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# SNELL'S OPTIMIZATION PROBLEM FOR SEQUENCES OF CONVEX COMPACT VALUED RANDOM SETS

BY

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*Abstract.* A random set analogue of the Snell problem is presented. In the original Snell's problem one observes a sequence of random variables $(\xi_n)$, say a gambler's capital at successive games. If the gambler leaves the game at a random time $v$, his expected capital at this time is $E\xi_v$. The objective is to stop at time $v$ (using information available up to this moment) such that the expected gambler's fortune $E\xi_v$ is maximal.

Here a multivalued analogue of this problem will be studied. Given a Banach space and a sequence of convex weakly or strongly compact valued random sets $(Z_n)$ in that space, the existence of a stopping time $v$ such that $EZ_v$ is maximal is investigated.

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## 1. PRELIMINARIES

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $(\mathscr{B}_n)$ be an increasing sequence of sub-$\sigma$-algebras of $\mathscr{A}$ such that $\mathscr{A}$ is the smallest $\sigma$-algebra containing the sequence $(\mathscr{B}_n)$. For any $n \in N$, let $\Lambda_n$ denote the family of all a.s. finite stopping times greater than $n$.

Let $X$ be a separable Banach space with the norm $\|\cdot\|$. By $B(0, r)$ we denote the closed ball in $X$, centered at the origin with radius $r$. Let $X^*$ denote the dual of $X$, and $\langle\cdot, \cdot\rangle$ the usual duality. The strong and weak topologies on $X$ will be denoted by $s$ and $w$, respectively. $B^*$ will denote the closed unit ball in $X^*$. For a set $A \subset X$, $\operatorname{cl}\operatorname{co} A$ will denote the closed convex hull of the set $A$. Let $\mathscr{P}(X)$ be the family of all closed subsets of $X$, $\mathscr{P}_{\text{wkc}}(X)$ and $\mathscr{P}_{\text{skc}}(X)$ will denote the family of all $w$-compact and, respectively, $s$-compact convex subsets of $X$.

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The *support function* of the set $C$ in $\mathscr{P}(X)$ will be defined in the following way:

$$s(x^*, C) := \sup_{x \in C} \langle x, x^* \rangle.$$

A sequence $(C_n)$ *scalarly converges* to some set $C$ if $s(x^*, C_n) \to s(x^*, C)$ for all $x^* \in X^*$. Topology of convergence of support functions will be denoted by $\mathscr{T}_{\text{scalar}}$. The *distance functional* is a mapping $d\colon X \times \mathscr{P}(X) \to \boldsymbol{R}$ such that

$$d(x, C) := \inf\{\|x-c\|\colon\ c \in C\}.$$

The *ε-envelope* of a set $C$ is defined as $C^\varepsilon := \{x \in X\colon\ d(x, C) \leqslant \varepsilon\}$. Define the *Hausdorff distance* between sets $C$ and $D$ as

$$\varrho_H(C, D) := \inf\{\varepsilon > 0\colon\ C \subset D^\varepsilon,\ D \subset C^\varepsilon\}.$$

On the space of closed sets, $\varrho_H$ is a metric. For a nonempty set $C$, we set also

$$\|C\| := \sup_{x \in C} \|x\|,$$

which is the Hausdorff distance of $C$ from $\{0\}$.

Let us now introduce a few topologies on the space of closed subsets of $X$. Given topology $\tau$ on $X$ the *lower τ-limit* of a sequence $(C_n) \subset X$ (denoted by $\tau\text{-Li}C_n$) is defined as the set of all $x \in X$ such that $x = \tau\text{-lim}_{n\to\infty} x_n$, where $x_n \in C_n$. The *upper τ-limit* of the sequence $(C_n)$ (denoted by $\tau\text{-Ls}\, C_n$) is the set of all $x \in X$ such that $x = \tau\text{-lim}_{k\to\infty} x_k$, where $x_k \in C_{n_k}$. We say that $(C_n) \subset \mathscr{P}(X)$ *Mosco converges* to $C \in \mathscr{P}(X)$ if

$$w\text{-Ls}_{n\to\infty} C_n \subset C \subset s\text{-Li}_{n\to\infty} C_n.$$

The Mosco convergence is an extension of the notion of the convergence in the Painlevé–Kuratowski sense (see [2]). Convergence with respect to the Hausdorff metric implies the Mosco convergence. In Euclidean spaces, the Mosco convergence and the Painlevé–Kuratowski convergence coincide. The Mosco convergence is not a topological notion. However, on the family of $w$-closed subsets of $X$, one may consider the *Mosco topology*, whose subbase consists of the families

$$\{V^-,\ V \text{ is } s\text{-open}\} \quad \text{and} \quad \{(K^c)^+,\ K \text{ is } w\text{-compact convex}\},$$

where $V^-$ is the collection of all closed sets which have a nonempty intersection with the set $V$ and $(K^c)^+$ is the collection of all closed sets contained in the complement of $K$ (for details see [2], Chapter 5). This topology will be denoted by $\mathscr{T}_{\text{Mosco}}$. In a slightly more general setting than the present one, a sequence of $w$-closed sets Mosco converges to a $w$-closed set if and only if it converges to that set in the Mosco topology ([2], Theorem 5.4.6).

## 2. RANDOM SETS AND AUXILIARY RESULTS ON MULTIVALUED MARTINGALES

A *multifunction* is any mapping $F\colon \Omega \to 2^X$. A multifunction $F$ is said to be (*Effros*) *measurable* if the preimage $F^- U = \{\omega \in \Omega\colon F(\omega) \cap U \neq \emptyset\}$ belongs to $\mathscr{A}$ for any $s$-open set $U \subset X$. Only measurable multivalued functions with closed values will be considered and the adjective will often be omitted. Measurable multivalued functions will be called *random sets*. This notion of measurability coincides with the usual measurability in the Borel sense when $\mathscr{P}(X)$ is equipped with the Effros $\sigma$-algebra generated by the families $G^-$, where $G$ are $s$-open sets.

$\mathscr{L}^1_{\boldsymbol{R}}$ will denote the family of real-valued integrable functions. Let $\mathscr{L}^1_{\mathscr{P}(X)}$ denote the space of all closed valued random sets $F$ such that $\|F\| \in \mathscr{L}^1_{\boldsymbol{R}}$. Those functions will be called *integrably bounded*. $\mathscr{L}^1_{\mathscr{P}_{\text{skc}}(X)}$ will denote the subspace of $\mathscr{P}_{\text{skc}}(X)$-valued random sets in $\mathscr{L}^1_{\mathscr{P}(X)}$. A set $\mathscr{H}$ in $\mathscr{L}^1_{\mathscr{P}(X)}$ is *bounded* if the set $\{\|F\|\colon F \in \mathscr{H}\}$ is bounded in $\mathscr{L}^1_{\boldsymbol{R}}$.

