

**ALMOST SURE CONVERGENCE
OF THE DISTRIBUTIONAL LIMIT THEOREM
FOR ORDER STATISTICS**

BY

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Abstract. Let X_n , $n \geq 1$, be a sequence of independent and identically distributed random variables and $X_{n,1} \leq X_{n,2} \leq \dots \leq X_{n,n}$ denote the order statistics of X_1, \dots, X_n . For any sequence of integers $\{k_n\}$ with $1 \leq k_n \leq n$ and $\lim_{n \rightarrow \infty} \min\{k_n, n - k_n + 1\} = \infty$, if there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ and some non-degenerate distribution function G such that $(X_{n,k_n} - b_n)/a_n$ converges in distribution to G , then with probability one

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I\left(\frac{X_{n,k_n} - b_n}{a_n} \leq x\right) = G(x) \quad \text{for all } x \in C(G),$$

where $C(G)$ is the set of continuity points of G .

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1. INTRODUCTION

Suppose we have a sequence of random variables $\{W_n\}$ which converges in distribution to a continuous random variable W , i.e., $W_n \xrightarrow{d} W$ as $n \rightarrow \infty$. Many authors have investigated almost sure versions of distributional limit theorem. One particular version is the following result:

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I(W_n \leq x) \stackrel{\text{a.s.}}{=} P(W \leq x) \quad \text{for all } x,$$

where $I(\cdot)$ denotes the indicator function. For example, Brosamler [4], Schatte [11]–[13], Lacey and Philipp [10], Berkes et al. [3] among others studied the case where W_n means the normalized sums of random variables. For the case where W_n means the maxima of i.i.d. random variables, we refer to Cheng et al. [5], [6], Fahrner [7], [8], and Fahrner and Stadtmüller [9]. A general method for dealing with logarithmic means is given by Berkes and Csáki [2].

Recently Stadtmüller [14] studied the case where W_n is order statistics. More details are as follows. Suppose $\{X_n\}$ is a sequence of i.i.d. random variables with common distribution function F . Let $X_{n,1} \leq \dots \leq X_{n,n}$ denote the order statistics of X_1, \dots, X_n .

Throughout this paper we assume that $\{k_n\}$ is a sequence of integers satisfying $1 \leq k_n \leq n$, and there exist constants $a_n > 0$, b_n and a distribution function G such that

$$(1.1) \quad \lim_{n \rightarrow \infty} P\left(\frac{X_{n,n-k_n} - b_n}{a_n} \leq x\right) = G(x) \quad \text{for all } x \in C(G),$$

where $C(G)$ is the set of continuity points of G . Assuming that either

$$(1.2) \quad k_n \uparrow \infty, \quad k_n/n \rightarrow 0, \quad \log k_n = O((\log n)^{1-\varepsilon}) \text{ for some } \varepsilon > 0,$$

or

$$(1.3) \quad \begin{aligned} &F \text{ has a differentiable density at } p \in (0, 1), \\ &(n - k_n)/n = p + O(1/\sqrt{n \log^{\varepsilon} n}) \text{ for some } \varepsilon > 0, \end{aligned}$$

Stadtmüller [14] showed that

$$(1.4) \quad \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I\left(\frac{X_{n,n-k_n} - b_n}{a_n} \leq x\right) \stackrel{\text{a.s.}}{=} G(x) \quad \text{for any } x.$$

In this paper we show that (1.4) holds under the following condition:

$$(1.5) \quad \min\{k_n, n - k_n + 1\} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which is weaker than the conditions (1.2) and (1.3); see Section 2 for our main results. All proofs are deferred till Section 3. A joint distribution of order statistics from nested samples is given in the Appendix, which may be of independent interest.

2. MAIN RESULTS

We first investigate the case of uniform order statistics. Let $\{U_n\}$ be a sequence of independent random variables with a uniform distribution over $(0, 1)$, and $U_{n,1} \leq U_{n,2} \leq \dots \leq U_{n,n}$ denote the order statistics of U_1, \dots, U_n . Throughout this section we assume that (1.5) is true. Hence

$$(2.1) \quad \begin{aligned} \lim_{n \rightarrow \infty} P\left(\frac{U_{n,k_n} - d_n}{c_n} \leq x\right) &= \Phi(x) \\ &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy \quad \text{for all } x \in R, \end{aligned}$$

where $c_n = \sqrt{k_n(n - k_n + 1)}/n^{3/2}$ and $d_n = k_n/(n + 1)$.

