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# LARGE DEVIATION THEOREMS FOR WEIGHTED COMPOUND POISSON SUMS

BY

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*Abstract.* In this paper we present two large deviation results for weighted compound sums $\sum_{i=1}^{N} a_i X_i$, where $X_i$'s are i.i.d. (possibly lattice) random variables, $a_i$'s are non-negative real numbers, and $N$ is a Poisson variable. These results are generalizations of approximations for non-weighted compound sums and for non-compound weighted sums.

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## 1. INTRODUCTION

The problem of approximating large values of a compound sum $\sum_{i=1}^{N} X_i$, where the $X_i$'s are i.i.d. and $N$ is a discrete random variable, has been initially motivated by the description of the total claim amount of an insurance company over a fixed time period (see [6]). In classical setups, $N$ is a Poisson or a Polyà variable (see for instance [5] and [10]). Among generalizations of the compound Poisson sum, the finite mixture model (see [1]) and a model that incorporates inflation on $X_i$'s (see [13]) have been developed.

In this paper we present an approximation for the probability $P\left(\sum_{i=1}^{N} a_i X_i > y\right)$, where the $X_i$'s are i.i.d. random variables, $y$ and $a_i$'s are non-negative real numbers, and $N$ is a Poisson variable independent of the $X_i$'s. Hence we have to study the weighted compound sum $Y = \sum_{i=1}^{N} a_i X_i$. In this paper, two approximations for this weighted compound sum are given. The first one is an approximation for the probability $P\left(\sum_{i=1}^{N} a_i X_i > y\right)$ when $y \to \infty$. It is a generalization of the compound sum approximation found in [5], obtained for all $a_i$'s equal to 1. The second one is an approximation for the probability $P\left(\sum_{i=1}^{N} a_i X_i > c \sum_{i=1}^{N} a_i\right)$ when $E(N) \to \infty$. It is a generalization of the non-compound weighted sum approximation found in [3] and [2].

In order to establish our theorems, we classically use an "exponentially tilted" variable (see [10], p. 11, for a general definition of tilting). We denote by $\phi_U(t) = E(e^{tU})$ the Laplace transform of a random variable $U$, we let $\phi = \phi_X$, where $X$ is a variable with the same distribution as the $X_i$'s, and $Q(t) = \phi'(t)/\phi(t)$ for $t \in \boldsymbol{R}$. We recall that $Q$ is an increasing function, and we denote by $Q^{-1}$ the inverse of $Q$ with respect to composition. For all $h$'s such that $\phi_{Y-y}(h) < +\infty$, we consider a random variable $Y_h$ with distribution function $H_h$ satisfying

$$\frac{dH_h}{dH_0}(x) = \frac{e^{hx}}{\phi_{Y-y}(h)},$$

where $H_0$ is the distribution function of $Y-y$. The parameter $h = h(y)$ of the exponentially tilted variable $Y_h$ is chosen to be a solution to the equation $E(Y_h) = 0$. Then an approximation for the centre of the distribution of $Y_h$ provides a good approximation for the tail of the distribution of $Y-y$. The approximation obtained by means of exponentially tilting and a normal approximation (or a local central limit theorem) for the tilted variable is often called a *saddlepoint approximation.* General references for saddlepoint approximations can be found in [10] and [11].

To establish the local central limit theorem for $Y_h$, we rewrite it as a compound sum

$$Y_h = \sum_{i=1}^{N_h} X_{hi} - y,$$

where the $X_{hi}$'s are independent and independent of $N_h$. Observe that, as shown in the forthcoming Lemma 4, we have $E(N_h) \to \infty$ for both asymptotics $E(N) \to \infty$ and $y \to \infty$. Related techniques for sum approximations can be found in [7] and [12]. However, note that the asymptotic $y \to \infty$ requires uniformity in $h$ for the local central limit theorem. We refer to Höglund [8] for pioneering work on uniform approximations. In the i.i.d. case, Höglund obtained uniform expansions for bounded lattice variables and for absolutely continuous variables such that

$$P(X_1 \geqslant x) = (x_0 - x)^{\alpha}\, l(x_0 - x), \qquad x < x_0 < \infty,\ \alpha > 0,$$

where $l$ is a slowly varying function.