Let $\mathscr{B} \subset \mathscr{A}$ be a sub-$\sigma$-algebra of $\mathscr{A}$ and let $\mathscr{L}^1_X(\mathscr{B})$ denote the family of Bochner integrable $\mathscr{B}$-measurable functions. Denote by

$$\mathscr{L}^1_F(\mathscr{B}) := \{f \in \mathscr{L}^1_X(\mathscr{B})\colon f(\omega) \in F(\omega) \text{ a.e.}\}$$

the set of all Bochner integrable $\mathscr{B}$-measurable selections of $F$. In particular, we use the following shorthand notation: $\mathscr{L}^1_X := \mathscr{L}^1_X(\mathscr{A})$ and $\mathscr{L}^1_F := \mathscr{L}^1_F(\mathscr{A})$. The integral of a $\mathscr{P}(X)$-valued function is defined as

$$\int_\Omega F dP := \{\int_\Omega f dP\colon f \in \mathscr{L}^1_F\}.$$

The integral $\int_A F dP$ of $F$ over a measurable subset $A$ of $\Omega$ is the integral of $1_A F$, where $1_A$ is the indicator function of the set $A$. Note (Theorem 4.2 of [10]) that if $(\Omega, \mathscr{A}, P)$ has no atom and $F \in \mathscr{L}^1_X$, then $\operatorname{cl} \int_\Omega F\, dP$ is convex. If $F$ is a $\mathscr{P}(X)$-valued function with $\mathscr{L}^1_F \neq \emptyset$, then (by Theorem 5.1 of [10]) there exists an almost surely unique $\mathscr{B}$-measurable $\mathscr{P}(X)$-valued function $E(F \mid \mathscr{B})$ satisfying

$$\mathscr{L}^1_{E(F\mid\mathscr{B})}(\mathscr{B}) = \operatorname{cl}\{E(f \mid \mathscr{B})\colon f \in \mathscr{L}^1_F\},$$

where the closure is taken in $\mathscr{L}^1_X$. The function $E(F \mid \mathscr{B})$ will be called the *conditional expectation* of $F$ with respect to the $\sigma$-algebra $\mathscr{B}$. Corollary 1.6 of [10] yields that if $F \in \mathscr{L}^1_{\mathscr{P}_{\text{skc}}(X)}$, then $E(F \mid \mathscr{B}) \in \mathscr{L}^1_{\mathscr{P}_{\text{skc}}(X)}$. In particular, $EF := E(F \mid \mathscr{A}) = \operatorname{cl} \int_\Omega F(\omega)\, dP(\omega)$. A similar result is true if $F \in \mathscr{L}^1_{\mathscr{P}_{\text{wkc}}(X)}$ (see [5]).

We say that an adapted sequence of random sets $(F_n)$ is a *submartingale* if for all $n \in N$, $E(F_{n+1} \mid \mathscr{B}_n)(\omega) \supset F_n(\omega)$ a.s. It is a *supermartingale* if $E(F_{n+1} \mid \mathscr{B}_n)(\omega) \subset F_n(\omega)$ a.s. for all $n \in N$. Finally, it is a *martingale* if it is a submartingale and supermartingale.

The lemmas presented below are multivalued analogues of the well-known results for real-valued martingales (see, for example, [11], Chapter II).

The following definitions play an important role not only in the proofs of this section but also in the proofs of main results. Let $D^* \subset X^*$ be a symmetric Mackey dense countable subset of the closed unit ball $B^* \subset X^*$. Let $H^*$ denote the set of all rational combinations of elements from $D^*$. This set is dense in the Mackey topology. The existence of such a set follows from Lemma III.32 of [3].

LEMMA 2.1. *Suppose $(F_n)$ is a $\mathscr{P}(X)$ supermartingale majorized by a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued random set $K$. Then for any $n \in N$ and any stopping time $\nu \in \Lambda_n$, $(F_{\nu \wedge n})$ is a supermartingale which a.s. Mosco converges to $F_\nu$.*

Proof. Since

$$E(F_{\nu\wedge n} \mid \mathscr{B}_{n-1}) = \sum_{m<n} F_m 1_{[\nu=m]} + E(F_n \mid \mathscr{B}_{n-1}) 1_{[\nu \geqslant n]}$$

$$\subset \sum_{m<n} F_m 1_{[\nu=m]} + F_{n-1} 1_{[\nu \geqslant n]} = F_{\nu \wedge (n-1)},$$

$(F_{\nu\wedge n})$ is a supermartingale. It is dominated by an integrably bounded $w$-compact random set $K$. For any $x^* \in H^*$, $\lim_{n\to\infty} s(x^*, F_{\nu\wedge n}(\omega)) = s(x^*, F_\nu(\omega))$ a.s. Now we proceed as in the proof of Proposition 5.8 of [8]. Let $D$ be a countable dense subset of $X$. For any $x \in D$,

$$d(x, F_{\nu\wedge n}) = \sup_{x^* \in X^*} \big(\langle x^*, x\rangle - s(x^*, F_{\nu\wedge n})\big).$$

Lemma V-2-9 of [11] yields that

$$d\big(x, F_\nu(\omega)\big) = \lim_{n\to\infty} d\big(x, F_{\nu\wedge n}(\omega)\big) \text{ a.s.} \tag{1}$$

for all $x \in D$. Since $(F_n(\omega))$ is contained in $K(\omega)$, the sequence $\big(d(\cdot, F_n^k(\omega))\big)$ is equicontinuous. Therefore, (1) holds for all $x \in X$. Lemma 5.5 of [8] yields that $(F_{\nu\wedge n})$ a.s. Mosco converges to $F_\nu$ as $n \to \infty$. ∎

LEMMA 2.2. *Let $(F_n)$ be a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued supermartingale majorized by an integrably bounded $\mathscr{P}_{\mathrm{wkc}}(X)$-valued random set $K$. Then for any pair $\nu_1, \nu_2$ of stopping times*

$$E(F_{\nu_2} \mid \mathscr{B}_{\nu_1}) \subset F_{\nu_1} \text{ a.s.}$$

Proof. It is sufficient to show that $F_n \supset E(F_\nu \mid \mathscr{B}_n)$ on $\{\omega \in \Omega\colon \nu(\omega) \geqslant n\}$. The sequence $(F_{\nu\wedge n})$ is a supermartingale which a.s. Mosco converges to $F_\nu$ (see the proof of Lemma 2.1). Corollary 5.13 of [8] implies that

$$E(F_\nu \mid \mathscr{B}_n) \subset F_{\nu\wedge n} \text{ a.s.}$$

Thus $E(F_\nu \mid \mathscr{B}_n) \subset F_n$ a.s. on the event $\{\omega \in \Omega\colon \nu(\omega) \geqslant n\}$. ∎