Our first theorem is the almost sure version of the distributional limit theorem for uniform order statistics.

THEOREM 1. *With probability one,*

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I\left(\frac{U_{n,k_n} - d_n}{c_n} \leq x\right) = \Phi(x) \quad \text{for all } x \in R.$$

Next we state the almost sure version for general order statistics.

THEOREM 2. *Suppose there exist constants $a_n > 0$, $b_n \in R$ and a non-degenerate distribution function G such that*

$$(2.2) \quad \lim_{n \rightarrow \infty} P\left(\frac{X_{n,k_n} - b_n}{a_n} \leq x\right) = G(x) \quad \text{for all } x \in C(G).$$

Then

$$(2.3) \quad \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I\left(\frac{X_{n,k_n} - b_n}{a_n} \leq x\right) \stackrel{\text{a.s.}}{=} G(x) \quad \text{for all } x \in C(G).$$

The condition (2.2) plays an essential role in the theorem. However, the discussion for the convergence of (2.2) is more complicated. We are not going further in this direction but will give some examples.

Besides the uniform distribution in Theorem 1, many distribution functions, for example, exponential and Gaussian distributions, are such that (2.2) holds under (1.5).

EXAMPLE 1. Let $F(x) = 1 - e^{-x}$ for $x > 0$. Then (2.2) holds with

$$a_n = \frac{\sqrt{n - k_n + 1}}{\sqrt{(n + 1)k_n}}, \quad b_n = \log \frac{n}{k_n} \quad \text{and} \quad G = \Phi.$$

Thus

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I\left(\frac{X_{n,k_n} - \log(n/k_n)}{\sqrt{(n - k_n + 1)/(n + 1)k_n}} \leq x\right) \stackrel{\text{a.s.}}{=} \Phi(x) \quad \text{for any } x \in R.$$

For central order statistics X_{n,k_n} with $k_n/n \rightarrow p \in (0, 1)$, a very weak condition is needed to guarantee (2.2).

EXAMPLE 2. For $p \in (0, 1)$ define $\xi_p = F^{-}(p)$, where F^{-} denotes the generalized inverse of F , as in the proof of Theorem 2. If F is differentiable at ξ_p and $F'(\xi_p) > 0$, then

$$P\left(\frac{F'(\xi_p)\sqrt{n}}{\sqrt{p(1-p)}}(X_{n,k_n} - F^{-}(k_n/n)) \leq x\right) \rightarrow \Phi(x)$$

under the condition $k_n/n \rightarrow p$. Therefore,

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \left(\frac{F'(\xi_p) \sqrt{n}}{\sqrt{p(1-p)}} (X_{n,k_n} - F^-(k_n/n)) \leq x \right) \stackrel{\text{a.s.}}{=} \Phi(x) \quad \text{for any } x \in \mathbb{R}.$$

3. PROOFS

First we list some notation and two lemmas. For integers $1 \leq n \leq m$ let $U(j; n; m)$ denote the j -th smallest order statistics of U_n, U_{n+1}, \dots, U_m . For convenience, let us set $U(j; n; m) = 0$ if $j \leq 0$, and $U(j; n; m) = 1$ if $j > m - n + 1$. Note that $U_{n,j} = U(j; 1; n)$ for $1 \leq j \leq n$. Let $[x]$ denote the integer part of x . For any $n < m$, let us set $l_{mn} = [nk_m/(m+1)]$ and define

$$V(n+1, m) = U(k_m - l_{mn}; n+1; m)$$

if $\min\{k_m, m - k_m + 1\} > (\log m)^2$, and

$$V(n+1, m) = \begin{cases} U(k_m; n+1; m) & \text{if } k_m \leq (\log m)^2, \\ U(k_m - n; n+1; m) & \text{if } m - k_m + 1 \leq (\log m)^2 \end{cases}$$

(in either case, $\min\{k_m, m - k_m + 1\} \leq (\log m)^2$).

