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ON NORMALIZERS AND CENTRALIZERS OF COMPACT LIE GROUPS. APPLICATIONS TO STRUCTURAL PROBABILITY THEORY

BY

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Abstract. The concept of operator stability on finite-dimensional vector spaces V was generalized in the past into several directions. In particular, operator-semistable and self-decomposable laws and self-similar processes were investigated and the underlying vector space V may be replaced by a simply connected nilpotent Lie group G. This motivates investigations of certain linear subgroups of $\mathrm{GL}(V)$ and $\mathrm{Aut}(G)$, respectively, the *decomposability group* of a full probability μ and its compact normal subgroup, the *invariance group.*

Using some basic properties of algebraic groups, the structure of normalizers and centralizers of compact matrix groups is analyzed and applied to the above-mentioned set-up, proving the existence and describing the shape of *exponents* and of *commuting exponents* of (operator-) semistable laws.

Further applications are mentioned, in particular for operator self-decomposable laws and self-similar processes.

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1. INTRODUCTION

A generalization of the concept of stable laws on $\boldsymbol{R}$ leads to operator stability [24] and operator semistability [14] on finite-dimensional vector spaces. For recent surveys see e.g. [16] and [21]. It turns out that a natural framework for these investigations are simply connected nilpotent Lie groups [9].

A continuous convolution semigroup $\mu_{\cdot}$ on a group G (or on a vector space) is called *(strictly) stable* if there exists a continuous one-parameter group $(a(t))_{t>0} \subseteq \mathrm{Aut}(G)$ such that the self-similarity property $a(t)(\mu_s) = \mu_{s\cdot t}$, $s \geqslant 0$, $t > 0$, is fulfilled. Recalling that $\mathrm{Aut}(G)$ may be considered as a closed subgroup of $\mathrm{GL}(\boldsymbol{R}, d)$, $a(t)$ has a representation $a(t) \approx t^E = \exp((\log t)\cdot E)$ for some endomorphism E. In fact, E is a derivation, $E \in \mathrm{Der}(G)$. The endomorphism E is called an *exponent* of $\mu_\cdot$. It can be shown for full $\mu_\cdot$ that *commuting* exponents exist, i.e. exponents E with (t^E) centralizing the invariance group of $\mu_\cdot$. The semigroup $\mu_\cdot$ is called *semistable* if for some $a \in \mathrm{Aut}(G)$ and $\alpha > 0$ ($\alpha \neq 1$) the weaker condition $a(\mu_s) = \mu_{\alpha \cdot s}$, $s \geqslant 0$, is fulfilled. A *semistable exponent* of $\mu_\cdot$ is a derivation $E \in \mathrm{Der}(G)$ fulfilling the relation $\alpha^E = a$. For vector spaces, the existence of semistable exponents is proved by Chorny [6], for more profound investigations and for applications see [21]; for groups see [9]. In fact, here we shall use a more restrictive definition (cf. 2.3) assuming in addition that t^E normalizes the invariance group $\mathrm{Inv}(\mu_\cdot)$ for all $t > 0$. (For the probabilistic background see e.g. the above-mentioned surveys [16] and [21] for vector spaces, and [9] for groups.)

The shape of the sets of *semistable exponents* and the existence of commuting semistable exponents have not been sufficiently investigated until now.

In [8] the existence of *commuting normalizations* was proved, so to say, a discrete version of commuting semistable exponents: For a compact subgroup K of an almost connected Lie group H and $a \in \mathrm{N}(K, H)$ there exist a natural number r and $\kappa \in K$ such that $a^r \kappa$ centralizes K. The proof relies on the structure of compact Lie groups, in fact, the essential result is the finiteness of $\mathrm{N}(K, H)/K_0 \cdot \mathrm{C}(K, H)$. In [23] these "finiteness results" are partially extended into different directions for reductive subgroups of almost connected Lie groups.

Using a few basic facts of the theory of algebraic groups we are able to prove the *existence* of semistable exponents (for some power a^p, respectively α^p) in the normalizer $\mathrm{N}(K, H)$, and then, by a general splitting theorem due to K.H. Hofmann, the existence of *commuting* exponents follows, thus completing and improving partially the results of [8] if the groups H under consideration are assumed to be algebraic, a condition which is fulfilled in all applications we have in mind.

The paper is organized as follows: As mentioned, motivated by probabilistic problems, we are first led to particular investigations of the structure of normalizers and centralizers of compact — hence algebraic — Lie groups. In particular, we investigate for $a \in \mathrm{N}(K, H)$ the embeddability into a continuous one-parameter group belonging to the normalizer and the centralizer of K, respectively. This is the content of Section 1. In Section 2 we apply these results to prove the existence of (commuting) semistable exponents of convolution semigroups of probabilities on a vector space or on a group and describe the shape of the set of (commuting) semistable exponents, and in Sections 3 and 4 we sketch some further applications of Section 1: canonical representations of semistable Lévy measures, (semi-)stable *hemi*-groups, operator self-similar processes and — in Section 4 — operator self-decomposable laws.

1. SOME BASIC RESULTS CONCERNING ALGEBRAIC SUBGROUPS OF GL$(\boldsymbol{R}, d)$

Let Γ be a subgroup of GL$(\boldsymbol{R}, d)$; the complexification is denoted by $\Gamma^{\boldsymbol{C}} \subseteq$ GL$(\boldsymbol{C}, d)$. We shall assume throughout that $\Gamma^{\boldsymbol{C}}$ is an algebraic subgroup of GL$(\boldsymbol{C}, d)$. Any $x \in \Gamma \subseteq$ GL$(\boldsymbol{R}, d)$ defines uniquely an element $\tilde{x} \in \Gamma^{\boldsymbol{C}} \subseteq$ GL$(\boldsymbol{C}, d)$.

Conversely, let σ denote the automorphism of GL$(\boldsymbol{C}, d)$ defined by complex conjugation. Then $y \in$ GL$(\boldsymbol{C}, d)$ is *real*, i.e., $y = \tilde{x}$ for some $x \in$ GL$(\boldsymbol{R}, d)$ iff $\sigma(y) = y$. For a subgroup $H \subseteq$ GL$(\boldsymbol{C}, d)$ let $H_{\boldsymbol{R}} = \{y\colon \sigma(y) = y\}$ denote the subgroup of real elements. In particular, $\Gamma = (\Gamma^{\boldsymbol{C}})_{\boldsymbol{R}}$.

For $x \in \Gamma$ and $\tilde{x} \in \Gamma^{\boldsymbol{C}}$ let $\langle x \rangle$ and $\langle \tilde{x} \rangle$ denote the subgroups generated by x and $\tilde{x}$, respectively.

Let $\mathscr{K} = \boldsymbol{R}$ or $\boldsymbol{C}$. For a subset $M \subseteq$ GL$(\mathscr{K}, d)$ let M^z denote the closure in the Zariski topology, for a subgroup M let M^0 denote the irreducible (i.e., Zariski-connected) component containing the unit e, and let M_0 denote the connected component (with respect to the given group topology).

Throughout we use the notation $\boldsymbol{C}^\times$ for the set of non-zero complex numbers. $\boldsymbol{R}^\times$ and $\Gamma^\times$ are defined analogously. Spec(a) will denote the spectrum of a vector space endomorphism a.

We shall only make use of a few basic properties of algebraic groups. The reader is referred e.g. to [22], [5] and [13]. To make the paper more selfcontained and accessible for the reader who is not familiar with algebraic groups we repeat some of the basic constructions. In particular, we make use of the following facts:

- Intersections of algebraic (i.e. Zariski-closed) subgroups of GL$(\mathscr{K}, d)$ are algebraic.

- The closure A^z of an Abelian subgroup A is an Abelian algebraic group. (This follows immediately by [4], 2.4, Proposition.)

- The normalizers and centralizers of algebraic subgroups of GL$(\mathscr{K}, d)$ are algebraic. (Cf. [22], Chapter I, 8.2, Corollary, or [3], 2.4c, d.)

- The Zariski topology is Noetherian; hence for an algebraic group H we obtain $[H\colon H^0] < \infty$. (Cf. [13], 7.3, Proposition.)

- Compact real Lie groups are algebraic groups. (Cf. [22], Chapter 3, §4, No. 4, Theorem 5.)

- Recall the (additive and multiplicative) Jordan decomposition: For an element $x \in$ End$(\boldsymbol{C}, d)$ let x_s, x_n and, for $x \in$ GL$(\boldsymbol{C}, d)$, x_u denote the semisimple, nilpotent and unipotent parts, respectively. x_s depends on the spectral decomposition of x; hence there exist polynomials P_s and P_n with $P_s(z)+P_n(z) = z$ for $z \in \boldsymbol{C}$ such that $x_s = P_s(x)$, $x_n = P_n(x)$, and, in particular, $x_s x_n = x_n x_s$; respectively, $x_s x_u = x_u x_s = x$, $x_u = (e + x_s^{-1} x_n)$. (Cf. e.g. [5], p. 80, Proposition 4.2.)

If x is real, the spectral decomposition is symmetric with respect to complex conjugation; therefore, the polynomials P_s, and hence also P_n belong to $\boldsymbol{R}[x]$.

For an algebraic subgroup $H \subseteq \mathrm{GL}(\boldsymbol{C}, d)$ we have: If $x \in H$, then x_s, and hence x_u belong to H. (Cf. [22], Chapter 3, No. 3, Theorem 6.) This applies in particular to $H = \Gamma^{\boldsymbol{C}}$ and the polynomial representation yields: if x is real, i.e. $x \in \Gamma$, then $x_s, x_u \in \Gamma$.

1.1. Lemma. *Let* $x \in \mathrm{GL}(\boldsymbol{R}, d)$ *be semisimple with* $\mathrm{Spec}(x) = \{\alpha, \bar{\alpha}\}$, $\Im\alpha \neq 0$, *and with the corresponding eigenspace decomposition* $\boldsymbol{C}^d = W = V \oplus \bar{V}$, $\tilde{x} = \alpha \cdot \mathrm{Id}_V + \bar{\alpha} \cdot \mathrm{Id}_{\bar{V}}$. (*That is, if* $u = u_1 + i \cdot u_2 \in V$, *then* $\tilde{x}(u) = \alpha \cdot u$, *and* $\tilde{x}(\bar{u}) = \tilde{x}(u_1 - i \cdot u_2) = \bar{\alpha} \cdot \bar{u}$.) *Define*

$$\tilde{x}(s, t) := s \cdot \alpha \cdot \mathrm{Id}_V + t \cdot \bar{\alpha} \cdot \mathrm{Id}_{\bar{V}} \quad \text{for } s, t \in \boldsymbol{C}^{\times}.$$

Then $\tilde{x}(s, t)$ *is real iff* $s = t$.