LEMMA 2.3. *For an integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued random set $F$ majorized by a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued random set $K$, the martingale $\big(E(F \mid \mathscr{B}_n)\big)$ a.s. Mosco converges to $F$.*

Proof. For any $x^* \in H^*$, $\big(E(s(x^*, F) \mid \mathscr{B}_n)\big)$ is a supermartingale. By Proposition II-2-11 of [11], for all $x^* \in H^*$,

$$\lim_{n\to\infty} E\big(s(x^*, F) \mid \mathscr{B}_n\big)(\omega) = s\big(x^*, F(\omega)\big) \text{ a.s.}$$

Thus $\big(E(F \mid \mathscr{B}_n)(\omega)\big)$ a.s. scalarly converges to $F(\omega)$. Noticing (as in the proof of Lemma 2.1 or Proposition 5.8 of [8]) that $\big(d\big(x, E(F \mid \mathscr{B}_n)\big)\big)$ constitutes a submartingale for any $x \in D$, and applying Lemma V-2-9 of [11] we can show that $\big(E(F \mid \mathscr{B}_n)(\omega)\big)$ a.s. Mosco converges to $F(\omega)$. ∎

## 3. MAIN RESULTS

Let us first state the problem which is a set valued analogue of the original Snell problem (see [4], [11]). We observe a sequence (finite or infinite) of random sets $(Z_n)$ satisfying some additional conditions. Our objective is to stop observation at a random time $v_0$ such that the expectation $EZ_{v_0}$ is maximal (in the sense of inclusion). The decision when to stop can be based only on what we have seen up to this time. This stopping time is the solution to the Snell problem.

**3.1. The smallest envelope of a family of random sets.** Before we present solutions to the optimization problem we start with the following proposition about the existence and some properties of the smallest essential closed convex envelope of a family of random sets. It will play an important role in the sequel.

PROPOSITION 3.1. *For every family $\mathscr{G}$ of random sets majorized by a w--compact convex valued random set $K$ there exists an a.s. unique random set $G\colon \Omega \to \mathscr{P}_{\mathrm{wkc}}(X)$ such that*

(i) *$F(\omega) \subset G(\omega)$ a.s. for all $F \in \mathscr{G}$;*

(ii) *if $H$ is a convex valued random set majorizing all $F \in \mathscr{G}$ a.s., then $H$ majorizes $G$ a.s.*

*The random set $G$ will be denoted by* $\operatorname{ess\,cl\,co}\mathscr{G}$. *Moreover, there exists a sequence $(F_n) \subset \mathscr{G}$ such that*

$$\operatorname{ess\,cl\,co}\mathscr{G} = \operatorname{cl\,co}\bigcup_{n=1}^{\infty} F_n \ \text{a.s.}$$

*If the family $\mathscr{G}$ is directed, the sequence $(F_n)$ can be chosen to be increasing and*

$$\operatorname{ess\,cl\,co}\mathscr{G} = \mathscr{T}_{scalar}\text{-}\lim_{n\to\infty} F_n \ \text{a.s.} \quad \text{and} \quad \operatorname{ess\,cl\,co}\mathscr{G} = \mathscr{T}_{Mosco}\text{-}\lim_{n\to\infty} F_n \ \text{a.s.}$$

*Moreover, if the random set $K$ is s-compact valued, then*

$$\operatorname{ess\,cl\,co}\mathscr{G} = \mathscr{T}_{\varrho H}\text{-}\lim_{n\to\infty} F_n \ \text{a.s.}$$

Let us remark here that (i) and (ii) have appeared in [13], where the existence of the essential supremum has been studied.

**3.1.1.** Proof of Proposition 3.1. Let $\mathfrak{J}$ denote the class of all countable subfamilies of $\mathscr{G}$. For every $\mathscr{F}\in\mathfrak{J}$ define

$$F_{\mathscr{F}}(\omega):=\operatorname{cl}\operatorname{co}\bigcup_{F\in\mathscr{F}}F(\omega).$$

Consider now the set

$$\hat{F}:=\operatorname{cl}\operatorname{co}\bigcup_{\mathscr{F}\in\mathfrak{J}}EF_{\mathscr{F}}. \tag{2}$$

The closed convex hull of the union on the left-hand side is attained in the sense that there exists a family $\hat{\mathscr{F}}$ such that $\hat{F}=EF_{\hat{\mathscr{F}}}$. Indeed, (2) yields that for each $x^*\in H^*$ there exists a sequence $(\mathscr{F}_n^{x^*})$ such that

$$\lim_{n\to\infty}Es(x^*,F_{\mathscr{F}_n^{x^*}})=s(x^*,\hat{F}). \tag{3}$$

Let $\hat{\mathscr{F}}:=\bigcup_{n=1}^{\infty}\bigcup_{x^*\in H^*}\mathscr{F}_n^{x^*}$. Obviously, $\hat{\mathscr{F}}\in\mathfrak{J}$. Thus

$$Es(x^*,F_{\mathscr{F}_n^{x^*}})\leqslant Es(x^*,F_{\hat{\mathscr{F}}})=s(x^*,EF_{\hat{\mathscr{F}}})\leqslant s\big(x^*,\bigcup_{\mathscr{F}\in\mathfrak{J}}EF_{\mathscr{F}}\big)=s(x^*,\hat{F}).$$

By (3), the above inequality yields that for all $x^*\in H^*$, $s(x^*,EF_{\hat{\mathscr{F}}})=s(x^*,\hat{F})$. Thus, applying Lemma III.34 of [3], we obtain $\hat{F}=EF_{\hat{\mathscr{F}}}$.

It will be shown that the set $G:=F_{\hat{\mathscr{F}}}$ satisfies the assertions of the proposition. Take any $F\in\mathscr{G}$. The set $F_{\mathscr{F}}$ corresponding to the countable family $\mathscr{F}=\hat{\mathscr{F}}\cup\{F\}$ equals $\operatorname{cl}\operatorname{co}(F_{\hat{\mathscr{F}}}\cup F)$. Thus, recalling (2), we get

$$\hat{F}=EF_{\hat{\mathscr{F}}}\subset E(F_{\hat{\mathscr{F}}}\cup F)\subset\hat{F}.$$

Therefore $G=F_{\hat{\mathscr{F}}}=\operatorname{cl}\operatorname{co}(F_{\hat{\mathscr{F}}}\cup F)$ a.s. This establishes the first claim.

If $H$ majorizes $\mathscr{G}$ then, almost surely, $F(\omega)\subset H(\omega)$ for all $F\in\mathscr{G}$. Thus for almost all $\omega\in\Omega$, $G(\omega)=F_{\hat{\mathscr{F}}}(\omega)=\operatorname{cl}\operatorname{co}\bigcup_{F\in\hat{\mathscr{F}}}F(\omega)\subset H(\omega)$.