LEMMA 1. If $n \leq m/(\log m)^2$ and $\min\{k_m, m - k_m + 1\} \leq (\log m)^2$, then

$$(3.1) \quad E|U_{m,k_m} - V(n+1, m)| \leq \frac{2n \min\{k_m, m - k_m + 1\}}{(m+1)^2}$$

for m large enough.

Proof. Note that

$$U(k_m - n; n+1; m) \leq U_{m,k_m} \leq U(k_m; n+1; m).$$

If $k_m \leq (\log m)^2$, then $V(n+1; m) = U(k_m; n+1; m)$. Since $n \leq m/(\log m)^2$ implies $\lim_{m \rightarrow \infty} (n/m) = 0$, we have, for large m ,

$$E|U_{m,k_m} - V(n+1, m)| = E(U(k_m; n+1; m) - U_{m,k_m}) = \frac{k_m}{m-n+1} - \frac{k_m}{m+1} \leq \frac{2nk_m}{(m+1)^2}$$

(we refer to the Appendix for the calculation of the expectation of uniform order statistics). Similarly, if $m - k_m + 1 \leq (\log m)^2$, then for large m

$$\begin{aligned} E|U_{m,k_m} - V(n+1, m)| &= E(U_{m,k_m} - U(k_m - n; n+1; m)) = \frac{k_m}{m+1} - \frac{k_m - n}{m - n + 1} \\ &\leq \frac{2n(m - k_m + 1)}{(m+1)^2}. \end{aligned}$$

Hence the lemma follows.

LEMMA 2. If $n \leq m/(\log m)^2$ and $\min\{k_m, m - k_m + 1\} > (\log m)^2$, then

$$(3.2) \quad E|U_{m,k_m} - V(n+1, m)| \leq \frac{6\sqrt{nk_m(m-k_m+1)}}{(m+1)^2} + \frac{4}{m+1}$$

for m large enough.

Proof. Note that $(V(n+1, m), U_{m,k_m})$ has the same distribution as $(U_{m-n,k_m-l_{mn}}, U_{m,k_m})$ in case $\min\{k_m, m - k_m + 1\} > (\log m)^2$. By applying Theorem 3 in the Appendix with $s = m - n$, $t = n$, $i = k_m - l_{mn}$, $j = k_m$ and $g(x, y) = |x - y|$ we get

$$\begin{aligned} E|U_{m,k_m} - V(n+1, m)| &= E|U_{m-n,k_m-l_{mn}} - U_{m,k_m}| \\ &= \sum_{k=0}^n \int_0^1 E|u(1 - V_{k_m-l_{mn}+k-1,k_m}) \\ &\quad + (1-u)V_{m-k_m+l_{mn}-k,l_{mn}-k}| f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du \\ &\leq \sum_{k=0}^n \int_0^1 E(1 - V_{k_m-l_{mn}+k-1,k_m}) u f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du \\ &\quad + \sum_{k=0}^n \int_0^1 E V_{m-k_m+l_{mn}-k,l_{mn}-k} (1-u) f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du, \end{aligned}$$

where $f_{s,t}(u)$ is defined in (A1) below. Note that $n \leq m/(\log m)^2$ implies that $k_m - l_{mn} > 0$ and $m - k_m + l_{mn} - k \geq m - k_m + l_{mn} - n > 0$ when $0 \leq k \leq n$ and m is large enough. Hence for large m we have

$$E(1 - V_{k_m-l_{mn}+k-1,k_m}) = \frac{k - l_{mn}}{k_m - l_{mn} + k} \vee 0 \leq \frac{k - l_{mn}}{k_m} I(k > l_{mn})$$

and

$$E V_{m-k_m+l_{mn}-k,l_{mn}-k} = \frac{l_{mn} - k}{m - k_m + l_{mn} - k} \vee 0 \leq \frac{l_{mn} - k}{m - k_m + 1} I(k < l_{mn})$$

