} x_0 = \inf S$ as $k \to \infty$, Theorem 1 implies the result. ∎

Remark 1. For $r = 1$ we obtain results of [4].

In the same way, but using Lemma 2 instead of Lemma 1, we can study limit behaviour of $k(Z_n^{(k)} - Z_{n+r}^{(k)})$.

THEOREM 3. *Let F be an absolutely continuous distribution function with density f and the interval $S \subset \boldsymbol{R}$ as the support, such that $\bar{h}(x) = f(x)/F(x)$ is* *a differentiable function with bounded first derivative*

(4.4) $$|\bar{h}'(x)| \leqslant M, \quad x \in S.$$

Let us fix $r \in N$ and assume that $\{F_k, k \geqslant 1\}$ is a sequence of distribution functions of the form

$$F_k(x) = 1 - \sum_{i=0}^{r-1} \frac{k^i}{i!} \int_S \left(-\log\frac{F(u-x/k)}{F(u)}\right)^i \left(\frac{F(u-x/k)}{F(u)}\right)^k dG_k(u)$$

for $x \geqslant 0$ and it equals 0 otherwise, where $\{G_k, k \geqslant 1\}$ is a sequence of distribution functions such that

(4.5) $$G_k \to G, \quad k \to \infty,$$

and G is a distribution concentrated at a point $x_0 \in \partial S$. Then

$$F_k \to F^*_{r,\bar{\lambda}}, \quad k \to \infty,$$

where

$$\bar{\lambda} = \begin{cases} \lim_{x \to x_0^+} \bar{h}(x) & \text{if } x_0 = \inf S, \\ \lim_{x \to x_0^-} \bar{h}(x) & \text{if } x_0 = \sup S. \end{cases}$$

Proof. Applying Taylor's formula to the function $\bar{s}(z) = \log F(z)$ we obtain

$$\log\frac{F(u-x/k)}{F(u)} = \bar{s}\left(u-\frac{x}{k}\right) - \bar{s}(u) = -\bar{h}(u)\frac{x}{k} + \bar{h}'\left(u-\frac{\theta x}{k}\right)\frac{x^2}{k^2},$$

where $0 < \theta < 1$. Therefore

$$1-F_k(x) = \sum_{i=0}^{r-1} \frac{1}{i!} \int_S \left(x\bar{h}(u) + \bar{h}'\left(u-\frac{\theta x}{k}\right)\frac{x^2}{k}\right)^i \times \exp(-x\bar{h}(u)) \exp\left(-\bar{h}'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right) dG_k(u).$$

By (4.4) we have

$$\bar{H}_{k,r}(x)\exp\left(-\frac{Mx^2}{k}\right) \leqslant 1-F_k(x) \leqslant \bar{H}_{k,r}(x)\exp\left(\frac{Mx^2}{k}\right),$$

where

$$\bar{H}_{k,r}(x) = \sum_{i=0}^{r-1} \frac{1}{i!} \int_S \left(x\bar{h}(u) - h'\left(u+\frac{\theta x}{k}\right)\frac{x^2}{k}\right)^i \exp(-x\bar{h}(u))\, dG_k(u).$$

Let us fix $i \in \{0, 1, \ldots, r-1\}$. Using the binomial formula we obtain

$$\int_S \left(x\bar{h}(u) - \bar{h}'\left(u + \frac{\theta x}{k}\right)\frac{x^2}{k}\right)^i \exp\left(-x\bar{h}(u)\right) dG_k(u)$$

$$= \int_S (x\bar{h}(u))^i \exp\left(-x\bar{h}(u)\right) dG_k(u)$$

$$+ \sum_{j=0}^{i-2} \binom{i-1}{j} \int_S (x\bar{h}(u))^j \left(-\bar{h}'\left(u + \frac{\theta x}{k}\right)\frac{x^2}{k}\right)^{i-1-j} \exp\left(-x\bar{h}(u)\right) dG_k(u)$$