Note. $\{\tilde{x}(s, t)\}$ is the Zariski closure of $\langle \tilde{x} \rangle$ if this group is Zariski-connected. (Cf. [22], Chapter 3, §2, Theorem 3.)

Proof. $\tilde{x}(s, t)$ is real if it is a fixed point with respect to complex conjugation σ. Fix $u = u_1 + i \cdot u_2 \in V$, $\bar{u} = u_1 - i \cdot u_2 \in \bar{V}$ $(u_2 \neq 0)$. Then

$$\tilde{x}(s, t) = s \cdot \alpha \cdot \mathrm{Id}_V + t \cdot \bar{\alpha} \cdot \mathrm{Id}_{\bar{V}} = \sigma(\tilde{x}(s, t))$$

iff

$$\tilde{x}(s, t) = \bar{s} \cdot \bar{\alpha} \cdot \mathrm{Id}_{\bar{V}} + \bar{t} \cdot \alpha \cdot \mathrm{Id}_V \Leftrightarrow s \cdot \alpha \cdot u + t \cdot \bar{\alpha} \cdot \bar{u} = \bar{s} \cdot \bar{\alpha} \cdot \bar{u} + \bar{t} \cdot \alpha \cdot u$$

$$\Leftrightarrow (s - \bar{t}) \cdot \alpha \cdot u = (\bar{s} - t) \cdot \bar{\alpha} \cdot \bar{u}.$$

Since $u = u_1 + i \cdot u_2$ and $u_2 \neq 0$, this is equivalent to $(s - \bar{t}) \cdot (\alpha - \bar{\alpha}) = 0$ and, consequently, to $s = \bar{t}$, as asserted. ■

1.2. Lemma. *Let* $x \in \mathrm{GL}(\boldsymbol{R}, d)$, $\tilde{x} \in \mathrm{GL}(\boldsymbol{C}, d)$ *as before. Let* $x = x_u$ *be unipotent. Then* $\tilde{x}(s)$ *is real iff* $s \in \boldsymbol{R}$.

Note. In this case $\{\tilde{x}(s) := \exp(s \cdot \log x_u) \colon s \in \boldsymbol{C}\}$ is the Zariski closure of $\langle \tilde{x} \rangle$. (Cf. [22], Chapter 3, §2, No. 2, Theorem 1.)

Proof. Since $e - \tilde{x}$ is nilpotent, it follows that $b := \log \tilde{x} = -\sum_1^{\infty} k^{-1} \cdot (e - \tilde{x})^k$ is a polynomial (with real coefficients) of $e - \tilde{x} \in \mathrm{End}(\boldsymbol{C}, d)$. Hence

$$\sigma(\tilde{x}(s)) = \sum \frac{\bar{s}^k}{k!} \cdot \tilde{b}^k$$

(since $\sigma(\tilde{b}) = \tilde{b}$). Therefore we infer immediately that $\tilde{x}(s)$ is real iff $s = \bar{s}$, i.e. iff $s \in \boldsymbol{R}$. ■

1.3. Lemma. *Let* $x \in \mathrm{GL}(\boldsymbol{R}, d)$ *be semisimple with* $\mathrm{Spec}(x) = \{\alpha\}$, $\alpha \in \boldsymbol{R}$. *Then* $\tilde{x}(s)$ *is real iff* $s \in \boldsymbol{R}^{\times}$.

Note. Here the Zariski closure is given by $\{\tilde{x}(s) := s \cdot \alpha \cdot \mathrm{Id}\colon s \in \boldsymbol{C}^\times\}$. (Cf. [22], Chapter 3, §2, Theorem 3.) The proof of Lemma 1.3 is obvious.

Since the projections of the eigenspaces are polynomials, combining Lemmas 1.1–1.3 we obtain

1.4. Proposition. *Let $x \in \mathrm{GL}(\boldsymbol{R}, d)$ with complexification $\tilde{x}$. Let $\Gamma := \langle x \rangle^z$ and $\Gamma^{\boldsymbol{C}} := \langle \tilde{x} \rangle^z$ denote the Zariski closures (over $\boldsymbol{R}$, respectively $\boldsymbol{C}$) of the subgroups generated by x and $\tilde{x}$, respectively. Then Γ (and $\Gamma^{\boldsymbol{C}}$) are Abelian (as mentioned above) and algebraic. Furthermore, we have $[\Gamma : \Gamma_0] < \infty$. That is, Γ is an almost connected real (Abelian) Lie group.*

Proof. 1. Since $\Gamma^{\boldsymbol{C}}$ is algebraic, it follows that, as mentioned above, $[\Gamma^{\boldsymbol{C}}\colon (\Gamma^{\boldsymbol{C}})^0] < \infty$. Therefore we may assume without loss of generality that $\Gamma^{\boldsymbol{C}}$ is irreducible (i.e. Zariski-connected).

2. Consequently, $\Gamma^{\boldsymbol{C}}$ is a torus, and $\Gamma^{\boldsymbol{C}} \approx (\boldsymbol{C}, +) \oplus \Sigma \oplus (\boldsymbol{C}^\times, \cdot)$ (cf. [22], Chapter 3, §2, No. 5, Theorem 8, Corollary).

More precisely, let $x = x_s \cdot x_u$ be the Jordan decomposition. Put $b := \log x_u$. Then

$$\mathrm{Spec}(x) = \mathrm{Spec}(x_s) = \{\alpha_j\colon j \in I_1\} \cup \{\alpha_j, \bar{\alpha}_j\colon j \in I_2\}$$

with $\alpha_j \in \boldsymbol{R}^\times, j \in I_1$, and $\Im \alpha_j > 0, j \in I_2$. Let V_j and $\bar{V}_j$ denote the corresponding eigenspaces. ($V_j = \bar{V}_j$ if $j \in I_1$.) Put $\vec{t} = (t_0, t_j, s_k, s_k')$, where $t_0 \in \boldsymbol{C}$, $t_j \in \boldsymbol{C}^\times, j \in I_1$, $s_k, s_k' \in \boldsymbol{C}^\times$, $k \in I_2$. Define

$$\tilde{x}(\vec{t}) := \exp(t_0 b) \cdot (\Sigma_{I_1} \oplus (t_j \cdot \alpha_j \cdot \mathrm{Id}_{V_j}) \oplus \Sigma_{I_2} \oplus (s_j \cdot \alpha_j \cdot \mathrm{Id}_V + s_j' \cdot \bar{\alpha}_j \cdot \mathrm{Id}_{\bar{V}_j})).$$

Then the Zariski closure $\langle \tilde{x} \rangle^z$ is given by $\{\tilde{x}(\vec{t})\colon \vec{t} \in (\boldsymbol{C}, +) \oplus \Sigma \oplus (\boldsymbol{C}^\times, \cdot)\}$. Applying Lemmas 1.1–1.3 we observe that $\tilde{x}(\vec{t})$ is real iff $t_0 \in \boldsymbol{R}$, $t_j \in \boldsymbol{R}^\times$ for $j \in I_1$, and $s_k' = \bar{s}_k \in \boldsymbol{C}^\times$ for $k \in I_2$; and, furthermore, $\tilde{x}(\vec{t}) \in \Gamma_0$ iff, in addition, $t_j \cdot \alpha_j > 0$ for $j \in I_1$. Therefore, we obtain immediately $[\Gamma\colon \Gamma_0] = 2^r$ for some $r \in \boldsymbol{Z}_+$. In fact, we have $x \in \Gamma_0$ iff $\alpha_i > 0$ for $i \in I_1$. Consequently, $x \in \Gamma$ yields $x^{2^r} \in \Gamma_0$. Thus, by the reduction step 1, the assertions follow. ∎

1.5. Corollary. (a) *With the notation of* 1.4, *there exist $p \in \boldsymbol{N}$, $p \mid [\Gamma\colon \Gamma_0]$, and a continuous one-parameter group $(y(t))_{t \in \boldsymbol{R}} \subseteq \Gamma_0$ such that $x^p = y(1)$.*

Indeed, we have $x^p \in \Gamma_0$. Since Γ_0 is a connected Abelian Lie group, the exponential mapping exp is surjective.

(b) In fact, the number p may be determined more precisely: If $\Gamma^{\boldsymbol{C}}$ is not irreducible, then as an Abelian agebraic group it is a quasi-torus, i.e. a direct product of a finite group F and the torus $(\Gamma^{\boldsymbol{C}})^0$. Thus, as immediately seen, $[\Gamma^{\boldsymbol{C}}\colon (\Gamma^{\boldsymbol{C}})^0] =: q$ is the order of the maximal finite subgroup F of Γ reflecting the symmetry properties of $\mathrm{Spec}(x)$. Indeed, put $y := x^q$, $\{\beta_j, j \in I_3\} = \mathrm{Spec}(y)$. Then $\beta_i^k \neq \beta_j^k$ for all $i \neq j$ and all $k \in \boldsymbol{N}$. Hence $(\tilde{y}(\vec{t}))$ is a torus, i.e. an irreducible subgroup. Thus step 2 of the proof of Proposition 1.4 applies to yield $p = 2^r \cdot q$, or $p = q$ if all real eigenvalues β_i are positive.

(c) The unipotent part $y(t)_u$ is uniquely determined by $b = \log x_u$, hence by x and also by x^p, and so are the logarithms of the semisimple parts $(\alpha_j)^p \cdot \mathrm{Id}_{V_j}$ of x^p for real α_j, i.e. for $j \in I_1$.

Remark. Proposition 1.4 is of course well known. In fact, if Γ^C is irreducible, then this group is connected (as a Lie group), and, as mentioned in [4], p. 5, for the real group Γ we have $[\Gamma : \Gamma_0] < \infty$. That is, Γ is an almost connected Lie group. Here however we preferred to have — for particular Abelian groups — a direct constructive proof which can be verified nearly without knowledge of algebraic groups, and which allows to calculate the number $p = p(x)$ explicitly (see 1.5). Note that Chorny's proof of the existence of semistable exponents (see [6]) is based on the fact that $\mathrm{GL}(\boldsymbol{R}, d)$ has two connected components; hence $p \leqslant 2$ for $\Gamma = \mathrm{GL}(\boldsymbol{R}, d)$.