Arrange now $\hat{\mathscr{F}}$ in a sequence $(\mathscr{F}_n)$. Then

$$\operatorname{ess}\operatorname{cl}\operatorname{co}\mathscr{G}(\omega)=G(\omega)=\operatorname{cl}\operatorname{co}\bigcup_{n=1}^{\infty}F_{\mathscr{F}_n}(\omega)\ \text{a.s.} \tag{4}$$

If the family $\mathscr{G}$ is directed upwards (i.e. for all $F_1, F_2\in\mathscr{G}$ there exists $F_3\in\mathscr{G}$ such that $F_1\cup F_2\subset F_3$ a.s.), then it is possible to construct an a.s. increasing sequence $(F_n')\subset\mathscr{G}$ such that

$$\mathscr{T}_{\text{Mosco}}\text{-}\lim_{n\to\infty}F_n'(\omega)=\operatorname{ess}\operatorname{cl}\operatorname{co}\mathscr{G}(\omega)\ \text{a.s.}$$

Indeed, put $F'_0 = F_{\mathscr{F}_0}$ and for $F'_{n+1}$ take a random set dominating $\mathrm{cl\,co}(F'_n \cup F_{\mathscr{F}_{n+1}})$. Then $\bigcup_{n=1}^{N} F_{\mathscr{F}_n}(\omega) \subset F'_N(\omega)$ a.s. Thus

$$\mathrm{cl\,co} \bigcup_{n=1}^{\infty} F_n(\omega) \subset \mathscr{T}_{\mathrm{Mosco}}\text{-}\lim_{n\to\infty} F'_n(\omega) \subset \mathrm{ess\,cl\,co}\,\mathscr{G}(\omega).$$

This follows from the fact that the random sets $(F'_n)$ are convex valued and that for an increasing sequence $(C_n)$, such that $\mathrm{cl}\big(\bigcup_{n=1}^{\infty} C_n\big)$ is $w$-closed, $\mathrm{cl}\big(\bigcup_{n=1}^{\infty} C_n\big) = \mathscr{T}_{\mathrm{Mosco}}\text{-}\lim_{n\to\infty} C_n$ (see [2], Exercise 5.4.3(c)). If a sequence of random sets is bounded by a $w$-compact random set, then the Mosco convergence implies the scalar convergence. Thus $\mathscr{T}_{\mathrm{scalar}}\text{-}\lim_{n\to\infty} F'_n(\omega) = \mathrm{ess\,cl\,co}\,\mathscr{G}(\omega)$ a.s. In fact, since $F'_n(\omega) \subset \mathrm{ess\,cl\,co}\,\mathscr{G}(\omega)$ a.s. for all $n \in N$, Proposition 11′ of [12] assures the equivalence of the scalar and Mosco convergence of the sequence $(F'_n(\omega))$ to $\mathrm{ess\,cl\,co}\,\mathscr{G}(\omega)$.

Suppose now that the random set $K$ is $\mathscr{P}_{\mathrm{skc}}(X)$-valued. Since the Mosco convergence of sets majorized by a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued sets implies convergence in the scalar topology, the first part of the proof implies that for all $x^* \in X^*$

$$\lim_{n\to\infty} s\big(x^*, F'_n(\omega)\big) = s\big(x^*, G(\omega)\big) \text{ a.s.} \tag{5}$$

Proceeding as in the proof of Lemma 3.2 of [1] it can be shown that the above convergence is uniform with respect to all $x^* \in B^*$. This proves the convergence in the Hausdorff distance. ∎

**3.2. Finite horizon.** The following theorem concerns finding an optimal stopping time for a finite sequence of random sets.

THEOREM 3.2. *Let $(Z_n)_{0\leqslant n\leqslant p}$ be a sequence of w-compact convex valued random sets. Suppose that $\sup \int_\Omega \|Z_n(\omega)\|\, dP(\omega)$ is finite and that the families $\big(E(Z_\nu \mid \mathscr{B}_n)\big)_{\nu\in\Lambda_n}$ are upwards directed (with respect to inclusion) for all $n = 0, \ldots, p$. Then the sequence $(X_n)_{0\leqslant n\leqslant p}$ defined by a backward induction*

$$X_{p-m} := \begin{cases} Z_p & \text{for } m = 0, \\ \mathrm{cl\,co}\big(Z_{p-m} \cup E(X_{p-m+1} \mid \mathscr{B}_{p-m})\big) & \text{for } 0 < m \leqslant p \end{cases} \tag{6}$$

*is the smallest integrably bounded supermartingale dominating the sequence $(Z_n)$. Moreover, the stopping time*

$$\nu_0 := \min\{n \in N\colon\ 0 \leqslant n \leqslant p,\ X_n = Z_n\}$$

*is the solution to the Snell problem associated with $(Z_n)$.*

The assumption that $\big(E(Z_\nu \mid \mathscr{B}_n)\big)_{\nu\in\Lambda_n}$ is upwards directed for any $n \in N$ is not required in the original Snell's result. In the set valued Snell's problem the usual relation $\leqslant$ in $\boldsymbol{R}$ is replaced by inclusion in some families of subsets of $X$. However, these families are not totally ordered by inclusion and the above--mentioned condition seems unavoidable.

**3.2.1.** Proof of Theorem 3.2. Since $Z_p = X_p$ and $(X_{\nu_0 \wedge n})$ is a martingale, $X_0 = E(X_{\nu_0} | \mathscr{B}_0) = E(Z_{\nu_0} | \mathscr{B}_0)$. On the other hand, by the definition of $(X_n)$ and the stopping time $\nu_0$, for every stopping time $\nu$, $X_0 \supset E(X_\nu | \mathscr{B}_0) \supset E(Z_\nu | \mathscr{B}_0)$. Thus $\nu_0$ is an optimal stopping time and

$$X_0 = \operatorname*{ess\,cl\,co}_{\nu \in \Lambda_0} E(Z_\nu | \mathscr{B}_0).$$

Similarly, we can show that for $0 \leqslant n \leqslant p$

$$X_n = \operatorname*{ess\,cl\,co}_{\nu \in \Lambda_n} E(Z_\nu | \mathscr{B}_n).$$

Thus $\nu_0$ is the desired stopping time. ∎

**3.3. Infinite horizon.** We begin here with the presentation of all results. The proofs are postponed till the end of this section.