$$:= I_1 + I_2,$$

say. Now, by (4.5) we have

$$I_1 \to (\bar{\lambda} x)^i \exp(-\bar{\lambda} x), \quad k \to \infty,$$

and by (4.4) again

$$|I_2| \leqslant \sum_{j=0}^{i-2} \binom{i-1}{j} \left(\frac{Mx^2}{k}\right)^{i-1-j} \int_S (x\bar{h}(u))^j \exp\left(-x\bar{h}(u)\right) dG_k(u) \to 0, \quad k \to \infty,$$

where $\bar{\lambda}$ is given by (4.3). This proves that for $x \geqslant 0$

$$\lim_{k \to \infty} \bar{H}_{k,r}(x) = \sum_{i=0}^{r-1} \frac{1}{i!} (\bar{\lambda} x)^i \exp(-\bar{\lambda} x),$$

which is the tail of $\Gamma(r, \bar{\lambda})$ distribution function. ■

THEOREM 4. *Suppose that* $\{X_n, n \geqslant 1\}$ *is a sequence of i.i.d. random variables with df* F *and pdf* f*, with the interval* $S \subset \mathbf{R}$ *as the support, and that* $\bar{h}(x) = f(x)/F(x)$ *is a differentiable function with bounded first derivative. Then for any fixed* n*,* $r \in N$

$$k\left(Z_n^{(k)} - Z_{n+r}^{(k)}\right) \xrightarrow{D} \Gamma(r, \bar{\lambda}), \quad k \to \infty,$$

where $\bar{\lambda} = \lim_{x \to x_0^-} \bar{h}(x)$ *and* $x_0 = \sup S$.

Proof. By Lemma 2 the df of $kD_{n,r}^{(k)}$ is

$$F_{D_{n,r}^{(k)}}(x) = 1 - \sum_{i=0}^{r-1} \frac{k^i}{i!} \int_S \left(-\log \frac{F(u - x/k)}{F(u)}\right)^i \left(\frac{F(u - x/k)}{F(u)}\right)^k dG_k(u),$$

where G_k is the distribution function of $Z_n^{(k)}$. Since $Z_n^{(k)} \xrightarrow{D} x_0 = \sup S$ as $k \to \infty$, Theorem 3 implies the result. ■

Remark 2. For $r = 1$ we obtain results of [2].

5. LIMIT DISTRIBUTIONS OF QUOTIENTS OF k-TH RECORD VALUES

THEOREM 5. *Let F be an absolutely continuous distribution function with density f and the interval $S \subset \mathbf{R}$ as the support and suppose that $q(s)$ is a differentiable function such that*

$$(5.1)\qquad |x^2 q'(x)| \leqslant M, \quad x \in S.$$

Let $\{G_n,\ n \geqslant 1\}$ be a sequence of distribution functions such that

$$G_n \to G, \quad n \to \infty,$$

and G is a distribution concentrated at a point $x_0 = \sup S$. Let us fix $r \in N$ and assume that $\{F_n,\ n \geqslant 1\}$ is a sequence of distribution functions of the form

$$(5.2)\quad F_n(x) = \begin{cases} p_n' - \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_0^{\infty} R_n^{n+r-1}(y, x)\big(R_n^{-1}(y, x)-1\big)^j dG_{n+r}(y) & \text{for } x \leqslant -n, \\ p_n' + \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_{-\infty}^{0} R_n^{n+r-1}(y, x)\big(R_n^{-1}(y, x)-1\big)^j dG_{n+r}(y) & \text{for } -n < x \leqslant 0, \\ p_n'' - \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_0^{\infty} R_n^{n+r-1}(y, x)\big(R_n^{-1}(y, x)-1\big)^j dG_{n+r}(y) & \text{for } x > 0, \end{cases}$$

where $p_n' = G_n(0) - G_{n+r}(0)$, $p_n'' = 1 + G_n(0) - G_{n+r}(0)$, and

$$R_n(y, x) = H\frac{\big(y/(1+x/n)\big)}{H(y)}.$$

Let

$$(5.3)\qquad \mu = \lim_{x \to x_0^-} xq(x).$$

Then:

(1) *if $x_0 > 0$, the sequence $\{F_n\}$ converges weakly to $F^*_{r,\mu}$ as $n \to \infty$, where $F^*_{r,\mu}$ is the df of $\Gamma(r, \mu)$ distribution*;