In the following let $\boldsymbol{H} \subseteq \mathrm{GL}(\boldsymbol{R}, d)$ be algebraic and let K be a compact subgroup of $\boldsymbol{H}$. Let $\mathrm{N}(K)$ and $\mathrm{C}(K)$,

$$\mathrm{N}(K, \boldsymbol{H}) := \mathrm{N}(K) \cap \boldsymbol{H}, \quad \mathrm{C}(K, \boldsymbol{H}) := \mathrm{C}(K) \cap \boldsymbol{H},$$

denote the *normalizer* and the *centralizer* of K, respectively.

K is algebraic (over $\boldsymbol{R}$) (cf. [22], Chapter 3, §4, No. 4, Theorem 5). Therefore $\mathrm{N}(K)$ and $\mathrm{C}(K)$, and hence also $\mathrm{N}(K, \boldsymbol{H})$ and $\mathrm{C}(K, \boldsymbol{H})$ are algebraic subgroups of $\mathrm{GL}(\boldsymbol{R}, d)$ and of $\boldsymbol{H}$, respectively (cf. [22], Chapter I, 8.2, Corollary, or [3], 2.4c, d). Consequently, we observe for $a \in \mathrm{N}(K, \boldsymbol{H})$ that

$$\langle a \rangle \subseteq \langle a \rangle^z \subseteq \mathrm{N}(K, \boldsymbol{H}). \tag{1.1}$$

Remark. Note that the group $\boldsymbol{H}$ endowed with the Zariski topology is *not* a topological group. Hence the above-mentioned properties of its subgroups are not obvious.

Using the previous preparatory results we obtain:

1.6. Proposition. *Let $a \in \mathrm{N}(K, \boldsymbol{H})$. There exist $p \in \boldsymbol{N}$ depending on a and a continuous one-parameter group $(y(t))_{t \in \boldsymbol{R}} \subseteq \mathrm{N}(K, \boldsymbol{H})$ such that $y(1) = a^p$.*

For the proof apply 1.4 and 1.5 to the algebraic Abelian group $\langle \tilde{a} \rangle^z \subseteq \mathrm{GL}(\boldsymbol{C}, d)$, respectively to $\langle a \rangle^z \subseteq \mathrm{GL}(\boldsymbol{R}, d)$. Using (1.1) we see that $y(t)$ (defined in 1.5) belongs to $(\mathrm{N}(K, \boldsymbol{H}))_0$.

Applying a splitting theorem of K.H. Hofmann we obtain, e.g. as in [9], 1.8.8, 2.8.8, or [8], Remark 1.9, the following

1.7. Theorem. *Let $\boldsymbol{H} \subseteq \mathrm{GL}(\boldsymbol{R}, d)$ be algebraic, let $K \subseteq \boldsymbol{H}$ be a compact subgroup and $a \in \mathrm{N}(K, \boldsymbol{H})$. Assume that $\langle a \rangle$ is not relatively compact in $\boldsymbol{H}$ (hence in $\mathrm{GL}(\boldsymbol{R}, d)$). Then there exist $p \in \boldsymbol{N}$ (depending on a) and continuous one-parameter groups $(y_c(t))_{t \in \boldsymbol{R}} \subseteq \mathrm{C}(K, \boldsymbol{H})$ and $(\kappa(t))_{t \in \boldsymbol{R}} \subseteq K$ such that $y_c(1) = a^p \cdot \kappa(1)^{-1}$ and $y(t) = y_c(t) \cdot \kappa(t)$, $t \in \boldsymbol{R}$.*

Proof. According to 1.6 there exists a continuous one-parameter group $(y(t))_{t\in\boldsymbol{R}} \subseteq \mathrm{N}(K, \boldsymbol{H})$ such that $y(1) = a^p$. Since $\langle a\rangle$ is not relatively compact, $(y(t)\colon t\in\boldsymbol{R})$ is not relatively compact; hence $\{y(t)\colon t\in\boldsymbol{R}\}\cap K = \{e\}$. Hence the splitting theorem (cf. [10], Proposition 9.4, or [11], Lemma 1.25; see also [8], 1.7, for a new proof) applies and yields the existence of a centralizing one-parameter group $(y_c(t))_{t\in\boldsymbol{R}} \subseteq \mathrm{C}(K, \boldsymbol{H})$ and $(\kappa(t))_{t\in\boldsymbol{R}} \subseteq K$ such that $y(t) = y_c(t)\cdot\kappa(t)$ for $t\in\boldsymbol{R}$. That is, we obtain a direct splitting

$$(y(t)\colon t\in\boldsymbol{R})\cdot K = (y_c(t)\colon t\in\boldsymbol{R})\otimes K.$$

Thus the assertion follows. ■

Remark. Note that the discrete version mentioned in the Introduction, i.e. the existence of commuting normalizations (see [8]), is slightly different: In [8] the finite number r depends on the structure of the compact subgroup K and its normalizer, whereas p in Theorem 1.7 depends on a, cf. Corollary 1.5.

2. APPLICATION TO SEMISTABLE CONVOLUTION SEMIGROUPS

As in [8], a probabilistic background of the previous considerations may be described as follows: Let $\boldsymbol{G}$ denote a simply connected step-r nilpotent Lie group of dimension d. (If $r = 0$, $\boldsymbol{G}$ is a vector space.) Let $\mathfrak{G}$ denote the Lie algebra ($\mathfrak{G} \approx \boldsymbol{R}^d$ as vector spaces).

$\mathcal{M}^1(\boldsymbol{G})$ denotes the set of probability measures on $\boldsymbol{G}$, endowed with convolution $\star$ and with topology of weak convergence.

$\mu_{\bullet} = (\mu_t\colon t \geqslant 0)$ denotes a continuous convolution semigroup, i.e. $\mu_t\in\mathcal{M}^1(\boldsymbol{G})$, $t \geqslant 0$, $t\mapsto\mu_t$ is weakly continuous, and $\mu_t\star\mu_s = \mu_{t+s}$, $t, s \geqslant 0$.

$\mathrm{Aut}(\boldsymbol{G})$ denotes the group of Lie group automorphisms. Note that $\mathrm{Aut}(\boldsymbol{G}) \approx \mathrm{Aut}(\mathfrak{G})$ (the group of Lie algebra automorphisms), since

$$\mathrm{Aut}(\boldsymbol{G})\ni a\mapsto a^{\circ}\in\mathrm{Aut}(\mathfrak{G}) \qquad \text{with } a^{\circ} := \exp^{-1}\circ a\circ\exp.$$

Therefore, $\mathrm{Aut}(\boldsymbol{G})$ is (isomorphic to) an algebraic subgroup of $\mathrm{GL}(\boldsymbol{R}, d)$. A multiplicatively parametrized one-parameter group in $\mathrm{Aut}(\boldsymbol{G})$ is a continuous map $\boldsymbol{R}_+^{\times}\ni t\mapsto a(t)\in\mathrm{Aut}(\boldsymbol{G})$ such that $a(t)a(s) = a(t\cdot s)$ for $t, s > 0$. (Then $s\mapsto b(s) := a(e^s)$ fulfils $b(t+s) = b(t)b(s)$ for all $t, s\in\boldsymbol{R}$.) In Sections 2 and 3 we assume throughout one-parameter groups to be multiplicatively parametrized. The differentials $a^{\circ}(t)$ are continuous operator semigroups in $\mathrm{Aut}(\mathfrak{G})$; hence $a^{\circ}(t) = t^E =: \exp((\log t)\cdot E)$ for some derivation $E\in\mathrm{Der}(\mathfrak{G}) \subseteq \mathrm{End}(\boldsymbol{R}, d)$.

2.1. Definition. (a) A continuous convolution semigroup $\mu_{\bullet}$ is called *stable* if there exists a continuous one-parameter group $(a(t))_{t>0} \subseteq \mathrm{Aut}(\boldsymbol{G})$ such that $a(t)(\mu_1) = \mu_t$, $t > 0$; equivalently,

$$a(t)(\mu_s) = \mu_{t\cdot s}, \quad t > 0,\ s \geqslant 0. \tag{2.1}$$

(b) $\mu_\cdot$ is *semistable* if there exist $a \in \mathrm{Aut}(G)$ and $\alpha \in (0, 1)$ such that

$$a(\mu_s) = \mu_{\alpha \cdot s}, \quad s \geqslant 0. \tag{2.2}$$

(c) For stable convolution semigroups $\mu_\cdot$ with corresponding groups $(a(t))_{t>0}$, $a^\circ(t) = t^E$, we call E an *exponent* of $\mu_\cdot$. Let $\mathrm{EXP}(\mu_\cdot) = \mathrm{EXP} \subseteq \mathrm{Der}(\mathfrak{G})$ denote the set of exponents.

(d) Let $\mathrm{Dec}(\mu_\cdot) := \{a \in \mathrm{Aut}(G);\ \exists\ \alpha \in \boldsymbol{R}_+^\times\colon a(\mu_s) = \mu_{\alpha \cdot s},\ s \geqslant 0\}$ and $\mathrm{Inv}(\mu_\cdot) := \{a\colon a(\mu_s) = \mu_s,\ s \geqslant 0\}$ denote the *decomposability group* and the *invariance group*, respectively. The mapping

$$\varphi\colon \mathrm{Dec}(\mu_\cdot) \to \boldsymbol{R}_+^\times, \quad \varphi(a) := \alpha$$

is called the *canonical homomorphism*. Define for fixed $\alpha \in \boldsymbol{R}_+^\times$:

$$\mathrm{Dec}_\alpha(\mu_\cdot) := \{a \in \mathrm{Dec}(\mu_\cdot)\colon \varphi(a) \in \{\alpha^k\colon k \in \boldsymbol{Z}\}\} = \varphi^{-1}\{\alpha^k\colon k \in \boldsymbol{Z}\}.$$

Remark. If $r = 0$, i.e. if G is a vector space ($\approx \mathfrak{G}$), then stable (respectively, semistable) semigroups defined as above are called *strictly operator stable* (respectively, *operator semistable*). (For more details cf. e.g. [16], [21]; respectively, [9], Chapter I). Furthermore, in this case $\mu_\cdot$ is uniquely determined by a single measure $\mu := \mu_1$. (For vector spaces this is folklore, for the group case see [9], 2.6.11, 2.6.11*.)

2.2. On vector spaces a probability measure is usually called "full" if it is not concentrated on a proper *affine* subspace. Here we shall call this property "S-full" and use a slightly different notation:

μ is called *full* if it is not concentrated on a proper connected subgroup (i.e. on a proper *linear* subspace if G is a vector space). Equivalently, μ is full iff $\mathrm{Inv}(\mu) := \{a\colon a(\mu) = \mu\}$ is compact.