for all $0 \leq k \leq n$. Therefore, for m large enough

$$\begin{aligned} &E|U_{m,k_m} - V(n+1, m)| \\ &\leq \sum_{k=0}^n \int_0^1 \frac{|k - l_{mn}|}{k_m} u f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du \\ &\quad + \sum_{k=0}^n \int_0^1 \frac{|k - l_{mn}|}{m - k_m + 1} (1-u) f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du \\ &\leq \frac{1}{k_m} \sum_{k=0}^n \int_0^1 (|k - nu| + |nu - l_{mn}|) u f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{m-k_m+1} \sum_{k=0}^n \int_0^1 (|k-nu| + |nu-l_{mn}|)(1-u) f_{m-n,k_m-l_{mn}}(u) \binom{n}{k} u^k (1-u)^{n-k} du \\
 \leq & \frac{1}{k_m} \int_0^1 u f_{m-n,k_m-l_{mn}}(u) \left\{ \sum_{k=0}^n |k-nu| \binom{n}{k} u^k (1-u)^{n-k} \right\} du \\
 & + \frac{1}{k_m} \int_0^1 |nu-l_{mn}| u f_{m-n,k_m-l_{mn}}(u) \left\{ \sum_{k=0}^n \binom{n}{k} u^k (1-u)^{n-k} \right\} du \\
 & + \frac{1}{m-k_m+1} \int_0^1 (1-u) f_{m-n,k_m-l_{mn}}(u) \left\{ \sum_{k=0}^n |k-nu| \binom{n}{k} u^k (1-u)^{n-k} \right\} du \\
 & + \frac{1}{m-k_m+1} \int_0^1 |nu-l_{mn}| (1-u) f_{m-n,k_m-l_{mn}}(u) \left\{ \sum_{k=0}^n \binom{n}{k} u^k (1-u)^{n-k} \right\} du \\
 =: & I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

Since $\binom{n}{k} u^k (1-u)^{n-k}$, $k = 0, 1, \dots, n$, is a binomial probability function, we have

$$\sum_{k=0}^n \binom{n}{k} u^k (1-u)^{n-k} = 1$$

and

$$\sum_{k=0}^n |k-nu| \binom{n}{k} u^k (1-u)^{n-k} \leq \sqrt{\sum_{k=0}^n |k-nu|^2 \binom{n}{k} u^k (1-u)^{n-k}} = \sqrt{nu(1-u)}$$

by Schwarz's inequality. Thus, in virtue of Hölder's inequality, for large m we obtain

$$\begin{aligned}
 I_1 & \leq \frac{\sqrt{n}}{k_m} \int_0^1 u^{3/2} (1-u)^{1/2} f_{m-n,k_m-l_{mn}}(u) du = \frac{\sqrt{n}}{k_m} EU_{m-n,k_m-l_{mn}}^{3/2} (1-U_{m-n,k_m-l_{mn}})^{1/2} \\
 & \leq \frac{\sqrt{n}}{k_m} (EU_{m-n,k_m-l_{mn}}^2)^{3/4} (E(1-U_{m-n,k_m-l_{mn}})^2)^{1/4} \\
 & = \frac{\sqrt{n}}{k_m} \left(\frac{(k_m-l_{mn})(k_m-l_{mn}+1)}{(m-n+1)(m-n+2)} \right)^{3/4} \left(\frac{(m-n-k_m+l_{mn}+1)(m-n-k_m+l_{mn}+2)}{(m-n+1)(m-n+2)} \right)^{1/4} \\
 & \leq \frac{\sqrt{nk_m(m-k_m+1)}}{(m-n+1)^2} \leq \frac{2\sqrt{nk_m(m-k_m+1)}}{(m+1)^2},
 \end{aligned}$$

and similarly

$$I_3 \leq \frac{\sqrt{n}}{m-k_m+1} \int_0^1 u^{1/2} (1-u)^{3/2} f_{m-n,k_m-l_{mn}}(u) du \leq \frac{2\sqrt{nk_m(m-k_m+1)}}{(m+1)^2}.$$