(2) *if $x_0 \leqslant 0$, the sequence $\{F_n\}$ converges weakly to $F^*_{r,-\mu}$ as $n \to \infty$, where $F^*_{r,\nu}$ is the df of $\mathrm{N}\Gamma(r, \nu)$ distribution.*

Proof. Let us consider the function $\underline{s}(x) = \log H(u)$. Applying Taylor's formula we obtain

$$\log R_n(y, x) = \underline{s}\left(\frac{y}{1+x/n}\right) - \underline{s}(u) = -yq(y)\frac{x}{n+x} + \frac{1}{2}q'\left(\frac{y}{1+\theta x/n}\right)\left(\frac{yx}{n+x}\right)^2,$$

where $0 < \theta < 1$. Therefore

$$R_n(y, x) = \exp\left(-yq(y)\frac{x}{n+x}\right)\exp\left(\frac{1}{2}y^2 q'\left(\frac{y}{1+\theta x/n}\right)\left(\frac{x}{n+x}\right)^2\right).$$

By (5.1) we have

$$\left|y^2 q'\left(\frac{y}{1+\theta x/n}\right)\right| \leqslant M\left(1+\frac{x}{n}\right)^2,$$

which gives

$$\exp\left(-yq(y)\frac{x}{n+x}\right)\exp\left(-\frac{Mx^2}{2n^2}\right) \leqslant R_n(y, x) \leqslant \exp\left(-yq(y)\frac{x}{n+x}\right)\exp\left(\frac{Mx^2}{2n^2}\right).$$

Now fix $j \in \{0, 1, \ldots, r-1\}$. Then

$$\begin{aligned}\exp\left(-\frac{Mx^2(n+r-1)}{n^2}\right) & E\underline{f}_n(Z_n) I_{[Y_{n+r}\in A]} \\ & \leqslant \binom{n+r-1}{j}\int_A R_n^{n+r-1}(y, x)\left(R_n^{-1}(y, x)-1\right)^j dG_{n+r}(y) \\ & \leqslant \exp\left(\frac{Mx^2(n+r-1)}{n^2}\right) E\bar{f}_n(Z_n) I_{[Y_{n+r}\in A]},\end{aligned}$$

where $Z_n = Y_{n+r}q(Y_{n+r})$, with Y_n having the df G_n,

$$\bar{f}_n(z) = \binom{n+r-1}{j}\exp\left(-\frac{zx(n+r-1)}{n+x}\right)\left(\exp\left(\frac{zx}{n+x}+\frac{Mx^2}{2n^2}\right)-1\right)^j$$

and

$$\underline{f}_n(z) = \binom{n+r-1}{j}\exp\left(-\frac{zx(n+r-1)}{n+x}\right)\left(\exp\left(\frac{zx}{n+x}-\frac{Mx^2}{2n^2}\right)-1\right)^j.$$

By (4.2) and (5.3) we have $Z_n \xrightarrow{D} \mu$ as $n \to \infty$. Moreover,

$$\bar{f}_n(z) \to e^{-zx}\frac{(zx)^j}{j!}, \qquad n \to \infty,$$

and

$$\underline{f}_n(z) \to e^{-zx}\frac{(zx)^j}{j!}, \qquad n \to \infty.$$

Consider two possible cases:

(1) $x_0 = \sup S > 0$. Then $p_n' \to 0$ as $n \to \infty$, $\mu > 0$ and, for n sufficiently large, $Y_n > 0$. Therefore $F_n(x) = 0$ for $x < 0$, and for $x > 0$

$$F_n(x) \to 1 - \sum_{j=0}^{r-1} e^{-\mu x}\frac{(\mu x)^j}{j!}, \qquad n \to \infty,$$

which is the df of $\Gamma(r, \mu)$.