In fact, for simply connected nilpotent Lie groups, in particular for vector spaces, we have $\mathrm{Inv}(\mu_\cdot) = \mathrm{Inv}(\mu_t)$ for any $t > 0$. (See e.g. [16] and [21] for vector spaces; respectively, [9], § 2.5, II, 2.5.13, for the group case; and the literature mentioned therein.)

Fullness has a strong impact on the structure of the decomposability group: In particular, let $\mu_\cdot$ be full and let $a \in \mathrm{Dec}_\alpha(\mu_\cdot)$, $\alpha \in (0, 1)$. Then a is contractive, i.e. $a^n(x) \to e$ as $n \to \infty$ for all $x \in G$. Equivalently, $\varrho(a^\circ) < 1$, where ϱ denotes the spectral radius. (Cf. e.g. [16] and [21] for vector spaces, or [9], 1.3.9, 2.1.9, 2.3.11d.) Hence, in particular, $\langle a \rangle$ is not relatively compact for any $a \in \mathrm{Dec}(\mu_\cdot) \backslash \mathrm{Inv}(\mu_\cdot)$, and consequently

$$\langle a \rangle \cap \mathrm{Inv}(\mu_\cdot) = \{e\}. \tag{2.3}$$

$\mu_\cdot$ is stable if $\mathrm{im}(\varphi) = \boldsymbol{R}_+^\times$; $\mu_\cdot$ is semistable if $\mathrm{im}(\varphi) \neq \{1\}$ (and hence $\mathrm{im}(\varphi) \supseteq \{\alpha^k\colon k \in \boldsymbol{Z}\}$ for some $0 < \alpha < 1$). (Cf. e.g. [9], § 1.5, § 2.5.)

The stable case. For full (operator-) stable $\mu_\cdot$ the structure of $\mathrm{EXP}(\mu_\cdot)$ is well known (cf. e.g. [16], [21], [9] and the literature mentioned therein):

$$\mathrm{EXP}(\mu_\cdot) = E_0 + \mathfrak{inv} \tag{2.4}$$

for some fixed exponent E_0, where $\mathfrak{inv}$ denotes the Lie algebra of the compact group $\mathrm{Inv}(\mu_\cdot)$.

Furthermore, there exist *commuting exponents*, i.e. $E_c \in \mathrm{EXP}(\mu_\cdot)$ such that

$$(t^{E_c}\colon t > 0) \subseteq \mathrm{C}(\mathrm{Inv}(\mu_\cdot), \mathrm{Aut}(G)).$$

Let $\mathrm{EXP}_c(\mu_\cdot)$ denote the set of commuting exponents. Then

$$\mathrm{EXP}_c(\mu_\cdot) = E_c + \mathfrak{inv}_c \tag{2.5}$$

for some fixed $E_c \in \mathrm{EXP}_c(\mu_\cdot)$, where $\mathfrak{inv}_c$ denotes the Lie algebra of the centre $\mathrm{Cent}(\mathrm{Inv}(\mu_\cdot))$.

The semistable case. Our aim is to obtain similar results for the *semistable case*, extending and improving partially the *discrete* versions of *commuting normalizations* obtained in [8], Theorem 3.9, see also [9], §1.11:

For all $a \in \mathrm{Dec}(\mu_\cdot)$ with $\varphi(a) = \alpha \neq 1$, there exist $r \in N$, depending on the compact group $\mathrm{Inv}(\mu_\cdot)$, and $\kappa \in \mathrm{Inv}(\mu_\cdot)$ such that $a^r \cdot \kappa \in \mathrm{C}(\mathrm{Inv}(\mu_\cdot), \mathrm{Dec}(\mu_\cdot))$.

2.3. Definition. (a) Let $\mu_\cdot$ be full and semistable. A derivation $E \in \mathrm{Der}(\mathfrak{G})$ is called a *semistable exponent* of $\mu_\cdot$, for short: $E \in \mathrm{SEXP}(\mu_\cdot)$ if

(i) $a(t)$, defined by $a^\circ(t) = t^E$, belongs to $\mathrm{N}(\mathrm{Inv}(\mu_\cdot), \mathrm{Aut}(G))$ for $t > 0$, and

(ii) $a(\alpha)(\mu_s) = \mu_{s\alpha}$ for some $\alpha \neq 1$ and for all $s \geqslant 0$.

That is, $a(t)_{t \geqslant 0}\ (\approx a^\circ(t) = t^E) \subseteq \mathrm{N}(\mathrm{Inv}(\mu_\cdot), \mathrm{Aut}(G))$ and $a(\alpha)\ (\approx \alpha^E) \in \mathrm{Dec}(\mu_\cdot)$ with $\varphi(a(\alpha)) = \alpha$.

Let $\mathrm{SEXP}_\alpha(\mu_\cdot) := \{E \in \mathrm{SEXP}(\mu_\cdot),\ a^\circ(t) = t^E \text{ and } \varphi(a(\alpha)) = \alpha\}$ for $\alpha \neq 1$. Note that in contrast to (stable) exponents it is not assumed that $(a(t))_{t > 0} \subseteq \mathrm{Dec}(\mu_\cdot)$. On the other hand, in contrast to previous investigations, here we assume in addition in condition (i) that $a(t)$ normalizes $\mathrm{Inv}(\mu_\cdot)$ for all $t > 0$.

(b) A semistable exponent E is called a *commuting exponent* if the corresponding automorphism group $(a(t)(\approx t^E))$ belongs to the centralizer $\mathrm{C}(\mathrm{Inv}(\mu_\cdot), \mathrm{N}(\mathrm{Inv}(\mu_\cdot)))$. Then we write $E = E_c$.

Let $\mathrm{SEXP}_c(\mu_\cdot)$ denote the set of commuting semistable exponents, and, if α is fixed, $\mathrm{SEXP}_{c,\alpha}(\mu_\cdot) := \mathrm{SEXP}_c(\mu_\cdot) \cap \mathrm{SEXP}_\alpha(\mu_\cdot)$.

Concerning the existence of (commuting) semistable exponents we first state the following:

2.4. Theorem. *Let $\mu_\cdot$ be full and semistable and let $\alpha \in \boldsymbol{R}_+^\times \setminus \{1\}$ with* $\mathrm{im}(\varphi) \supseteq \{\alpha^k,\ k \in \boldsymbol{Z}\}$.

(a) *For some* $p \in N$ *depending on a we have* $\mathrm{SEXP}_{\alpha^p}(\mu_\cdot) \neq \emptyset$. *That is, if we put* $\beta := \alpha^p$, *then there exists a one-parameter group* $(a(t)) \subseteq \mathrm{N}(\mathrm{Inv}(\mu_\cdot), \mathrm{Aut}(G))$ *such that* $a(\beta) \in \mathrm{Dec}(\mu_\cdot)$ *with* $\varphi(a(\beta)) = \beta$.

(b) *Furthermore,* $\mathrm{SEXP}_{c,\alpha^p} \neq \emptyset$. *That is, there exists a one-parameter group* $(a_c(t))_{t>0} \subseteq \mathrm{N}(\mathrm{Inv}(\mu_\cdot), \mathrm{Aut}(G))$, *as in* (a), *commuting elementwise with* $\mathrm{Inv}(\mu_\cdot)$.

Proof. We apply the results of Section 1, in particular 1.6 and 1.7, to the groups $H = \mathrm{Aut}(G)$ and $K = \mathrm{Inv}(\mu_\cdot)$, respectively to the corresponding subgroups of $\mathrm{Aut}(\mathfrak{G}) \subseteq \mathrm{GL}(R, d)$. Reparametrizing $y(\cdot)$ in 1.6, we put $\beta := \alpha^p$, $a(t) := y(\log_\beta(t))$ and obtain a continuous multiplicatively parametrized group. Thus (a) follows by 1.6.

Now, Hofmann's splitting theorem mentioned in 1.7 yields (again by reparametrization) the existence of $(a_c(t)) \subseteq \mathrm{C}(\mathrm{Inv}(\mu_\cdot), \mathrm{N}(\mathrm{Inv}(\mu_\cdot)))$ such that $a_c(t) = a(t)\cdot\kappa(t)$ for $t > 0$, with $(\kappa(t)) \subseteq \mathrm{Inv}(\mu_\cdot)$. Consequently (b) follows. ∎

Next we investigate the shape of SEXP_α and $\mathrm{SEXP}_{c,\alpha}$. (For the structure of the Lie algebras of $\mathrm{N}(K, H)$ and $\mathrm{C}(K, H)$ see [8], Corollary B.) To simplify the notation we shall identify $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\mathfrak{G})$ as well as $\mathrm{Der}(G)$ and $\mathrm{Der}(\mathfrak{G})$: let $E \in \mathrm{Der}(\mathfrak{G})$. Then t^E means an element $t^E = \exp((\log t)E) \in \mathrm{Aut}(\mathfrak{G})$, and, by abuse of language, we also write t^E for the corresponding element in $\mathrm{Aut}(G)$ (since $a \mapsto a^\circ$ is a Lie group isomorphism). Hence, with this notation, multiplicative one-parameter groups $(a(t))_{t>0} \subseteq \mathrm{Aut}(G)$ have the representation $a(t) = t^E$, $t > 0$, for $E \in \mathrm{Der}(G)$.

In the following, E and E_c will denote elements of $\mathrm{SEXP}_\alpha(\mu_\cdot)$ and $\mathrm{SEXP}_{c,\alpha}(\mu_\cdot)$, respectively, and T and T_c elements of $\mathfrak{inv}$ and $\mathfrak{inv}_c$. Obviously, $\alpha^E \mathrm{Inv}(\mu_\cdot) \subseteq \mathrm{Dec}_\alpha(\mu_\cdot)$. Hence, in particular, $\alpha^E t^T \in \mathrm{Dec}_\alpha(\mu_\cdot)$ for all $t \geqslant 0$. On the other hand, we observe that the following proposition holds.

2.5. Proposition. *We have* $\alpha^{E+T} \in \mathrm{Dec}_\alpha(\mu_\cdot)$, *hence* $E + \mathfrak{inv} \subseteq \mathrm{SEXP}_\alpha(\mu_\cdot)$.