THEOREM 3.3. *Let $(Z_n)$ be a sequence, adapted to $(\mathscr{B}_n)$, of closed convex random sets majorized by a $\mathscr{P}_{wkc}(X)$-valued integrably bounded random set $K$ and such that, for any $n \in N$, the family $\big(E(Z_\nu | \mathscr{B}_n)\big)_{\nu \in \Lambda_n}$ is upwards directed. Then the random sets*

$$X_n = \operatorname*{ess\,cl\,co}_{\nu \in \Lambda_n} E(Z_\nu | \mathscr{B}_n)$$

*form an adapted sequence of integrably bounded random sets such that, for all $n \in N$,*

$$X_n = \operatorname{cl\,co}\big(Z_n \cup E(X_{n+1} | \mathscr{B}_n)\big). \tag{7}$$

*The sequence $X_n$ is the smallest (with respect to inclusions) supermartingale dominating the sequence $(Z_n)$. Finally,*

$$EX_n = \operatorname{cl\,co} \bigcup_{\nu \in \Lambda_n} EZ_\nu. \tag{8}$$

The next theorem gives the optimal stopping time for the multivalued Snell problem.

THEOREM 3.4. *Let $(Z_n)$ be a sequence of w-compact valued convex valued random sets satisfying the following conditions:*

(i) *$(Z_n)$ is majorized by a $\mathscr{P}_{wkc}(X)$-valued integrably bounded random set $K$;*

(ii) *for any $n \in N$, the family $\big(E(Z_\nu | \mathscr{B}_n)\big)_{\nu \in \Lambda_n}$ is upwards directed.*

*In order that $\operatorname{cl\,co} \bigcup_{\nu \in \Lambda_0} EZ_\nu$ is attained it is necessary and sufficient that the stopping time $\nu_0$ defined in terms of the supermartingale $(X_n)$ by*

$$\nu_0(\omega) := \begin{cases} \inf\{n \in N:\ X_n(\omega) = Z_n(\omega)\}, \\ +\infty \quad \text{if } X_n(\omega) \setminus Z_n(\omega) \neq \emptyset \text{ for all } n \in N \end{cases}$$

*is a.s. finite. When this condition is satisfied,* $EZ_{\nu_0} = \mathrm{cl\,co} \bigcup_{\nu \in \Lambda_0} EZ_\nu$ *and* $\nu_0$ *is the smallest finite stopping time satisfying this equation. Moreover, if* $K$ *is* $\mathscr{P}_{\mathrm{skc}}(X)$-*-valued, then, for all* $\varepsilon > 0$,

$$\nu_\varepsilon(\omega) = \inf\{n \in N\colon\ X_n(\omega) \subset Z_n(\omega) + B(0, \varepsilon)\}$$

*always defines an a.s. finite stopping time such that*

$$\bigcup_{\nu \in \Lambda_0} EZ_\nu \subset EZ_{\nu_\varepsilon} + B(0, \varepsilon).$$

*The stopping time* $\bar{\nu}_0$ *is said to be* optimal *when it is a.s. finite. The stopping times* $\bar{\nu}_\varepsilon$ *are called* $\varepsilon$-optimal.

The following corollary gives conditions which assure finiteness of the optimal stopping time $\nu_0$ defined in Theorem 3.4.

COROLLARY 3.5. *Let* $(Z_n)$ *be a sequence of* $\mathscr{P}_{\mathrm{wkc}}(X)$*-valued random sets such that*

(i) $(Z_n)$ *is majorized by a* $\mathscr{P}_{\mathrm{wkc}}(X)$*-valued integrably bounded random set* $K$,

(ii) *the set* $\big(E(Z_\nu \mid \mathscr{B}_n)\big)_{\nu \in \Lambda_n}$ *is upwards directed for any* $n \in N$,

(iii) *there exists a set* $C$ *such that* w-$\mathrm{Ls}_{n \to \infty} Z_n(\omega) \subset C$ *a.s.; moreover, there exists a subsequence* $(Z_{n_k})$ *such that* $C \subset Z_{n_k}$ *a.s.*

*Then the* $\mathrm{cl\,co} \bigcup_{\nu \in \Lambda_0} EZ_\nu$ *is attained.*

EXAMPLE 3.6. Consider a Euclidean space $R^d$. Let $(f_i)_{i=1,\ldots,d}$ be a collection of real-valued increasing bounded functions. Let $(\xi_n)$ be i.i.d. random variables with positive mean. For any $n \in N$ define

$$Z_n := \Big\{(x_1, \ldots, x_d) \in R^d\colon\ |x_i| \leqslant \Big|f_i\Big(n^{-1} \sum_{k=1}^{n} \xi_k\Big)\Big|,\ i = 1, \ldots, d\Big\}.$$

According to Corollary 3.5 there exists a finite stopping time $\nu_0$ such that

$$EZ_{\nu_0} = \mathrm{cl\,co} \bigcup_{\nu \in \Lambda_0} EZ_\nu.$$

The following corollary shows how the solution of the infinite horizon problem can be approximated by the solutions of the finite horizon problems.

COROLLARY 3.7. *Let* $(Z_n)$ *be as in Theorem* 3.3. *For any* $p \in N$, *let* $(X_n^p)_{0 \leqslant n \leqslant p}$ *is the smallest supermartingale dominating the sequence* $(Z_n)$ *(defined as in* (6)). *For all* $\omega \in \Omega$ *and* $n \in N$ *define*

$$X_n^\infty(\omega) := \mathscr{T}_{Mosco}\text{-}\lim_{p \to \infty} X_n^p(\omega).$$

*Then* $X_n^\infty(\omega) = \mathrm{ess\,cl\,co}_{\nu \in \Lambda_n^B} E(Z_\nu \mid \mathscr{B}_n)$, *where* $\Lambda_n^B$ *is the set of all bounded stopping times in* $\Lambda_n$. *Moreover, the supermartingale* $(X_n)$ *defined in Theorem* 3.3 *coincides with* $(X_n^\infty)$.