Since g is bounded, we assume for simplicity that $|g(x)| \leq 1$. Let $D \geq 1$ be a constant such that $|g(x) - g(y)| \leq D|x - y|$. In fact, we have

$$(3.4) \quad |g(x) - g(y)| \leq 2D(|x - y| \wedge 1) \quad \text{for all } x, y \in R.$$

Let us set

$$\xi_n = g\left(\frac{U_{n,k_n} - d_n}{c_n}\right) - Eg\left(\frac{U_{n,k_n} - d_n}{c_n}\right).$$

For integers $n \leq m$, it follows from the independence of ξ_n and $V(n+1, m)$ and the condition (3.4) that

$$\begin{aligned} |E\xi_n \xi_m| &= \left| E\xi_n g\left(\frac{U_{m,k_m} - d_m}{c_m}\right) \right| \\ &= \left| E\xi_n g\left(\frac{V(n+1, m) - d_m}{c_m}\right) + E\xi_n \left\{ g\left(\frac{U_{m,k_m} - d_m}{c_m}\right) - g\left(\frac{V(n+1, m) - d_m}{c_m}\right) \right\} \right| \\ &= \left| E\xi_n \left[g\left(\frac{U_{m,k_m} - d_m}{c_m}\right) - g\left(\frac{V(n+1, m) - d_m}{c_m}\right) \right] \right| \\ &\leq 2DE \frac{|U_{m,k_m} - V(n+1, m)|}{c_m} \wedge 1. \end{aligned}$$

Therefore, it follows from (3.1) and (3.2) that there exists $m_1 > 0$ such that for $1 \leq n < m/(\log m)^2$ and $m \geq m_1$

$$(3.5) \quad |E\xi_n \xi_m| \leq \frac{4Dn}{m+1} \quad \text{if } \min\{k_m, m - k_m + 1\} \leq (\log m)^2$$

and

$$\begin{aligned} (3.6) \quad |E\xi_n \xi_m| &\leq \frac{12D\sqrt{n}}{\sqrt{m+1}} + \frac{8D\sqrt{m+1}}{\sqrt{k_m(m - k_m + 1)}} \\ &\leq \frac{12D\sqrt{n}}{\sqrt{m+1}} + \frac{8D\sqrt{m+1}}{\sqrt{[(m+1)/2] \min\{k_m, m - k_m + 1\}}} \\ &\leq \frac{12D\sqrt{n}}{\sqrt{m+1}} + \frac{8\sqrt{2}D}{\sqrt{\min\{k_m, m - k_m + 1\}}} \end{aligned}$$

in case $\min\{k_m, m - k_m + 1\} > (\log m)^2$. Note that the inequality

$$k_m(m - k_m + 1) \geq \frac{m+1}{2} \min\{k_m, m - k_m + 1\}$$

was also used in the estimation of (3.5).

Note that

$$\begin{aligned}
 I_2 &= \frac{1}{k_m} \int_0^1 |nu - l_{mn}| u f_{m-n, k_m - l_{mn}}(u) du \\
 &= \frac{n}{k_m} E |U_{m-n, k_m - l_{mn}} - (l_{mn}/n)| U_{m-n, k_m - l_{mn}} \\
 &\leq \frac{n}{k_m} \sqrt{E |U_{m-n, k_m - l_{mn}} - (l_{mn}/n)|^2 E U_{m-n, k_m - l_{mn}}^2} \\
 &= \frac{n}{k_m} \sqrt{(\text{Var}(U_{m-n, k_m - l_{mn}}) + (E(U_{m-n, k_m - l_{mn}}) - l_{mn}/n)^2) E U_{m-n, k_m - l_{mn}}^2} \\
 &= \frac{n}{k_m} \sqrt{\left\{ \frac{(k_m - l_{mn})(m - n - k_m + l_{mn} + 1)}{(m - n + 1)^2 (m - n + 2)} + \left(\frac{k_m - l_{mn}}{m - n + 1} - \frac{l_{mn}}{n} \right)^2 \right\} \frac{(k_m - l_{mn})(k_m - l_{mn} + 1)}{(m - n + 1)(m - n + 2)}} \\
 &\leq \frac{n}{k_m} \frac{k_m}{m - n + 1} \sqrt{\frac{k_m(m - k_m + 1)}{(m - n + 1)^3} + \left(\frac{nk_m - (m + 1)l_{mn}}{n(m - n + 1)} \right)^2}.
 \end{aligned}$$