Proof. Let $t > 0$. We have

$$t^{E+T} = \lim_{n\to\infty} (t^{(1/n)E}\, t^{(1/n)T})^n = \lim (t^{(1/n)E}\, t^{(1/n)T})(t^{-(1/n)E}\, t^{(2/n)E} \ldots t^E\, t^{(1/n)T})$$

$$= \lim i_{t^{(1/n)E}}(t^{(1/n)T}) \ldots i_{t^{(1-1/n)E}}(t^{(1/n)T})\, t^E\, t^{(1/n)T},$$

where i_X denotes the inner automorphism $i_X\colon Y \mapsto XYX^{-1}$. Therefore, for $t = \alpha$ and $s \geqslant 0$ we obtain

$$\alpha^{E+T}(\mu_s) = \lim \left(i_{\alpha^{(1/n)E}}(\alpha^{(1/n)T}) \ldots i_{\alpha^{(1-1/n)E}}(\alpha^{(1/n)T})\, \alpha^E\, \alpha^{(1/n)T}\right)(\mu_s) = \mu_{\alpha\cdot s},$$

since $t^T \in \mathrm{Inv}(\mu_\cdot)$ and $t^E \in \mathrm{N}(\mathrm{Inv}(\mu_\cdot))$ for $t > 0$. Hence $E + T \in \mathrm{SEXP}_\alpha(\mu_\cdot)$, as asserted. ∎

2.6. Proposition. *We have*

$$\mathrm{SEXP}_\alpha(\mu_\cdot) = \mathrm{SEXP}_{c,\alpha}(\mu_\cdot) + \mathfrak{inv}. \tag{2.6}$$

Proof. According to Theorem 2.4 it follows that (with $a(t) = t^E$, $a_c(t) = t^{E_c}$, $\kappa(t) = t^T$) for any exponent $E \in \mathrm{SEXP}_\alpha(\mu_\cdot)$ there exist $E_c \in \mathrm{SEXP}_{c,\alpha}(\mu_\cdot)$ and $T \in \mathfrak{inv}$ such that $t^E = t^{E_c} t^{-T}$, $t > 0$. Since t^{E_c} centralizes $\mathrm{Inv}(\mu_\cdot)$, we conclude that $t^{E_c} t^{-T} = t^{E_c - T}$. Thus the part "$\subseteq$" follows.

Conversely, let $E_c \in \mathrm{SEXP}_{c,\alpha}(\mu_\cdot)$ and $T \in \mathfrak{inv}$. Then, since (t^{E_c}) and (s^T) commute, we have

$$\alpha^{E_c + T}(\mu_s) = (\alpha^{E_c} \alpha^T)(\mu_s) = \alpha^{E_c}(\mu_s) = \mu_{s \cdot \alpha}.$$

Consequently, $E_c + \mathfrak{inv} \subseteq \mathrm{SEXP}_\alpha(\mu_\cdot)$ follows. ∎

2.7. Proposition. *Let $E = E_c + T$ be as in 2.6. Then (t^E) and (s^{E_c}) commute.*

Proof. We have $s^{E_c} t^E = s^{E_c} t^{E_c + T}$, which (as in the proof of Proposition 2.6) equals $s^{E_c} t^{E_c} t^T = (st)^{E_c} t^T$. Since E_c is a commuting exponent, we have $(st)^{E_c} t^T = t^T (ts)^{E_c} = t^E s^{E_c}$. ∎

Therefore, by analogy with (2.5) we obtain

2.8. Theorem. *We have*

$$\mathrm{SEXP}_{c,\alpha}(\mu_\cdot) = E_c + \mathfrak{inv}_c \tag{2.7}$$

for a particular commuting exponent $E_c \in \mathrm{SEXP}_{c,\alpha}(\mu_\cdot)$.

Proof. Let $E_c, F_c \in \mathrm{SEXP}_{c,\alpha}(\mu_\cdot)$. According to 2.6, we have $E_c - F_c =: T \in \mathfrak{inv}$. Since F_c is commuting, $t^{F_c} t^T = t^{F_c + T} = t^{E_c}$. On the other hand, E_c is commuting. Hence for any $\kappa \in \mathrm{Inv}(\mu_\cdot)$ we have $t^{E_c}\kappa = \kappa t^{E_c}$. Thus $t^{F_c} t^T \kappa = \kappa t^{F_c} t^T = t^{F_c} \kappa t^T$. Therefore, $\kappa t^T = t^T \kappa$ for $t > 0$, and hence $T \in \mathfrak{inv}_c$ follows. ∎

Now we have the means to improve Proposition 2.5 and to obtain the analogue of (2.4):

2.9. Theorem. *We have*

$$\mathrm{SEXP}_\alpha(\mu_\cdot) = E + \mathfrak{inv} \tag{2.8}$$

for a particular exponent $E \in \mathrm{SEXP}_\alpha(\mu_\cdot)$.

Proof. According to 2.6 and 2.7 we have

$$\mathrm{SEXP}_\alpha(\mu_\cdot) = \mathrm{SEXP}_{c,\alpha}(\mu_\cdot) + \mathfrak{inv} \quad \text{and} \quad \mathrm{SEXP}_{c,\alpha}(\mu_\cdot) = E_c + \mathfrak{inv}_c,$$

whence $\mathrm{SEXP}_\alpha(\mu_\cdot) = E_c + \mathfrak{inv}_c + \mathfrak{inv} = E_c + \mathfrak{inv}$ for any commuting exponent $E_c \in \mathrm{SEXP}_{c,\alpha}(\mu_\cdot)$. Therefore, for $E, F \in \mathrm{SEXP}_\alpha(\mu_\cdot)$ we have $E - E_c =: T \in \mathfrak{inv}$ and $F - E_c = S \in \mathfrak{inv}$. Thus $E - F \in \mathfrak{inv}$ follows. ∎

2.10. Remark. *Affine normalizations.* As mentioned above, stability properties of probabilities on vector spaces are usually described in terms of affine transformations instead of linear transformations. (See e.g. [16] and [21] for a survey of the literature, see also [9].) Since the groups we have in mind will in general be non-Abelian, we preferred to simplify the notation and to restrict the considerations to the case of normalization by automorphisms. But it is not hard to extend the considerations above to the more general set-up.

For example, $\mathrm{Dec}(\mu_.)$ has to be replaced then by $\{a \in \mathrm{Aut}(G)\colon \exists \alpha > 0$, a function $t \mapsto b_t \in G$ such that $a(\mu_t) = \mu_{\alpha \cdot t} \star \varepsilon_{b_t} = \varepsilon_{b_t} \star \mu_{\alpha \cdot t}\}$, and fullness has to be replaced by S-fullness, equivalently, by compactness of the *symmetry group* $\mathrm{Sym}(\mu_.) = \{a \in \mathrm{Aut}(G)\colon \exists b \in G\colon a(\mu) = \mu \star \varepsilon_b\}$. (See e.g. the discussion in [8].)

It is natural to assume that the shift terms ε_b commute with μ, i.e., $\mu \star \varepsilon_b = \varepsilon_b \star \mu$. This is trivially fulfilled for probabilities on vector spaces. But also in the group case – under the S-fullness assumption – this is not a serious restriction. As the convergence of types theorem easily shows, it is equivalent to the assumption that $b \in \mathrm{Cent}(G)$.

The applications mentioned in Section 2 generalize almost verbatim to the case of affine normalizations. We omit the details.

3. SOME FURTHER APPLICATIONS AND ILLUSTRATIONS

A. Canonical representations of semistable Lévy measures. Let $\mu_.$ denote a full (a, α)-semistable continuous convolution semigroup with Lévy measure

$$\eta = \lim_{t \to 0} t^{-1} \cdot \mu_t|_{G \setminus \{e\}},$$

an unbounded non-negative measure which is bounded outside any neighbourhood of e and fulfils the semistability relation

$$\alpha \cdot \eta = a(\eta). \tag{3.1}$$

Note that a is contractive since $\mu_.$ is full.

A Borel set L is called a *cross-section* with respect to the action of the discrete group $(a^k\colon k \in \mathbb{Z})$ if $G \setminus \{e\} = \bigcup_{k \in \mathbb{Z}} a^k(L)$ (disjoint union). Define $\tau := \eta|_L$. Then we obtain the representation

$$\eta = \sum_{k \in \mathbb{Z}} \alpha^{-k} \cdot a^k(\tau).$$

Without loss of generality we may assume that $a = \alpha^E$ for some exponent $E \in \mathrm{SEXP}_\alpha(\mu_.)$ (with the notation described above). Our aim is to prove the existence of a "natural" cross-section L which is independent of the particular choice of the exponent E. The proof follows by a sequence of steps 3.1–3.3 which are of independent interest.

We consider first the case $G = V$, a finite-dimensional vector space. Let $\mu_.$ be (a, α)-semistable with compact invariance group $\mathrm{Inv}(\mu)$. Fix $E \in \mathrm{SEXP}_\alpha(\mu_.)$.

3.1. PROPOSITION. *There exists a Euclidean norm* $\|\cdot\|$ *on* V *satisfying*:
(a) $\mathcal{O}(V) \supseteq \mathrm{Inv}(\mu_.)$, *whence* $\{t^U\colon U \in \mathfrak{inv},\ t > 0\} \subseteq \mathcal{O}(V)$;
(b) $t \mapsto \|t^E \kappa\|$ *is strictly increasing for all* $\kappa \neq 0$;

(c) *the sphere* $K := \{x: \|x\| = 1\}$ *is a (compact) cross-section for the action of the continuous group* $(t^E)_{t>0}$.

Proof. Since $\mathrm{Inv}(\mu_.)$ is a compact subgroup of $\mathrm{GL}(V)$, we may assume without loss of generality that $\mathrm{Inv}(\mu_.)$ consists of isometries. Hence (a) follows.
For (b) and (c) see e.g. the construction in [16], 3.4.3, [21], 6.1.15, or [9], 1.1.15–1.1.17. ■

3.2. Proposition. *Let* K, E, *and* $\|\cdot\|$ *be as in* 3.1. *Then for any* $F \in \mathrm{SEXP}_\alpha(\mu_.)$ *we have*:

(a) $\|t^E \kappa\| = \|t^F \kappa\|$ *for all* $t > 0$, $\kappa \in V$.

(b) *In particular,* K *is a cross-section for the action of the continuous group* (t^F) *for any exponent* $F \in \mathrm{SEXP}_\alpha$ *and* $t \mapsto \|t^F \kappa\|$ *is strictly increasing for all* $\kappa \neq 0$ *and all* $E, F \in \mathrm{SEXP}_\alpha$.