(2) $x_0 = \sup S \leqslant 0$. Then $p_n = 0$, $\mu < 0$ and $F_n(x) = 1$ for $x > 0$. Therefore for $-n \leqslant x \leqslant 0$

$$F_n(x) \to \sum_{j=0}^{r-1} e^{-\mu|x|} \frac{(-\mu|x|)^j}{j!}, \qquad n \to \infty,$$

which is the df of $\mathrm{N}\Gamma(r, -\mu)$. ∎

THEOREM 6. *Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. random variables with df F and pdf f, with the interval $S \subset \boldsymbol{R}$ as the support, and that $q(x)$ is a differentiable function satisfying* (5.1). *Then for any fixed k, $r \in N$*

$$n\left(\frac{Y_{n+r}^{(k)}}{Y_n^{(k)}} - 1\right) \xrightarrow{D} \begin{cases} \Gamma(r, \mu) & \text{if } \sup S > 0, \\ \mathrm{N}\Gamma(r, -\mu) & \text{if } \sup S \leqslant 0 \end{cases}$$

as $n \to \infty$, where μ is given by (5.3).

Proof. By Lemma 4 the distribution function of $n(Y_{n+r}^{(k)}/Y_n^{(k)} - 1)$ is of the form (5.2) with G_n being the df of $Y_n^{(k)}$. Since $Y_n^{(k)} \xrightarrow{D} x_0 = \sup S$ as $n \to \infty$, the result follows from Theorem 5. ∎

In the same way we can study limit behaviour of quotients of k-th lower records.

THEOREM 7. *Let F be an absolutely continuous distribution function with density f and the interval $S \subset \boldsymbol{R}$ as the support and assume that $\bar{q}(x)$ is a differentiable function such that*

$$|x^2 \bar{q}'(x)| \leqslant M, \qquad x \in S. \tag{5.4}$$

Let $\{G_n, n \geqslant 1\}$ be a sequence of distribution functions such that

$$G_n \to G, \qquad n \to \infty,$$

and G is a distribution concentrated at a point $x_0 = \inf S$. Let us fix $r \in N$ and assume that $\{F_n, n \geqslant 1\}$ is a sequence of distribution functions of the form

$$F_n(x) = \begin{cases} \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_{-\infty}^{0} \bar{R}_n^{n+r-1}(y, x)(\bar{R}_n^{-1}(y, x) - 1)^j \, dF_{Z_{n+r}^{(k)}}(y) & \text{for } x < 0, \\ 1 - \sum_{j=0}^{r-1} \binom{n+r-1}{j} \int_{0}^{\infty} \bar{R}_n^{n+r-1}(y, x)(\bar{R}_n^{-1}(y, x) - 1)^j \, dF_{Z_{n+r}^{(k)}}(y) & \text{for } x \geqslant 0, \end{cases}$$

where

$$\bar{R}_n(y, x) = \frac{\bar{H}(y(1 + x/n))}{\bar{H}(y)}.$$

Let

(5.5) $$\bar{\mu} = \lim_{x \to x_0^-} x\bar{q}(x).$$

Then:

(1) *if* $x_0 > 0$, *the sequence* $\{F_n\}$ *converges weakly to* $F^*_{r,\bar{\mu}}$ *as* $n \to \infty$;
(2) *if* $x_0 \leqslant 0$, *the sequence* $\{F_n\}$ *converges weakly to* $F^*_{r,-\bar{\mu}}$ *as* $n \to \infty$.

THEOREM 8. *Suppose that* $\{X_n, n \geqslant 1\}$ *is a sequence of i.i.d. random variables with df* F *and pdf* f, *with the interval* $S \subset \mathbf{R}$ *as the support, and that* $\bar{q}(x)$ *is a differentiable function satisfying* (5.4). *Then for any fixed* k, $r \in N$

$$n\left(\frac{Z_n^{(k)}}{Z_{n+r}^{(k)}} - 1\right) \xrightarrow{D} \begin{cases} \Gamma(r, \bar{\mu}) & \text{if } \inf S > 0, \\ \mathrm{N}\Gamma(r, -\bar{\mu}) & \text{if } \inf S \leqslant 0 \end{cases}$$

as $n \to \infty$, *where* $\bar{\mu}$ *is given by* (5.5).

Remark 3. For $r = 1$ we obtain results of [2].

6. EXAMPLES

In this section we give examples of asymptotic behaviour of differences and quotients of k-th record values from particular distributions.