Since

$$0 \leq nk_m - (m + 1)l_{mn} \leq nk_m - (m + 1) \left(\frac{nk_m}{m + 1} - 1 \right) \leq m + 1,$$

we get

$$I_2 \leq \frac{n}{m - n + 1} \left\{ \sqrt{\frac{k_m(m - k_m + 1)}{(m - n + 1)^3}} + \frac{m + 1}{n(m - n + 1)} \right\} \leq \frac{\sqrt{nk_m(m - k_m + 1)}}{(m + 1)^2} + \frac{2}{m + 1}$$

for all large m . Likewise, we can show that

$$I_4 \leq \frac{\sqrt{nk_m(m - k_m + 1)}}{(m + 1)^2} + \frac{2}{m + 1}$$

holds for all large m . Hence the lemma follows from the above estimates of I_1, \dots, I_4 .

Proof of Theorem 1. The main point of the proof comes from that of Theorem 1 of Berkes and Csáki [2], which provides a very powerful tool to solve this kind of problem. Unfortunately, the theorem cannot be applied directly in our case because it is hard to verify the condition (1.9) in Berkes and Csáki [2].

Like in Berkes and Csáki [2] (or see, e.g., Lacey and Philipp [10]), it suffices to show that for any bounded Lipschitz 1 function $g: R \rightarrow R$

$$(3.3) \quad \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \left\{ g \left(\frac{U_{n, k_n} - d_n}{c_n} \right) - E g \left(\frac{U_{n, k_n} - d_n}{c_n} \right) \right\} = 0 \text{ a.s.}$$

Additionally, the trivial estimation $|E\xi_n \xi_m| \leq 4$ holds for all $1 \leq n \leq m$. Now define

$$S_N = \{m \in [m_1 + 1, N] : \min \{k_m, m - k_m + 1\} > (\log m)^2\}$$

and

$$T_N = \{m \in [m_1 + 1, N] : \min \{k_m, m - k_m + 1\} \leq (\log m)^2\}.$$

Let us set

$$j_m = \min \{k_m, m - k_m + 1\}, \quad i_m = m/j_m \quad \text{and} \quad q_m = [m/(\log m)^2].$$

Then, as N is large enough,

$$\begin{aligned} \text{Var} \left(\sum_{n=1}^N \frac{1}{n} \left\{ g \left(\frac{U_{n,k_n} - d_n}{c_n} \right) - Eg \left(\frac{U_{n,k_n} - d_n}{c_n} \right) \right\} \right) &= \text{Var} \left(\sum_{n=1}^N \frac{1}{n} \xi_n \right) \\ &\leq 2 \sum_{m=1}^N \sum_{n=1}^m \frac{1}{mn} |E\xi_n \xi_m| \leq 2 \sum_{m=1}^{m_1} \sum_{n=1}^m \frac{1}{mn} |E\xi_n \xi_m| + 2 \sum_{m=m_1+1}^N \sum_{n=1}^m \frac{1}{mn} |E\xi_n \xi_m| \\ &\leq 8 \sum_{m=1}^{m_1} \sum_{n=1}^m \frac{1}{mn} + 8 \sum_{m=m_1+1}^N \sum_{n=q_m+1}^m \frac{1}{mn} + 2 \sum_{m=m_1+1}^N \sum_{n=1}^{q_m} \frac{1}{mn} |E\xi_n \xi_m| \\ &\leq 8m_1^2 + 8 \sum_{m=m_1+1}^N \frac{\log(m/q_m)}{m} \\ &\quad + 2 \sum_{m \in T_N} \sum_{n=1}^{q_m} \frac{1}{mn} |E\xi_n \xi_m| + 2 \sum_{m \in S_N} \sum_{n=1}^{q_m} \frac{1}{mn} |E\xi_n \xi_m| \\ &\leq 9(\log N)(\log \log N) + 2 \sum_{m \in T_N} \sum_{n=1}^{q_m} \frac{1}{mn} |E\xi_n \xi_m| + 2 \sum_{m \in S_N} \sum_{n=1}^{q_m} \frac{1}{mn} |E\xi_n \xi_m| \\ &=: 9(\log N)(\log \log N) + A_1 + A_2. \end{aligned}$$