Proof. 1. Assume first that $EF = FE$. Then we observe that $\|t^E \kappa\| = \|(t^{E-F}) t^F \kappa\| = \|t^F \kappa\|$ since $E - F \in \mathrm{inv}$ according to Proposition 2.9, and hence t^{E-F} is isometric.

2. Let $F \in \mathrm{SEXP}_\alpha(\mu_.)$ and let E_c be a commuting exponent, $E_c \in \mathrm{SEXP}_{c,\alpha}(\mu_.)$. Applying step 1 to both (E, E_c) and (E_c, F), we obtain $\|t^E \kappa\| = \|t^{E_c} \kappa\| = \|t^F \kappa\|$ for all $t > 0$, $\kappa \in V$. Consequently we get the assertion. ■

3.3. Proposition. *Let* a *and* K *be as above. For* $E \in \mathrm{SEXP}_\alpha(\mu_.)$ *define*

$$L^E := \{t^E \kappa: \alpha \leqslant t < 1, \kappa \in K\}.$$

Then

(a) L^E *is a cross-section for the action of the discrete group* $(a^k: k \in Z)$ *and*

(b) L^E *is independent of the particular choice of the exponent* E, *i.e.* $L^E = L^F =: L$ *for all* $E, F \in \mathrm{SEXP}_\alpha(\mu_.)$.

Proof. Obviously, $a^k(L^E) = \alpha^{k \cdot E} = \{t^E \kappa: \kappa \in K, \alpha^{k+1} \leqslant t < \alpha^k\}$. Hence, by Proposition 3.1(b), $V^\times = V \backslash \{0\} = \bigcup \alpha^{k \cdot E}(L^E)$ is a disjoint union. Let $F \in \mathrm{SEXP}_\alpha(\mu_.)$ and assume again first that $EF = FE$. Then

$$L^F = \bigcup (\alpha^{k \cdot E}(L^E) \cap L^F).$$

Assume that $\alpha^{k \cdot E}(L^E) \cap L^F \neq \emptyset$. Hence for some $\kappa, \kappa_1 \in K$ and $\alpha^{k+1} \leqslant s < \alpha^k$, $\alpha \leqslant t < 1$ we have

$$s^F \kappa_1 = t^E \kappa, \text{ and hence } \kappa_1 = t^{E-F} (t/s)^F \kappa. \tag{3.2}$$

Considering norms on both sides, by Proposition 3.1(b) and the relation $t^{E-F} \in \mathcal{O}(V)$ we obtain $s = t$, and hence $k = 0$. Furthermore, we have $t^F \kappa_1 = t^E (t^{F-E} \kappa) \in L^E$ since $t^{F-E} \kappa \in K$. Consequently, $L^F \subseteq L^E$ follows, and, by symmetry, $L^E = L^F$. In general, the assertion immediately follows when considering again (E, E_c) and (E_c, F). ■

Putting things together we obtain:

3.4. Theorem. *Let $\mu_\cdot$ be a full (a, α)-semistable continuous convolution semigroup on a vector space or on a simply connected nilpotent Lie group G with Lévy measure η. Assume without loss of generality that $a = \alpha^E$ for some exponent $E \in \mathrm{SEXP}_\alpha(\mu_\cdot)$.*

(a) *There exists a cross-section L for the discrete action $(a^k = \alpha^{k \cdot E})_{k \in \mathbf{Z}}$ which is independent of the particular choice of E. In fact, $L = \{t^E \kappa\colon \alpha \leqslant t < 1, \kappa \in K\}$, where K is a compact cross-section with respect to (t^E) for all exponents $E \in \mathrm{SEXP}_\alpha(\mu_\cdot)$.*

(b) *Let $\tau = \eta|_L$. Then*

$$\eta = \sum_{k \in \mathbf{Z}} \alpha^{-k} \cdot t^{k \cdot E}(\tau). \tag{3.3}$$

This representation is independent of the particular choice of E. Conversely, let τ be a bounded positive measure concentrated on L. Then η defined by (3.3) is an (a, α)-semistable Lévy measure.

Proof. For vector spaces the assertions are proved in 3.1–3.3.

Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{G}$. We apply the "translation procedure" developed in [9], Chapter II, especially 2.1, I ff. Therefore, the exponential map $\exp\colon \mathfrak{G} \to G$ is a topological isomorphism. $\eta^\circ := \exp^{-1}(\eta)$ is an (a°, α)-semistable Lévy measure on the tangent space $\mathfrak{G}$. Let $K^\circ, L^\circ, \tau^\circ, \eta^\circ$ denote the corresponding objects on the tangent space. According to step 1 the assertions (a) and (b) hold for $K^\circ, L^\circ, \tau^\circ, \eta^\circ$ on $\mathfrak{G}$. Applying the exponential map exp yields the corresponding assertions for K, L, τ, η on the group G. ∎

Remark. Let $\mu_\cdot$ be a full *stable* continuous convolution semigroup on a vector space V with Lévy measure η. Furthermore, let E and K denote an exponent and a compact cross-section for $(t^E)_{t>0}$, respectively. Then η admits a desintegration $\eta = \int_K \eta_\kappa \, d\sigma(\kappa)$, where η_κ is a (t^E)-stable Lévy measure concentrated on the orbit $(t^E \kappa\colon t > 0)$. In fact, we have $\eta_\kappa = \int_{(0,\infty)} t^{-2} \cdot \varepsilon_{t^E \kappa} \, dt$. (Cf. e.g. [9], 1.4.5, [21], 7.2.5.) If K is chosen independently of the particular exponent E, the existence of commuting exponents shows that the mixing measure σ is also independent of E. (Cf. e.g. [9], 1.4.11, 1.4.16 and 1.8.13 for vector spaces V, and 2.8.12 for groups G.)

In view of our previous considerations, for full (a, α)-*semi*-stable $\mu_\cdot$ we obtain an analogous desintegration of the measure τ, and hence of the Lévy measure $\eta = \int_K \eta_\kappa^E \, d\sigma^E(\kappa)$ with Lévy measures η_κ concentrated on the orbits $(t^E \kappa)$ and with semistable exponent E. (Cf. e.g. [9], 1.8.14–1.8.17 for V, and 2.8.16 for G).

Furthermore, K may be chosen independently of the exponent, and commuting exponents exist. But it is still an open problem whether the mixing measure is independent of the exponent also in the semistable case. In fact, the desintegration formulas differ in an essential point: For fixed $\kappa \in K$ in the stable case, as mentioned above,

$$\eta_\kappa = \int_{(0,\infty)} t^{-2} \cdot \varepsilon_{tE_\kappa} dt$$

is — up to normalization — uniquely determined by E and κ, whereas in the semistable case the set of orbital Lévy measures is — again up to normalization — isomorphic to $\mathscr{M}^1([\alpha, 1))$ in view of the representation

$$\eta_\kappa = \sum_{k \in \mathbf{Z}} \alpha^{-k} \cdot \alpha^{k \cdot E}\Big(\int_\alpha^1 \varepsilon_{tE_\kappa} d\varrho(t)\Big) \quad \text{for some } \varrho = \varrho_\kappa^E \in \mathscr{M}^1([\alpha, 1)).$$

(Cf. [9], 1.8.15, 1.8.16, or [21], 7.1.14 for vector spaces V; a group version can be found in [9], 2.8.16.)

B. Semistable convolution hemigroups. Continuous convolution semigroups in $\mathscr{M}^1(G)$ are distributions of *stationary independent increment processes* (Lévy processes) taking values in a group G. If stationarity is not assumed, the distributions of the corresponding *additive processes* are convolution *hemigroups*:

3.5. Definition. A family $(\mu_{s,t})_{0<s\leqslant t} \subseteq \mathscr{M}^1(G)$ is called a *continuous convolution hemigroup* if $\mu_{s,s} = \varepsilon_e$, $s \geqslant 0$, $\mu_{s,t} \star \mu_{t,r} = \mu_{s,r}$, $0 < s \leqslant t \leqslant r$, and if $(s, t) \mapsto \mu_{s,t}$ is continuous. We always assume that $\mu_{s,t} \neq \varepsilon_x$ for $x \in G$, for all $s < t$. Put $\mathscr{H} := \{\mu_{s,t}\colon 0 \leqslant s \leqslant t\}$.

As for convolution semigroups we define

$$\operatorname{Dec}(\mathscr{H}) := \{a \in \operatorname{Aut}(G);\ \exists \alpha > 0\colon a(\mu_{s,t}) = \mu_{\alpha s, \alpha t} \text{ for all } s < t\}$$

and

$$\operatorname{Inv}(\mathscr{H}) := \{a \in \operatorname{Aut}(G)\colon a(\mu_{s,t}) = \mu_{s,t} \text{ for all } s \leqslant t\},$$

and define the canonical homomorphism $\psi\colon \operatorname{Dec}(\mathscr{H}) \to \mathbf{R}_+^\times$ by $\psi(a) = \alpha$ iff $a(\mu_{s,t}) = \mu_{\alpha s, \alpha t}$ for all $s \leqslant t$.

Remark. Hemigroups $\mathscr{H}$ are defined mostly for the time parameters $s \leqslant t$ belonging to an interval, e.g. to $(0, 1]$. But if ψ is not trivial, i.e. if there exists $\alpha < 1$ with $\psi(a) = \alpha$, then $\mathscr{H}$ may be canonically extended to $0 < s \leqslant t$, $s, t \in \mathbf{R}_+^\times$, defining

$$\mu_{\alpha^k \cdot s, \alpha^k \cdot t} := a^k(\mu_{s,t}) \quad \text{for } 0 < s \leqslant t \leqslant 1 \text{ and } k \in \mathbf{Z}.$$

We call $\mathscr{H}$ *full* if $\mu_{s,t}$ is full for all $s < t$. In this case, $\operatorname{Inv}(\mu_{s,t})$, and hence $\operatorname{Inv}(\mathscr{H}) := \bigcap_{s<t} \operatorname{Inv}(\mu_{s,t})$ are compact subgroups. A convolution hemigroup $\mathscr{H} = (\mu_{s,t})$ is called *semistable* if $\operatorname{Dec}(\mathscr{H}) \setminus \operatorname{Inv}(\mathscr{H}) \neq \emptyset$, and $\mathscr{H}$ is called *stable* if there exists a continuous one-parameter group $(a(t))_{t>0} \subseteq \operatorname{Dec}(\mathscr{H})$ such that $\psi(a(t)) = t$, $t > 0$. In this case $a^\circ(t) = t^E$ for some $E \in \operatorname{Der}(\mathfrak{G})$ — we adopt again the notation $a(t) = t^E$ — and E is called an *exponent* of (the stable hemigroup) $\mathscr{H}$. By analogy with convolution semigroups we use the notation $\mathrm{EXP}(\mathscr{H}) := \{E\colon E \text{ is an exponent of } \mathscr{H}\}$.