EXAMPLE 1. Consider exponential $\operatorname{Exp}(\lambda)$ and negative exponential $\operatorname{NExp}(\lambda)$ distribution functions given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0 \end{cases}$$

and

$$G(x) = \begin{cases} e^{\lambda x} & \text{for } x \leqslant 0, \\ 1 & \text{for } x > 0, \end{cases}$$

respectively. Then using (2.5) we see that the density function of $\Delta_{n,r}^{(k)}$ from F is

$$f_{\Delta_{n,r}^{(k)}}(x) = \frac{\lambda^r}{(r-1)!}(kx)^{r-1} e^{-k\lambda x}, \quad x \geqslant 0.$$

This implies that $k\Delta_{n,r}^{(k)}$ from the df F has the gamma $\Gamma(r, \lambda)$ distribution for all $k \in N$. Similarly, using (2.7) we see that $kD_{n,r}^{(k)}$ from the df G has also the gamma $\Gamma(r, \lambda)$ distribution for all $k \in N$.

Further on, using Theorems 2 and 4 we see that $k\Delta_{n,r}^{(k)}$ from G as well as $kD_{n,r}^{(k)}$ from F both converge, as $k \to \infty$, to improper distribution concentrated at ∞.

EXAMPLE 2. Let f_1 and f_2 be probability density functions. Write

$$f(x) = pf_1(x)+qf_2(x),$$

where $p = 1-q \in (0, 1)$ with

$$f_1(x) = ae^{ax} I_{(-\infty,0)}(x),$$

$$f_2(x) = \frac{pa}{q}\exp\left(-\frac{pa}{q}x\right) I_{[0,\infty)}(x), \quad a > 0.$$

Note that $f(x) = (f_1 * f_2)(x)$, i.e. f_1 and f_2 satisfy the Dugué condition (cf. [6]). Now

$$q(x) = \frac{pa}{pax-q\log q} \text{ for } x > 0 \quad \text{and} \quad \lim_{x\to\infty} xq(x) = 1,$$

and by Theorem 6 we have

$$n(Y_{n+r}^{(k)}/Y_n^{(k)}-1) \xrightarrow{D} \Gamma(r, 1), \quad n \to \infty.$$

Moreover,

$$\bar{q}(x) = \frac{a}{ax-\log p} \text{ for } x < 0 \quad \text{and} \quad \lim_{x\to-\infty} x\bar{q}(x) = 1,$$

which by Theorem 8 gives

$$n(Z_n^{(k)}/Z_{n+r}^{(k)}-1) \xrightarrow{D} \Gamma(r, 1), \quad n \to \infty.$$

On the other hand,

$$\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} \bar{h}(x) = 0,$$

and by Theorems 2 and 4 the limit distributions of differences $k\Delta_{n,r}^{(k)}$ and $kD_{n,r}^{(k)}$ are improper distributions concentrated at ∞.

EXAMPLE 3. Define the following distribution functions:

$$F_1(x) = \begin{cases} 1-e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0, \end{cases} \quad \lambda > 0;$$

$$F_2(x) = \begin{cases} 1-e^{\mu/x} & \text{for } x < 0, \\ 1 & \text{for } x \geqslant 0, \end{cases} \quad \mu > 0.$$

Then $H_1(x) = \lambda x$ for $x > 0$, and by Lemma 4 we see that the distribution function of quotients of k-th upper record values from F_1 is $F_{U_{n,r}^{(k)}}(z) = 0$ for

$z \leqslant 1$ and

$$F_{U_{n,r}^{(k)}}(z) = 1 - \sum_{j=0}^{r-1} \binom{n+r-1}{j} z^{-(n+r-1)} (z-1)^j \quad \text{for } z > 1.$$

Therefore for $x \geqslant 0$

$$F_{n(U_{n,r}^{(k)}-1)}(x) = 1 - \left(1+\frac{x}{n}\right)^{-(n+r-1)} \sum_{j=0}^{r-1} \frac{x^j}{j!} \frac{(n+r-1)!}{n^j (n-j-1)!} \to 1 - e^{-x} \sum_{j=0}^{r-1} \frac{x^j}{j!}, \quad n \to \infty,$$

or $n(U_{n,r}^{(k)}-1) \xrightarrow{D} \Gamma(r, 1)$, $n \to \infty$.