It follows from (3.5) that

$$A_1 \leq 8D \sum_{m \in T_N} \sum_{n=1}^{q_m} \frac{1}{mn} \frac{n}{m+1} \leq 8D \sum_{m \in T_N} \frac{q_m}{m(m+1)},$$

which is bounded for all N . Finally, from (3.6)

$$\begin{aligned} A_2 &= 4D \sum_{m \in S_N} \sum_{n=1}^{q_m} \frac{1}{mn} \left(\frac{6\sqrt{n}}{\sqrt{m+1}} + \frac{4\sqrt{2}}{\sqrt{j_m}} \right) \\ &= 4D \sum_{m \in S_N} \sum_{n=1}^{q_m} \left(\frac{6}{m\sqrt{m+1}} \frac{1}{\sqrt{n}} + \frac{4\sqrt{2}}{m\sqrt{j_m}} \frac{1}{n} \right) \\ &\leq 24D \sum_{m \in S_N} \frac{\sqrt{m}}{m\sqrt{m+1}} + 16\sqrt{2}D \sum_{m \in S_N} \frac{\log m}{m\sqrt{j_m}} \\ &\leq 24D \log N + 16\sqrt{2}D \sum_{m \in S_N} \frac{1}{m} \leq (24 + 16\sqrt{2})D \log N \end{aligned}$$

for N large enough. Therefore, as N is large enough,

$$\text{Var} \left(\sum_{n=1}^N \frac{1}{n} \left\{ g \left(\frac{U_{n,k_n} - d_n}{c_n} \right) - Eg \left(\frac{U_{n,k_n} - d_n}{c_n} \right) \right\} \right) \leq 10(\log N)(\log \log N).$$

The rest follows by the arguments in Lacey and Philipp [10].

Proof of Theorem 2. Define the generalized inverse of F by

$$F^-(u) = \inf \{x: F(x) \geq u\} \quad \text{for } u \in (0, 1).$$

Then $\{X_i, i \geq 1\}$ is distributed in the same way as $\{F^-(U_i), i \geq 1\}$. For simplicity we assume that $X_i = F^-(U_i)$ for $i \geq 1$. In this case, $X_{n,k} = F^-(U_{n,k})$. It follows from (2.1) that

$$\limsup_{n \rightarrow \infty} \sup_x \left| P \left(\frac{U_{n,k_n} - d_n}{c_n} \leq x \right) - \Phi(x) \right| = 0.$$

Since

$$I \left(\frac{X_{n,k_n} - b_n}{a_n} \leq x \right) = I \left(\frac{U_{n,k_n} - d_n}{c_n} \leq \frac{F(a_n x + b_n) - d_n}{c_n} \right),$$

the condition (2.2) implies that

$$\lim_{n \rightarrow \infty} \Phi \left(\frac{F(a_n x + b_n) - d_n}{c_n} \right) = G(x) \quad \text{for any } x \in C(G).$$

This, together with Theorem 1, gives Theorem 2.

APPENDIX: JOINT DISTRIBUTION OF ORDER STATISTICS FROM NESTED SAMPLES

Let $\{U_n\}$ be a sequence of independent random variables with a uniform distribution over $(0, 1)$ and $U_{n,1} \leq U_{n,2} \leq \dots \leq U_{n,n}$ denote the order statistics of U_1, \dots, U_n . It is well known that the density function of $U_{n,k}$ is given by

$$(A1) \quad f_{n,k}(x) = n \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} \quad \text{for } x \in (0, 1)$$

and we have

$$E(U_{n,k}) = \frac{k}{n+1}, \quad E(U_{n,k}^2) = \frac{k(k+1)}{(n+1)(n+2)} \quad \text{and} \quad \text{Var}(U_{n,k}) = \frac{k(n-k+1)}{(n+1)^2(n+2)}.$$

See, e.g., Balakrishnan and Cohen [1].