For the structure of the decomposability group $\mathrm{Dec}(\mathscr{H})$ see e.g. [7], for vector spaces cf. [1] and [2]. If $\mathscr{H}$ is full, we observe that $\mathrm{Inv}(\mathscr{H}) = \ker(\psi)$ is a compact normal subgroup of $\mathrm{Dec}(\mathscr{H})$. Therefore, in particular, as in the case of convolution semigroups, for *stable hemigroups* we obtain

$$(3.4) \qquad \mathrm{EXP}(\mathscr{H}) = E + \mathfrak{inv}$$

for a particular exponent E ($\mathfrak{inv}$ denotes again the Lie algebra of $\mathrm{Inv}(\mathscr{H})$). Furthermore, there exist *commuting exponents* E_c, i.e. $E_c \in \mathrm{EXP}(\mathscr{H})$, such that $(t^{E_c}) \subseteq C(\mathrm{Inv}(\mathscr{H}), \mathrm{Dec}(\mathscr{H}))$. Let $\mathrm{EXP}_c(\mathscr{H})$ denote the set of commuting exponents.

In the following theorem, E is called again a *semistable* (*hemigroup*) *exponent*, and $\mathrm{SEXP}_\beta(\mathscr{H})$ will denote the set of semistable exponents for $\beta := \alpha^p$. Moreover, commuting exponents $E_c \in \mathrm{SEXP}_\beta(\mathscr{H})$ are called *commuting semistable exponents.*

3.6. THEOREM. *Let $\mathscr{H}$ be full and semistable, $a \in \mathrm{Dec}(\mathscr{H})$ with $\psi(a) = \alpha$.*

(a) *There exist $p \in N$ and a one-parameter group $(a(t) = t^E)_{t>0} \subseteq N(\mathrm{Inv}(\mathscr{H}), \mathrm{Aut}(G))$ such that $a(\alpha^p) = \alpha^{p \cdot E} = a^p$.*

(b) *There exist commuting semistable exponents $E_c \in \mathrm{SEXP}_\beta(\mathscr{H})$ such that*

$$(t^{E_c}) \subseteq C(\mathrm{Inv}(\mathscr{H}), \mathrm{Aut}(G)) \quad \text{with } \alpha^{p \cdot E_c} = a^p \in \mathrm{Dec}(\mathscr{H}) \text{ and } \psi(\alpha^{p \cdot E_c}) = \alpha^p.$$

(c) *Furthermore, the structures of $\mathrm{SEXP}_\beta(\mathscr{H})$ and $\mathrm{SEXP}_{c,\beta}(\mathscr{H})$ obtained for convolution semigroups in Propositions 2.5 and 2.6 generalize to the hemigroup case.*

The proof of Theorem 3.6 is an almost verbatim repetition of that of Theorem 2.4. ∎

We shall continue the investigations of stable hemigroups in Section 4.

C. Semi-self-similar processes. As mentioned above, stable continuous convolution semigroups correspond to stable G-valued stationary *independent increment* processes. If this condition is not fulfilled, we obtain (stationary) self-similar processes, i.e. G-valued stationary stochastic processes $(X_s)_{s \geq 0}$ fulfilling the self-similarity condition (*equality of distributions of finite-dimensional marginals*):

$$(3.5) \qquad a(t)(X_s) \overset{\mathscr{D}}{=} X_{s \cdot t} \quad \text{for } s \geq 0, \text{ and for all } t > 0.$$

In fact, G is C^∞-isomorphic to the tangent space $\mathfrak{G} \simeq R^d$, and now — as independence of *increments* is not supposed — the algebraic structure of the state space is not involved. Hence it is sufficient to define self-similarity for R^d-valued processes. All processes are assumed to be continuous in distribution.

3.7. DEFINITION. (a) A process $X_{\bullet} = (X_t)_{t \geq 0}$ taking values in $\boldsymbol{R}^d$ is called a *strictly operator self-similar* process if $X_0 \equiv 0$ a.e., and if for a one-parameter group $(a(t) = t^E)_{t>0} \subseteq \mathrm{GL}(\boldsymbol{R}^d)$ for all $t > 0$ we have $t^E(X_s) \stackrel{\mathscr{D}}{=} X_{s \cdot t}$, $s \geq 0$ *(equality of finite-dimensional marginals). E* is called an *exponent* of $X_{\bullet}$, and $\mathrm{EXP}(X_{\bullet})$ denotes the set of exponents.

(b) Let $X_{\bullet}$ be a process in $\boldsymbol{R}^d$. Then the *decomposability and invariance groups* are defined as follows:

$$\mathrm{Dec}(X_{\bullet}) := \{a \in \mathrm{GL}(\boldsymbol{R}^d)\colon\ a(X_t) \stackrel{\mathscr{D}}{=} X_{\alpha \cdot t} \text{ for some } \alpha > 0 \text{ and all } t \geq 0\},$$

$$\mathrm{Inv}(X_{\bullet}) := \{a \in \mathrm{GL}(\boldsymbol{R}^d)\colon\ a(X_t) \stackrel{\mathscr{D}}{=} X_t\},$$

and the map

$$\varphi\colon \mathrm{Dec}(X_{\bullet}) \to \boldsymbol{R}_+^{\times}, \quad \varphi(a) := \alpha,$$

is called again the *canonical homomorphism.*

(c) $X_{\bullet}$ is called a *strictly operator semi-self-similar* process if

$$\mathrm{Dec}(X_{\bullet}) \backslash \mathrm{Inv}(X_{\bullet}) \neq \varnothing.$$

As before we obtain:

3.8. PROPOSITION. *Let $\varphi(a) = \alpha$ for $a \in \mathrm{Dec}(X_{\bullet})$. Then for some $p \in \boldsymbol{N}$ there exists a "semi-exponent" $E \in \mathrm{End}(\boldsymbol{R}^d)$ such that $a^p = \alpha^{p \cdot E}$ and $(t^E)_{t>0} \subseteq \mathrm{N}(\mathrm{Inv}(X_{\bullet}))$. Furthermore, there exist commuting exponents E_c, i.e. exponents E_c such that (t^{E_c}) and $\mathrm{Inv}(X_{\bullet})$ commute elementwise.*

For investigations of operator self-similar processes the reader is referred e.g. to [12] or to more recently published papers [19], [20], [18] and the literature mentioned therein.

4. COMMUTING EXPONENTS OF SELF-DECOMPOSABLE LAWS

In the following we shall restrict our considerations to finite-dimensional vector spaces $G = V$, since self-decomposability on groups is not yet sufficiently investigated. For the vector space case see, in particular, [16], Chapter 3; for groups see e.g. [9], §2.14.

$\mu \in \mathscr{M}^1(G)$ is called an *operator self-decomposable* measure if there exists a one-parameter group $(a(t) = \exp(-t \cdot E))_{t \in \boldsymbol{R}} \subseteq \mathrm{GL}(\boldsymbol{R}, d)$ – here and in the sequel we use an *additive* parametrization – such that for all $t \geq 0$:

$$\mu = a(t)(\mu) * \nu(t) \tag{4.1}$$

for some measure $\nu(t) \in \mathscr{M}^1(V)$, called a *cofactor*. In this case, we see immediately that the cofactors define a stable hemigroup. In fact, the hemigroup

$$\mathscr{H} := \{\mu_{s,t} := a(s)(\nu(t-s)),\ 0 \leq s \leq t\} \tag{4.2}$$

has the stability property

(4.3) $$a(r)(\mu_{s,t}) = \mu_{s+r,t+r} \quad \text{for } 0 \leqslant s \leqslant t,\ r \geqslant 0.$$

Hence by reparametrization $\lambda_{u,v} := \mu_{-\log(v),-\log(u)}$, $0 < u \leqslant v \leqslant 1$, we obtain a hemigroup which is stable with respect to $\big(b(t) := a(-\log(t)) = t^E\big)_{t>0}$. We shall always assume that $a(\cdot)$ is contractive; equivalently, $\Re(\alpha) > 0$ for $\alpha \in \operatorname{Spec}(E)$. For these exponents we observe immediately that

(4.4) $$\lim_{t \mapsto \infty} \mu_{0,t} = \mu.$$

Conversely, let $\mathscr{H}$ be a stable hemigroup (as in Section 3) such that (4.4) holds. Then μ is an operator self-decomposable measure. But note that the hemigroup $\mathscr{H}$ is not uniquely determined by the limit measure μ.

We assume that the measures $\mu_{s,t}$ and μ are S-full. Hence the following objects are compact subgroups:

$$I_{s,t} := \operatorname{Inv}(\mu_{s,t}), \quad K_0(\mathscr{H}) := \operatorname{Inv}(\mathscr{H}) = \bigcap_{s<t} I_{s,t}, \quad K_1(\mathscr{H}) := \bigcap_{0<t} I_{0,t},$$
$$K_2(\mathscr{H}) := \operatorname{Inv}(\mu).$$

Hence we define $E \in \operatorname{End}(\boldsymbol{R}, d)$ to be an *exponent of the self-decomposable law* μ if (4.1) holds with $a(t) = \exp(-t \cdot E)$, and let $\operatorname{EXP}_u(\mu)$ denote the set of those exponents.

Note that, according to 3.2, there exist *exponents of the hemigroup $\mathscr{H}$ commuting with $K_0(\mathscr{H})$*. However, the question if there exist exponents in $\operatorname{EXP}_u(\mu)$ commuting with $\operatorname{Inv}(\mu)(= K_2(\mathscr{H}))$ is not answered by the results of Section 3.

First we make the following observations:

(4.5) $$I_{r,r+t} = a(r)\, I_{0,t}\, a(-r)$$

and $\{\bigcup_{t>0} I_{0,t}\}$ is relatively compact in $\operatorname{GL}(V)$ with

(4.6) $$\operatorname*{LIM}_{t \to \infty} I_{0,t} \subseteq \operatorname{Inv}(\mu),$$

where LIM denotes the set of accumulation points.