For quotients of k-th records from F_2 we have $H_2(x) = -\lambda/x$ for $x < 0$ and by Lemma 4 we see that $F_{n(U_{n,r}^{(k)}-1)}(x) = 1$ for $x \geqslant 0$ while for $-n \leqslant x < 0$

$$F_{n(U_{n,r}^{(k)}-1)}(x) = \left(1+\frac{x}{n}\right)^{n+r-1} \sum_{j=0}^{r-1} \frac{(n+r-1)!}{j!\, n^j (n-j-1)!} \left(\frac{n|x|}{n+x}\right)^j \to e^x \sum_{j=0}^{r-1} \frac{|x|^j}{j!}, \quad n \to \infty,$$

or $n(U_{n,r}^{(k)}-1) \xrightarrow{D} \mathrm{N}\Gamma(r, 1)$, $n \to \infty$.

EXAMPLE 4. Consider the Pareto distribution function given by

$$F(x) = \begin{cases} 1 - 1/x^\alpha & \text{for } x \geqslant 1, \\ 0 & \text{for } x < 1. \end{cases}$$

Then

$$\frac{\bar{F}(u+x)}{\bar{F}(u)} = \left(\frac{u}{u+x}\right)^\alpha$$

and

$$\left(\frac{\bar{F}(u+x/k)}{\bar{F}(u)}\right)^k = \left(1+\frac{x}{ku}\right)^{-\alpha k} \to e^{-(\alpha x)/u}, \quad k \to \infty.$$

Therefore, since $Y_n^{(k)} \xrightarrow{D} 1$, $k \to \infty$, we have

$$F_{k\Delta_{n,r}^{(k)}}(x) = 1 - \sum_{i=0}^{r-1} \frac{1}{i!} \int_S \left(-\log\left(\frac{\bar{F}(u+x/k)}{\bar{F}(u)}\right)^k\right)^i \left(\frac{\bar{F}(u+x/k)}{\bar{F}(u)}\right)^k dF_{Y_n^{(k)}}(u)$$

$$\to 1 - \sum_{i=0}^{r-1} \frac{(\alpha x)^i}{i!} e^{-\alpha x}, \quad k \to \infty,$$

or equivalently $k(Y_{n+r}^{(k)} - Y_n^{(k)}) \xrightarrow{D} \Gamma(r, \alpha)$, $k \to \infty$.

Similarly it can be shown that for k-th lower records from the negative Pareto distribution NPareto(α) with pdf $f(x) = \alpha |x|^{-\alpha-1} I_{(-\infty,-1)}(x)$, where $\alpha > 0$, we have

$$k(Z_n^{(k)} - Z_{n+r}^{(k)}) \xrightarrow{D} \Gamma(r, \alpha) \quad \text{as } k \to \infty.$$

Moreover, consider inverse Pareto distribution InvPareto(α, σ) and negative inverse Pareto distribution NInvPareto(α, σ) given by distribution functions

$$F_1(x) = (x/\sigma)^\alpha, \qquad x \in (0, \sigma),$$

$$F_2(x) = 1-(-x/\sigma)^\alpha, \qquad x \in (-\sigma, 0),$$

respectively. Let $\alpha = 1$. Then using Theorems 2 and 4 it can be shown that $k\Delta_{n,r}^{(k)}$ from F_1 as well as $kD_{n,r}^{(k)}$ from F_2 both converge, as $k \to \infty$, to $\Gamma(r, 1/\sigma)$ distribution.

EXAMPLE 5. We know that the limit distribution of differences $kD_{n,1}^{(k)}$ of k-th lower records from Gumbel distribution is not a proper distribution (cf. [2]); it may be considered as the distribution concentrated at ∞. This fact can also be shown for $kD_{n,r}^{(k)}$. In a similar way the sequence of differences $k\Delta_{n,r}^{(k)}$ of k-th upper records from the negative Gumbel distribution

$$F(x) = 1-\exp(-e^x), \qquad x \in \boldsymbol{R},$$

converge to the distribution concentrated at ∞.

The limit properties of $k\Delta_{n,r}^{(k)}$, $kD_{n,r}^{(k)}$, $n(U_{n,r}^{(k)}-1)$ and $n(T_{n,r}^{(k)}-1)$ for the distributions of the above examples are shown in the Table.