Let $s \geq 1$ and $t \geq 1$ be integers. Then $U_{s,i}$ and $U_{s+t,j}$ are order statistics based on the sample U_1, \dots, U_s and $U_1, \dots, U_s, U_{s+1}, \dots, U_{s+t}$, where $1 \leq i \leq s$ and $1 \leq j \leq s+t$. We are interested in the joint distribution

of $U_{s,i}$ and $U_{s+t,j}$. Since both the marginal distributions of $U_{s,i}$ and $U_{s+t,j}$ are known, we derive the conditional distribution function of $U_{s+t,j}$ given $U_{s,i}$.

Let $K(u) = \sum_{j=s+1}^{s+t} I(U_j \leq u)$. Then $0 \leq K(u) \leq t$ if $u \in (0, 1)$. Obviously, $K(u)$ is a binomial random variable. Let $H(u, k) = P(U_{s,i} \leq u, K(U_{s,i}) = k)$. Then

$$H(du, k) = f_{s,i}(u) P(K(u) = k) = f_{s,i}(u) \binom{s}{k} u^k (1-u)^{s-k} du \quad \text{for } k = 0, 1, \dots, t.$$

For any integers n and r with $1 \leq r \leq n$, let $V_{n,r}$ be independent of U_m , $m \geq 1$, but have the same distribution as $U_{n,r}$.

Given $\{U_{s,i} = u, K(U_{s,i}) = k\} = \{U_{s,i} = u, K(u) = k\}$, we observe that $U_{s+t,j}$ has the same distribution as $uV_{i+k,j}$ in case $j < i+k$ since $U_{s+t,j}$ is the j -th smallest term among the $i+k$ random variables whose values are less than u , $U_{s+t,j} = u$ in case $j = i+k$, and $U_{s+t,j}$ has the same distribution as $u + (1-u)V_{i+k,j-k-i}$ in case $j > i+k$ since $U_{s+t,j}$ is the $(j-i-k)$ -th smallest term among the $s+t-i-k$ random variables whose values are greater than u . So the conditional distribution of $U_{s+t,j}$ given $\{U_{s,i} = u, K(U_{s,i}) = k\}$ is

$$(A2) \quad U_{s+t,j} \stackrel{d}{=} uV_{i+k-1,j} + (1-u)V_{s+t-i-k,j-i-k}.$$

Here we employ the conventional notation: $V_{n,m} = 0$ whenever $m \leq 0$ and $V_{n,m} = 1$ if $m > n$. Note that only one of $V_{i+k-1,j}$ and $V_{s+t-i-k,j-i-k}$ is random. We remark that (A2) is extremely useful for the calculation of the expectation of any function related to both $U_{s,i}$ and $U_{s+t,j}$ such as the covariance.

We conclude immediately the following theorem which is a generalization of Lemma A1 in Stadtmüller [14] when the underlying distribution is uniform.

THEOREM 3. For any bounded function $g(x, y)$ defined over $(0, 1)$ we have

$$\begin{aligned} & Eg(U_{s,i}, U_{s+t,j}) \\ &= \sum_{k=0}^t \int_0^1 Eg(u, uV_{i+k-1,j} + (1-u)V_{s+t-i-k,j-i-k}) f_{s,i}(u) \binom{t}{k} u^k (1-u)^{t-k} du, \end{aligned}$$

where $f_{s,i}$ is defined as in (A1). Further,

$$\begin{aligned} & Eg(u, uV_{i+k-1,j} + (1-u)V_{s+t-i-k,j-i-k}) \\ &= \begin{cases} \int_0^1 g(u, uv) f_{i+k-1,j}(v) dv & \text{if } j < i+k, \\ g(u, u) & \text{if } j = i+k, \\ \int_0^1 g(u, u+(1-u)v) f_{s+t-i-k,j-i-k}(v) dv & \text{if } j > i+k. \end{cases} \end{aligned}$$

If one selects $g(x, y) = I(x \leq w_1, y \leq w_2)$, then the joint distribution of $U_{s,i}$ and $U_{s+t,j}$ is obtained.

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