**3.3.1.** *Proofs*

Proof of Theorem 3.3. The random sets $X_n$ are $\mathscr{B}_n$-measurable and

$$Z_n(\omega) \subset X_n(\omega) \subset E(K \mid \mathscr{B}_n)(\omega) \text{ a.s.,} \tag{9}$$

so the sequence $(X_n)$ is bounded in $\mathscr{L}^1_{\mathscr{P}_{\mathrm{wkc}}(X)}$. For a fixed $n \in N$, Proposition 3.1 yields the existence of a sequence $(\nu_k)$ such that

$$\mathscr{T}_{\mathrm{Mosco}}\text{-}\lim_{k\to\infty} E(Z_{\nu_k} \mid \mathscr{B}_n)(\omega) = X_n(\omega) \text{ a.s.}$$

It will be shown that $(X_n)$ is a supermartingale dominating a.s. the sequence $(Z_n)$. Since $\big(E(Z_{\nu_k} \mid \mathscr{B}_n)(\omega)\big)$ converges a.s. in the Mosco topology, the application of Fatou's lemma ([9], Theorem 2.3) yields that for almost all $\omega \in \Omega$

$$E(X_n \mid \mathscr{B}_{n-1})(\omega) = E\big(\mathscr{T}_{\mathrm{Mosco}}\text{-}\lim_{k\to\infty} E(Z_{\nu_k} \mid \mathscr{B}_n) \mid \mathscr{B}_{n-1}\big)(\omega)$$

$$\subset s\text{-}\operatorname*{Li}_{k\to\infty} E\big(E(Z_{\nu_k} \mid \mathscr{B}_n) \mid \mathscr{B}_{n-1}\big)(\omega) = s\text{-}\operatorname*{Li}_{k\to\infty} E(Z_{\nu_k} \mid \mathscr{B}_{n-1})(\omega) = X_{n-1}(\omega).$$

Thus $(X_n)$ is a supermartingale dominating a.s. the sequence $(Z_n)$. Therefore,

$$\operatorname{cl}\operatorname{co}\big(Z_n(\omega) \cup E(X_{n+1} \mid \mathscr{B}_n)(\omega)\big) \subset X_n(\omega) \text{ a.s.}$$

Let us now establish the opposite inclusion. For all $\omega \in \Omega$ and all $\nu \in \Lambda_n$,

$$Z_\nu(\omega) = Z_n(\omega)\, 1_{[\nu=n]}(\omega) + Z_{\nu\vee(n+1)}(\omega)\, 1_{[\nu>n]}(\omega).$$

Since $E(Z_{\nu\vee(n+1)} \mid \mathscr{B}_{n+1})(\omega) \subset X_{n+1}(\omega)$ a.s., we get, a.s. for all $\nu \in \Lambda_n$,

$$E(Z_\nu \mid \mathscr{B}_n)(\omega) = Z_n(\omega)\, 1_{[\nu=n]}(\omega) + E(Z_{\nu\vee(n+1)} \mid \mathscr{B}_n)(\omega)\, 1_{[\nu>n]}(\omega)$$

$$\subset Z_n(\omega)\, 1_{[\nu=n]}(\omega) + E(X_{n+1} \mid \mathscr{B}_n)(\omega)\, 1_{[\nu>n]}(\omega) \subset \operatorname{cl}\operatorname{co}\big(Z_n(\omega) \cup E(X_{n+1} \mid \mathscr{B}_n)(\omega)\big).$$

Thus, recalling the definition of $(X_n)$ and Proposition 3.1, we obtain

$$X_n(\omega) \subset \operatorname{cl}\operatorname{co}\big(Z_n(\omega) \cup E(X_{n+1} \mid \mathscr{B}_n)(\omega)\big) \text{ a.s.}$$

Invoking again Fatou's lemma ([9], Theorem 2.3), we get

$$E \operatorname*{ess\,cl\,co}_{\nu\in\Lambda_n} E(Z_\nu \mid \mathscr{B}_n) = E\big(\mathscr{T}_{\mathrm{Mosco}}\text{-}\lim_{k\to\infty} E(Z_{\nu_k} \mid \mathscr{B}_n)\big) \subset s\text{-}\operatorname*{Li}_{k\to\infty} EZ_{\nu_k} \subset \operatorname{cl}\operatorname{co} \bigcup_{\nu\in\Lambda_n} EZ_\nu.$$

On the other hand, for all $\nu \in \Lambda_n$ we have

$$EZ_\nu \subset E \operatorname*{ess\,cl\,co}_{\Lambda_n} E(Z_\nu \mid \mathscr{B}_n).$$

Hence (8) follows.

Suppose that $(X'_n)$ is another supermartingale dominating the sequence $(Z_n)$. Lemma 2.2 yields that for any $\nu \in \Lambda_n$

$$E(Z_\nu \mid \mathscr{B}_n) \subset E(X'_\nu \mid \mathscr{B}_n) \subset X'_n \text{ a.s.}$$

Thus $X_n \subset X'_n$ a.s. by Proposition 3.1. ∎

Proof of Theorem 3.4. The proof of Theorem 3.4 will follow from a few lemmas. In these lemmas we assume the settings of Theorem 3.4.

LEMMA 3.8. *For every stopping time $\nu$ dominated by $\nu_0$ the stopped sequence $(X_{\nu\wedge n})$ is a martingale. When $\nu$ is a.s. finite as well, it follows that $EX_\nu \supset EX_0$.*

Proof. On the event $\{\omega\in\Omega\colon \nu_0(\omega) > n\}$ and on the smaller event $\{\omega\in\Omega\colon \nu(\omega) > n\}$ we have $X_n \supset Z_n$ a.s. Theorem 3.3 (see (7)) implies that $X_n = E(X_{n+1}\mid\mathscr{B}_n)$ on these events. Thus

$$E(X_{\nu\wedge(n+1)}\mid\mathscr{B}_n) = X_\nu 1_{[\nu\leqslant n]} + E(X_{n+1}\mid\mathscr{B}_n)1_{[\nu>n]} = X_\nu 1_{[\nu\leqslant n]} + X_n 1_{[\nu\leqq n]} = X_{\nu\wedge n}.$$

Thus the sequence $(X_{\nu\wedge n})$ is a martingale. Lemma 2.1 yields $X_\nu = \mathscr{T}_{\text{Mosco}}\text{-}\lim_{n\to\infty} X_{\nu\wedge n}$ a.s. Since $(X_{\nu\wedge n})$ is bounded by an integrably bounded $w$-compact valued random set $K$, the Mosco convergence implies convergence in the scalar topology. Application of Fatou's lemma and properties of support functions and Lemma 3.2 (b) of [7] yield

$$\limsup_{n\to\infty} s(x^*, EX_{\nu\wedge n}) \leqslant E\limsup_{n\to\infty} s(x^*, X_{\nu\wedge n}) \leqslant Es(x^*, w\text{-Ls}\, X_{\nu\wedge n}) = Es(x^*, X_\nu).$$

Thus, by Lemma 1.1 of [9] we obtain

$$w\text{-}\underset{n\to\infty}{\text{Ls}}\ EX_{\nu\wedge n} \subset EX_\nu.$$

Since $(X_{\nu\wedge n})$ is a martingale, we get the result. ■

Lemma 3.8 implies that if the stopping time $\nu_0$ is finite then, since $Z_{\nu_0}(\omega) = X_{\nu_0}(\omega)$ a.s.,

$$EZ_{\nu_0} = EX_{\nu_0} \supset EX_0.$$

Proposition 3.3 yields $EX_0 = \text{cl}\,\text{co}\bigcup_{\nu\in\Lambda_0} EZ_\nu$, thus

$$EZ_{\nu_0} = \text{cl}\,\text{co}\bigcup_{\nu\in\Lambda_0} EZ_\nu.$$

This yields optimality of $\nu_0$.