Table

F	$k\Delta_{n,r}^{(k)}$	$kD_{n,r}^{(k)}$	$n(U_{n,r}^{(k)}-1)$	$n(T_{n,r}^{(k)}-1)$
	$k \to \infty$		$n \to \infty$	
Exp (λ)	$\sim \Gamma(r, \lambda)$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \Gamma(r, 1)$	$\xrightarrow{D} \infty$
NExp (λ)	$\xrightarrow{D} \infty$	$\sim \Gamma(r, \lambda)$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \mathrm{N}\Gamma(r, 1)$
Exp (pa/q)*NegExp (a)	$\xrightarrow{D} \infty$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \Gamma(r, 1)$	$\xrightarrow{D} \mathrm{N}\Gamma(r, 1)$
InvExp	$\xrightarrow{D} \infty$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \Gamma(r, 1)$
NInvExp	$\xrightarrow{D} \infty$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \mathrm{N}\Gamma(r, 1)$	$\xrightarrow{D} \infty$
Pareto (α)	$\xrightarrow{D} \Gamma(r, \alpha)$	$\xrightarrow{D} 0$	$\xrightarrow{D} \infty$	$\xrightarrow{D} 0$
NPareto (α)	$\xrightarrow{D} 0$	$\xrightarrow{D} \Gamma(r, \alpha)$	$\xrightarrow{D} 0$	$\xrightarrow{D} \infty$
InvPareto (1, σ)	$\xrightarrow{D} \Gamma(r, 1/\sigma)$	$\xrightarrow{D} \infty$	$\xrightarrow{D} 0$	$\xrightarrow{D} \infty$
NInvPareto (1, σ)	$\xrightarrow{D} \infty$	$\xrightarrow{D} \Gamma(r, 1/\sigma)$	$\xrightarrow{D} \infty$	$\xrightarrow{D} 0$
NGumbel	$\xrightarrow{D} \infty$	$\xrightarrow{D} \infty$	$\xrightarrow{D} 0$	$\xrightarrow{D} \mathrm{N}\Gamma(r, 1)$
Gumbel	$\xrightarrow{D} \infty$	$\xrightarrow{D} \infty$	$\xrightarrow{D} \Gamma(r, 1)$	$\xrightarrow{D} 0$

7. DISCUSSION

In Sections 4 and 5 we studied limit distributions of k-th record values following the approach of Gajek [4] and its extension from [2]. It leads to theorems yielding gamma distributions as limit distributions for large classes of sequences of distribution functions. In particular, from Theorems 1, 3, 5 and 7 we are able to derive easily limit distributions of differences and quotients of non-adjacent k-th record values. In this section we note that Theorems 2, 4, 6 and 8 (being consequences of Theorems 1, 3, 5 and 7) can be also obtained by arguments based on some distribution properties of record values.

Let $\{Y_n^{(k)}, n \geqslant 1\}$ be the sequence of k-th record values of the sequence $\{X_n, n \geqslant 1\}$ of i.i.d. random variables with the df F and the pdf f. Moreover, let $\{\bar{Y}_n^{(k)}, n \geqslant 1\}$ be the sequence of k-th record values of the sequence $\{\bar{X}_n, n \geqslant 1\}$ of i.i.d. random variables with the df G and the pdf g. Define the function

$$H_G(x) = F^{-1}(G(x)), \quad x \in \boldsymbol{R},$$

where F^{-1} is the pseudo-inverse of F. We use the fact that the sequences $\{Y_n^{(k)}, n \geqslant 1\}$ and $\{H_G(\bar{Y}_n^{(k)}), n \geqslant 1\}$ have the same finite-dimensional distributions. In particular, for $n \geqslant 1$, $k \geqslant 1$

$$Y_n^{(k)} \stackrel{d}{=} H_G(\bar{Y}_n^{(k)}),$$

where $\stackrel{d}{=}$ denotes equality in distribution. Therefore

$$Y_{n+r}^{(k)} - Y_n^{(k)} \stackrel{d}{=} H_G(\bar{Y}_{n+r}^{(k)}) - H_G(\bar{Y}_n^{(k)}) = H'_G(\theta_{n,r}^{(k)})(\bar{Y}_{n+r}^{(k)} - \bar{Y}_n^{(k)}), \tag{7.1}$$

by the mean value theorem, where

$$\bar{Y}_n^{(k)} \leqslant \theta_{n,r}^{(k)} \leqslant \bar{Y}_{n+r}^{(k)}. \tag{7.2}$$