In fact, (4.5) is an obvious consequence of (4.3), and (4.6) follows immediately by the convergence of types theorem. Hence in particular we observe that

(4.7) $$K_0(\mathscr{H}) \subseteq K_1(\mathscr{H}) \subseteq K_2(\mathscr{H}).$$

4.1. Proposition. $K_0(\mathscr{H}) = K_1(\mathscr{H})$.

Proof. Let $b \in K_1(\mathscr{H})$. Then

$$\mu_{0,t+r} = \mu_{0,r} * a(r)(\mu_{0,t}) = b(\mu_{0,t+r}) = b(\mu_{0,r}) * ba(r)(\mu_{0,t}) = \mu_{0,r} * ba(r)(\mu_{0,t}).$$

The cofactors are infinitely divisible, so the mapping $\varrho \mapsto \mu_{0,t} * \varrho$ is injective for any t. Therefore we conclude that $\mu_{r,r+t} = a(r)(\mu_{0,t}) = ba(r)(\mu_{0,t}) = b(\mu_{r,r+t})$, whence $b \in I_{r,r+t}$ follows for all $r, t \geqslant 0$, i.e. $b \in K_2(\mathscr{H})$. ∎

4.2. Remark. Let μ be a self-decomposable measure with exponent $E \in \mathrm{EXP}_u(\mu)$ and corresponding stable hemigroup $\mathscr{H} = \mathscr{H}^E = \{\mu_{s,t+s} = a(s)(\mu_{0,t}),\ s, t \geqslant 0\}$ with $a(s) = e^{-s \cdot E}$ and cofactors $\mu_{0,t}$ ($= \mu^E_{0,t}$, depending on E) fulfilling the condition $\mu = \lim_{t \mapsto \infty} \mu_{0,t}$. For $b \in \mathrm{Inv}(\mu)$ let us put $E^b := bEb^{-1}$. Hence

$$a^b(s) := ba(s)b^{-1}, \qquad \mathscr{H}^b := \{\mu^b_{s,t} := b(\mu_{s,t}) = a^b(s)\, b(\mu_{0,t-s})\}.$$

Then $\mathscr{H}^b$ is a stable hemigroup with exponent E^b fulfilling also the condition $\lim_{t \mapsto \infty} \mu^b_{0,t} = \mu$.

4.3. Proposition. *Assume that μ is an operator self-decomposable measure with commuting exponent E_c, i.e. $(\exp(-t \cdot E_c) =: a_c(t))_{t \geqslant 0}$ centralizing* $\mathrm{Inv}(\mu)$, *and with corresponding hemigroup $\mathscr{H}_c = \{\mu^c_{s,t}\}$. Then*

$$K_0(\mathscr{H}_c) = K_1(\mathscr{H}_c) = K_2(\mathscr{H}_c) = \mathrm{Inv}(\mu). \tag{4.8}$$

Proof. Let $b \in \mathrm{Inv}(\mu)$. Then $\mu = a_c(r)(\mu) * \mu^c_{0,r} = b(\mu) = ba_c(r)(\mu) * b(\mu^c_{0,r})$, which, by assumption, equals $a_c(r)\, b(\mu) * b(\mu^c_{0,r}) = a_c(r)(\mu) * b(\mu^c_{0,r})$. Consequently, again as in the proof of Proposition 4.1, $b \in \mathrm{Inv}(\mu^c_{0,t})$ follows. ∎

4.4. Remark. The considerations should be compared with Lemma 4 and Theorem 5 of [17]. Namely, Jurek [15] and Luczak [17] proved, by different methods, the existence of commuting exponents $E_c \in \mathrm{EXP}_u(\mu)$ of S-full operator self-decomposable measure μ, and in particular in [17] it is shown that there exist exponents such that $\exp(-t \cdot E_c)$ is contractive, and hence $\mu^c_{0,t} \to \mu$.

This result is similar to the investigations in our Sections 2 and 3. However, it cannot be proved by those methods: $\mathrm{Inv}(\mu)$ need not to be normalized by $\exp(-t \cdot E)$ for $E \in \mathrm{EXP}_u(\mu)$. (In contrast, $\exp(-t \cdot E)$ normalizes $\mathrm{Inv}(\mathscr{H}^E)$.) Thus, it is an open problem to find a proof depending only on the underlying group structures.

We illustrate the investigations by two examples:

4.5. Example. *Space-time processes* (cf. e.g. [9], §2.14, III, for more details.) Let $\mathscr{H} = \{\mu_{s,t}\colon s \leqslant t\}$ be a stable hemigroup with exponent E and $a(t) = e^{-t \cdot E}$, $t \geqslant 0$. Let $\boldsymbol{H} := V \rtimes \mathrm{GL}(V)$ (a non-Abelian Lie group). Define

$$\mathscr{M}_* := \{\lambda \otimes \varepsilon_a\colon \lambda \in \mathscr{M}^1(V),\ a \in \mathrm{GL}(V)\} \subseteq \mathscr{M}^1(\boldsymbol{H}).$$

Convolution on $\boldsymbol{H}$ yields, for $\beta_i := \lambda_i \otimes \varepsilon_{a_i}$, $i = 1, 2$,

$$\beta_1 * \beta_2 = (\lambda_1 \star a_1(\lambda_2)) \otimes \varepsilon_{a_1 a_2},$$

where $*$ and $\star$ denote convolutions on $\boldsymbol{H}$ and $\boldsymbol{G}$, respectively. Then it follows that

($\lambda_t := \mu_{0,t} \otimes \varepsilon_{a(t)}$: $t \geq 0$) is a continuous convolution semigroup in $\mathcal{M}_* \subseteq \mathcal{M}^1(H)$. As immediately seen, $\beta := \varepsilon_0 \otimes \varepsilon_b$ commutes with λ_t, i.e.

(4.9) $\qquad \beta * \lambda_t = \lambda_t * \beta \quad$ iff $\quad b \in \mathrm{Inv}(\mu_{0,t})$ and $ba(t) = a(t)b$.

Hence, let E be a commuting exponent of the hemigroup $\mathscr{H}$ with $a(t) = \exp(-t \cdot E)$, and corresponding space-time semigroup (λ_t). Then

(4.10) $\quad \mathrm{Inv}(\mathscr{H}) = \{b$: for $\beta = \varepsilon_0 \otimes \varepsilon_b \in \mathcal{M}_*$, $\beta * \lambda_t = \lambda_t * \beta$ for all $t\}$.

Furthermore, assume that μ is an operator self-decomposable and full measure with *commuting* exponent $E_c \in \mathrm{EXP}_u(\mu)$ and corresponding hemigroup $\mathscr{H}_c = \{\mu^c_{s,t}\}$ and such that $\mu^c_{0,t} \to \mu$. Then, by (4.8), for the corresponding space-time semigroup (λ^c_t) we obtain

(4.11) $\quad \mathrm{Inv}(\mathscr{H}_c) = \mathrm{Inv}(\mu)$

$$= \{b\text{: for } \beta := \varepsilon_0 \otimes \varepsilon_b,\ \beta * \lambda^c_t = \lambda^c_t * \beta \text{ for all } t \geq 0\}.$$

4.6. Example. *Self-decomposable Gaussian laws* (for details see e.g. [16], 3.3.6 ff.). Let $\mu := N_{0,I}$ be the standard Gaussian distribution (with covariance operator I). Let γ denote a symmetric Gaussian law with covariance operator S. Then the set of exponents is given by

$$\mathrm{EXP}_u(\gamma) = \{E \in \mathrm{End}(V)\text{: } ESE^* \geq 0 \text{ (positive semidefinite)}\}.$$

Furthermore, the decomposability semigroup is given by

$$D(\gamma) := \{a\text{: } \gamma = a(\gamma) * v(a) \text{ for a cofactor } v(a)\} = \{a\text{: } aSa^* \leq I\},$$

and we have $\mathrm{Inv}(\gamma) = \{a\text{: } aSa^* = S\}$. In particular, for $\gamma = \mu$ and $S = I$ we obtain $\mathrm{Inv}(\mu) = \mathcal{O}(V)$, the group of orthogonal transformations, and $D(\mu) = \{a\text{: } aa^* \leq I\}$.

Let $E \in \mathrm{EXP}_u(\mu)$. Then $\mu = e^{-t \cdot E}(\mu) * \mu_{0,t}$ yields

$$\exp\left(-\tfrac{1}{2}\left(\langle (I - e^{-t \cdot E} e^{-t \cdot E^*}) y, y \rangle\right)\right) = \hat{\mu}_{0,t}(y).$$

Hence $\mu_{0,t} = N_{0,A(t)}$ with $A(t) := I - e^{-t \cdot E} e^{-t \cdot E^*}$, or $\mu_{0,t} = B(t)(\mu)$ with $B(t) = A(t)^{1/2}$. If $e^{-t \cdot E}$ is contractive, then $A(t) \to I$ as $t \to \infty$. Hence $\mu_{0,t} \to \mu$. It follows that $\mu_{0,t} = B(t)(\mu)$ yields

$$\mathrm{Inv}(\mu_{0,t}) = I_{0,t} = B(t)\,\mathrm{Inv}(\mu)\,B(t)^{-1} \qquad \text{for } t > 0.$$

Thus, if for example we consider $E = E^*$ with one-dimensional eigenspaces and if $d = \dim(V) = 2k+1$, we obtain:

- $I_{0,t} = B(t)\,\mathcal{O}(V)\,B(t)^{-1}$.
- $\mathrm{LIM}_{t \to \infty} I_{0,t} = \mathcal{O}(V) (= \mathrm{Inv}(\mu))$.
- $I_{0,t} \neq I_{0,s}$, $s \neq t$; in fact, $I_{0,t} \cap I_{0,s} = \Delta$, the finite subgroup of $\mathcal{O}(V)$ with diagonal entries ± 1.

- $\mathrm{Inv}(\mathscr{H}) = \bigcap I_{0,t} = \Delta$, a proper (finite) subgroup of $\mathcal{O}(V) = \mathrm{Inv}(\mu)$.

- For semistable convolution *semi*-groups the invariance groups $\mathrm{Inv}(\mu_t)$ coincide for all $t > 0$. (See 2.2; as this example shows, this is not true for stable *hemi*-groups.)

- *Commuting exponents.* We have: $(e^{-t\cdot E}) \subseteq \mathrm{C}(\mathrm{Inv}(\mu), \mathrm{GL}(V))$ iff $\mu_{0,t} = N_{0,f(t)\cdot I}$ for some real function f, i.e., iff $I - e^{-t\cdot E} e^{-t\cdot E^*} = f(t)^2 \cdot I$; hence iff $E = c \cdot I$ for some positive c. In this case, obviously, the invariance groups $I_{0,t}$, $I_{s,t}$, $K_i(\mathscr{H})$, $i = 0, 1, 2, 3$, coincide (in accordance with Proposition 4.3).

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