In order to show that $\nu_0$ is the smallest optimal stopping time we will use the following lemma.

LEMMA 3.9. *For every finite stopping time $\nu_1$ we have*

$$X_{\nu_1} = \underset{\nu\in\Lambda_{\nu_1}}{\text{ess}\,\text{cl}\,\text{co}}\, E(Z_\nu\mid\mathscr{B}_{\nu_1}), \tag{10}$$

*where $\Lambda_{\nu_1}$ is the set of all a.s. finite stopping times not less than $\nu_1$. Consequently,*

$$EX_{\nu_1} = \text{cl}\,\text{co}\bigcup_{\nu\in\Lambda_{\nu_1}} EZ_\nu.$$

Proof. Since $\nu_1 \in \Lambda_{\nu_1}$, the set $\Lambda_{\nu_1}$ is nonempty. Let $X_{(\nu_1)}$ denote the ess cl co in (10). For every stopping time $\nu \in \Lambda_{\nu_1}$ the random variable $\nu \vee n$ is a stopping time in $\Lambda_n$ and $\nu = \nu \vee n$ on the event $\{\omega \in \Omega\colon \nu_1(\omega) = n\}$. On this event we have

$$E(Z_\nu \mid \mathscr{B}_{\nu_1}) = E(Z_\nu \mid \mathscr{B}_n) = E(Z_{\nu \vee n} \mid \mathscr{B}_n) \subset X_n = X_{\nu_1}.$$

Therefore $X_{(\nu_1)} \subset X_{\nu_1}$ a.s. Conversely, if $\nu'$ is a stopping time belonging to $\Lambda_n$, then the stopping time $\nu' \vee \nu_1 \in \Lambda_{\nu_1}$ and $\nu' = \nu' \vee \nu_1$ on $\{\omega \in \Omega\colon \nu_1(\omega) = n\}$. On this event we have

$$E(Z_{\nu'} \mid \mathscr{B}_n) = E(Z_{\nu' \vee \nu_1} \mid \mathscr{B}_n) = E(Z_{\nu' \vee \nu_1} \mid \mathscr{B}_{\nu_1}) \subset \overline{X_{\nu_1}} \text{ a.s.}$$

Thus one the above-mentioned event $X_n \subset X_{(\nu_1)}$ a.s. Therefore $X_{\nu_1} \subset X_{(\nu_1)}$ a.s.

The family $\{E(Z_\nu \mid \mathscr{B}_\eta),\ \nu \in \Lambda_\eta\}$ is directed upwards. Indeed, on the event $\{\omega \in \Omega\colon \eta(\omega) = n\}$ the set $(E(Z_\nu \mid \mathscr{B}_\eta))$ is upwards directed, i.e. for $\nu_1, \nu_2 \in \Lambda_\eta$ there exists $\nu^{(n)}$ such that

$$E(Z_{\nu_1} \mid \mathscr{B}_\eta) \cup E(Z_{\nu_2} \mid \mathscr{B}_\eta) \subset E(Z_{\nu^{(n)}} \mid \mathscr{B}_\eta).$$

Thus we can set $\nu = \sum_{n=1}^{\infty} 1_{[\eta = n]} \nu^{(n)}$, so that $E(Z_{\nu_1} \mid \mathscr{B}_\eta) \cup E(Z_{\nu_2} \mid \mathscr{B}_\eta) \subset E(Z_\nu \mid \mathscr{B}_\eta)$ a.s.

Suppose that $\nu^*$ is an optimal finite stopping time, i.e. $\operatorname{cl\,co} \bigcup_{\nu \in \Lambda_0} EZ_\nu = EZ_{\nu^*}$. The preceding lemma yields

$$EX_{\nu^*} = \operatorname{cl\,co} \bigcup_{\nu \in \Lambda_{\nu^*}} EZ_\nu \subset \operatorname{cl\,co} \bigcup_{\nu \in \Lambda_0} EZ_\nu \subset EZ_{\nu^*}.$$

On the other hand, by the definition of $X_{\nu^*}$, $X_{\nu^*} \supset Z_{\nu^*}$ a.s. Therefore $EX_{\nu^*} \supset EZ_{\nu^*}$. Hence $EX_{\nu^*} = EZ_{\nu^*}$ a.s. Now the definition of $\nu_0$ implies that $\nu_0 \leqslant \nu^*$ and $\nu_0$ is finite a.s.

In order to show finiteness and $\varepsilon$-optimality of the stopping times $\nu_\varepsilon$, we will use the following lemma:

LEMMA 3.10. *The following relation holds*:

$$w\text{-}\operatorname*{Ls}_{n \to \infty} X_n(\omega) = w\text{-}\operatorname*{Ls}_{n \to \infty} Z_n(\omega) \ \text{a.s.}$$

Proof. For every stopping time $\nu \in \Lambda_m$ (and in $\Lambda_n$ for any $n \geqslant m$)

$$Z_\nu(\omega) \subset \operatorname{cl\,co} \bigcup_{i=m}^{\infty} Z_i(\omega) \text{ a.s.}$$

Therefore

$$X_n = \operatorname*{ess\,cl\,co}_{\nu \in \Lambda_n} E(Z_\nu \mid \mathscr{B}_n) \subset E\Big(\bigcup_{i=m}^{\infty} Z_i \mid \mathscr{B}_n\Big).$$

For any $x^* \in H^*$, $m \in N$, the sequences $\big(E\big(s\big(x^*, \bigcup_{i=m}^{\infty} Z_i\big) \mid \mathscr{B}_n\big)\big)$ are martingales.

Theorem II-2-11 of [11] implies that these martingales converge a.s. and in $L^1$ to $s(x^*, \bigcup_{i=m}^{\infty} Z_i)$. Thus, by Lemma 3.2 of [7] and Fatou's lemma, a.s. for all $x^* \in H^*$,

$$s(x^*, w\text{-}\underset{n\to\infty}{\mathrm{Ls}}\, X_n(\omega)) \leqslant \limsup_{n\to\infty} s(x^*, X_n(\omega))$$
$$\leqslant \lim_{n\to\infty} E\big(s(x^*, \bigcup_{i=m}^{\infty} Z_i) \mid \mathscr{B}_n\big)(\omega) = s(x^*, \bigcup_{i=m}^{\infty} Z_i(\omega)).$$

Therefore

$$w\text{-}\underset{n\to\infty}{\mathrm{Ls}}\, X_n(\omega) \subset \overline{\bigcup_{i=m}^{\infty} Z_i(\omega)} \text{ a.s.}$$