Note that

$$H'_G(x) = (F^{-1})'(G(x))\, g(x) = \frac{g(x)}{f(H_G(x))}. \tag{7.3}$$

Now the statement of Theorem 2 is a consequence of the following arguments. Let $G(x) = 1 - e^{-x}$, $x \geqslant 0$. By (7.1) we have

$$k(Y_{n+r}^{(k)} - Y_n^{(k)}) \stackrel{d}{=} H'_G(\theta_{n,r}^{(k)})\, k(\bar{Y}_{n+r}^{(k)} - \bar{Y}_n^{(k)}).$$

But $k(\bar{Y}_{n+r}^{(k)} - \bar{Y}_n^{(k)})$ has the gamma $\Gamma(r, 1)$ distribution and by (7.2) we obtain $\theta_{n,r}^{(k)} \xrightarrow{P} 0$ as $k \to \infty$, and then

$$H'_G(\theta_{n,r}^{(k)}) \xrightarrow{P} \lambda^{-1}, \quad k \to \infty,$$

where

$$\lambda = \lim_{x \to F^{-1}(0)} f(x),$$

which is the same as λ given in (4.3). Therefore

$$k(Y_{n+r}^{(k)} - Y_n^{(k)}) \xrightarrow{D} \Gamma(r, \lambda), \quad k \to \infty.$$

Remark 4. Theorem 4 can be proved in the same way, but with $G(x) = e^x$, $x \leqslant 0$, which is a negative exponential distribution function.

Similarly, Theorem 6 can be established as follows. Assume that $G(x) = 1-\exp(-e^{x})$, $x \in \boldsymbol{R}$, is a negative Gumbel distribution function. By (7.1) we have

$$n\left(\frac{Y_{n+r}^{(k)}}{Y_{n}^{(k)}}-1\right) = \frac{n(Y_{n+r}^{(k)}-Y_{n}^{(k)})}{Y_{n}^{(k)}} = \frac{H'_G(\theta_{n,r}^{(k)})}{H_G(Y_{n}^{(k)})}\, n(\bar{Y}_{n+r}^{(k)}-\bar{Y}_{n}^{(k)}).$$

From Lemma 1 we see that $n(\bar{Y}_{n+r}^{(k)}-\bar{Y}_{n}^{(k)})$ has the gamma $\Gamma(r, 1)$ distribution and by (7.2) we obtain $\theta_{n,r}^{(k)} \xrightarrow{P} 0$ as $n \to \infty$, which implies

$$\frac{H'_G(\theta_{n,r}^{(k)})}{H_G(Y_{n}^{(k)})} \xrightarrow{P} \frac{1}{\mu}, \qquad n \to \infty,$$

where

$$\mu = \lim_{x \to \infty} \frac{H_G(x)}{H'_G(x)} = -\lim_{x \to F^{-1}(1)} \frac{xf(x)}{\bar{F}(x)\log \bar{F}(x)},$$

which is the same as μ given in (5.3). Note that $\mu \geqslant 0$ if $F^{-1}(1) > 0$ and $\mu \leqslant 0$ if $F^{-1}(1) \leqslant 0$. Therefore

$$n\left(\frac{Y_{n+r}^{(k)}}{Y_{n}^{(k)}}-1\right) \xrightarrow{D} \begin{cases} \Gamma(r, \mu) & \text{if } F^{-1}(1) > 0, \\ \mathrm{N}\Gamma(r, -\mu) & \text{if } F^{-1}(1) \leqslant 0, \end{cases} \quad n \to \infty.$$

Remark 5. Theorem 8 can be proved in the same way, but with $G(x) = \exp(-e^{-x})$, $x \in \boldsymbol{R}$, which is a Gumbel distribution function.

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