Letting $m \to \infty$ and using the fact that for almost all $\omega \in \Omega$ the sequence $(\mathrm{cl\,co} \bigcup_{i=m}^{\infty} Z_i(\omega))$ is decreasing, we get almost surely

$$w\text{-}\underset{n\to\infty}{\mathrm{Ls}}\, X_n(\omega) \subset \bigcap_{m=1}^{\infty} \mathrm{cl\,co} \bigcup_{i=m}^{\infty} Z_i(\omega).$$

Applying Proposition 3.10 of [6] we conclude that

$$w\text{-}\underset{n\to\infty}{\mathrm{Ls}}\, X_n(\omega) \subset w\text{-}\underset{n\to\infty}{\mathrm{Ls}}\, Z_n(\omega) \text{ a.s.}$$

The opposite inequality is obvious. ∎

Now we come back to the proof of Theorem 3.4. Consider the stopping time $\nu_\varepsilon$ defined in the statement of Theorem 3.4. Since $\nu_\varepsilon \leqslant \nu_0$, Lemma 3.8 implies that the sequence $(X_{\nu_\varepsilon \wedge n})_{n\in N}$ is an integrable martingale. Moreover, by the convergence theorem for martingales ([8], Proposition 5.8), there exists a $\mathscr{P}_{\mathrm{skc}}(X)$-valued random set $X_\infty$ such that $X_\infty = \mathscr{T}_{\mathrm{Mosco}}\text{-}\lim_{n\to\infty} X_n$ a.s. on the event $\{\omega \in \Omega: \nu_\varepsilon = \infty\}$. However, almost everywhere on this event we have

(11) $$\varrho_H(X_n(\omega), Z_n(\omega)) \geqslant \varepsilon.$$

Fix an $\omega$ in the subset of $\Omega$, where (11) is valid. For every $n \in N$ it is possible to choose $x_n \in X_n(\omega) \backslash Z_n(\omega)$ such that

(12) $$d(x_n, Z_n(\omega)) > \varepsilon/2.$$

Since $(Z_n(\omega))$ is majorized by the $s$-compact set $K(\omega)$, there exists a subsequence $(x_{n_k})$ which $s$-converges to some $x \in X_\infty(\omega)$. According to Lemma 3.10, we have

$$\varrho_H(X_\infty(\omega), w\text{-}\underset{n\to\infty}{\mathrm{Ls}}\, Z_n(\omega)) = 0.$$

Thus there exists a sequence $(z_{n_k})$ such that $z_{n_k} \in Z_{n_k}(\omega)$ for all $k \in N$ and $x = w\text{-}\lim_{k\to\infty} z_{n_k}$. Again, since this sequence is bounded by an $s$-compact set, one can choose a subsequence $(z_{n_{k'}})$ such that $x = s\text{-}\lim_{k'\to\infty} z_{n_{k'}}$. This, together with the fact that $x = s\text{-}\lim_{k'\to\infty} x_{n_{k'}}$ and (12), yields $\{\omega\in\Omega\colon\ \nu_\varepsilon = \infty\} = \emptyset$. Thus $\nu_\varepsilon$ is a.s. finite. Moreover, $X_{\nu_\varepsilon}(\omega) \subset Z_{\nu_\varepsilon}(\omega)+B(0, \varepsilon)$ a.s. Theorem 3.3 and Lemma 3.8 imply now that

$$\operatorname{cl}\operatorname{co}\bigcup_{\nu\in\Lambda_0} EZ_\nu = EX_0 \subset EX_{\nu_\varepsilon} \subset EZ_{\nu_\varepsilon}+B(0, \varepsilon). \ \blacksquare$$

Proof of Corollary 3.5. As was done in Lemma 3.8 one can show that $(X_{\nu_0\wedge n})$ is a martingale. It is majorized by an integrably bounded $w$-compact valued random set. Proposition 5.8 of [8] yields the existence of a random set $X_\infty$ such that

$$\mathscr{T}_{\text{Mosco}}\text{-}\lim_{n\to\infty} X_n(\omega) = X_\infty(\omega) \text{ a.s.} \quad \text{on } \{\omega\in\Omega\colon\ \nu_0(\omega) = \infty\}.$$

Since $w\text{-Ls}_{n\to\infty} Z_n(\omega) \subset C$ a.s., Lemma 3.10 yields $X_\infty(\omega) \subset C$ a.s. on $\{\omega\in\Omega\colon\ \nu_0(\omega) = \infty\}$. Recalling that $(X_{\nu_0\wedge n})$ is a martingale, we obtain $X_n(\omega) \subset E(C\,|\,\mathscr{B}_n) = C$ a.s. on this set. Since $Z_{n_k}(\omega) \subset X_{n_k}(\omega)$ a.s. for all $k\in N$, $Z_{n_k}(\omega) = C$ a.s. on $\{\omega\in\Omega\colon\ \nu_0(\omega) = \infty\}$. Then $Z_{n_1}(\omega) = X_{n_1}(\omega)$ a.s. on the above set. This contradicts the definition of $\nu_0$, at least when $\{\omega\in\Omega\colon\ \nu_0(\omega) = \infty\} \neq \emptyset$. Thus $\nu_0 < \infty$ a.s. $\blacksquare$

Proof of Corollary 3.7. Observe first that for all $\omega\in\Omega$, $X_n^p(\omega) \subset X_n^{p+1}(\omega)$ for $n = 0, \ldots, p$. Thus, taking into account that $X_n^p$ are majorized by a $w$-compact valued random set, we infer that $X_n^\infty(\omega) := \mathscr{T}_{\text{Mosco}}\text{-}\lim_{p\to\infty} X_n^p(\omega)$ is well defined. As in the proof Theorem 3.3 one can show that $(X_n^\infty)$ is the smallest supermartingale dominating the sequence $(Z_n)$. Moreover,

$$X_n^\infty(\omega) = \operatorname*{ess\,cl\,co}_{\nu\in\Lambda_n^B} E(Z_\nu\,|\,\mathscr{B}_n)(\omega) \quad \text{for all } \omega\in\Omega.$$

Obviously, the supermartingale $(X_n^\infty)$ is dominated by the supermartingale $(X_n)$ defined in Theorem 3.3. However, since both supermartingales are majorized by a $w$-compact set, the Fatou lemma ([9], Theorem 2.3) implies that for any $n\in N$ and almost all $\omega\in\Omega$

$$E(Z_\nu\,|\,\mathscr{B}_n)(\omega) \subset s\text{-}\operatorname*{Li}_{k\to\infty} E(Z_{\nu\wedge k}\,|\,\mathscr{B}_n)(\omega) \subset X_n^\infty(\omega)$$

for every $\nu\in\Lambda_n$. Thus also $(X_n)$ is a.s. dominated by $(X_n^\infty)$. $\blacksquare$

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