## 2. RESULTS

First we enumerate four conditions that are necessary to establish our results (see [3] and [2]). Set $s^2 = \operatorname{Var}(Y_h)$ and $\mu_i = E(Y_h^i)$ for $i = 3, 4$. Note that, as will be shown in the forthcoming Lemma 4, we have $s \to \infty$ as $y \to \infty$. Moreover, let $\phi_h$ be the Laplace transform of $Y_h$ and $\omega_h(\zeta) = \phi_h(i\zeta)$ be the characteristic function of $Y_h$. Finally, set $\sigma_1 = \min\{a_k\colon k \in N^*\}$, $\sigma_2 = \max\{a_k\colon k \in N^*\}$, $p_n = P(N = n)$ for $n \in N$, and $\lambda = E(N)$.

CONDITION I. There exist $\alpha$ and $\theta$ with $0 < \alpha \leqslant 1, 0 < \theta \leqslant 1$, such that, for all $n$, at least $\alpha n$ of the $a_k$'s, $k \in \{1, \ldots, n\}$, exceed or equal $\theta\sigma_2$.

Condition II is required for the asymptotic $y \to \infty$, and Condition II′ for the asymptotic $\lambda \to \infty$.

CONDITION II. The support of $X$ contains positive values and $\sup\{t\colon \phi(t) < \infty\} = +\infty$.

CONDITION II′. $\phi(t)$ is finite on an $\mathscr{I}$ containing $(-B, B)$ for some $B > 0$. Moreover, $Q$ assumes the value $c/(\alpha\theta)$ at some point and $B_0 = \theta^{-1} Q^{-1}(c/(\alpha\theta)) \in \mathscr{I}$.

When $X$ is absolutely continuous, Condition III is required for the asymptotic $y \to \infty$.

CONDITION III. For all real numbers $a$ and $\delta$ such that $a > \delta > 0$, and for a solution $h$ to the equation $E(Y_h) = 0$, we assume that $\omega_h(\zeta) = o(1/s^2)$ as $y \to \infty$ for $as > \zeta > \delta > 0$.

In the lattice case, we always need to assume the following

CONDITION IV. The $a_i$'s are such that $Y$ is a lattice with span $d$.

Condition IV is satisfied for instance if the $X_i$'s are defined on $N$ and the $a_i$'s can be written in the form $a_i = r_i/s$, where $r_i$, $s$ belong to $N$.

We now state our theorems. First we consider the asymptotic $y \to \infty$.

THEOREM 1. *Let us assume that Conditions* I *and* II *are satisfied and that* $\sigma_1 > 0$. *Let $h$ be a solution to the equation $E(Y_h) = 0$. Then, as $y \to \infty$, we have $s \to \infty$. Let us assume moreover that $(\mu_4 - 3s^4)/s^4 = O(1/s^2)$ and that $h/s \to 0$ as $y \to \infty$. Then*

$$P(Y > y) = \frac{1}{\sqrt{2\pi}} \frac{1}{sh} e^{-hy} \big(\phi_Y(h) - p_0\big)\big(1 + o(1)\big)$$

*as $y \to \infty$ if $X$ is absolutely continuous and Condition* III *is fulfilled, and*

$$P(Y > y) = \frac{1}{\sqrt{2\pi}} \frac{de^{-hd}}{s(1-e^{-hd})} e^{-hy} \phi_Y(h)\big(1 + o(1)\big)$$

*as $y \to \infty$ if $X$ is a lattice and Condition* IV *is fulfilled.*

To state our results for the asymptotic $\lambda \to \infty$, we now consider rather a random $y$ defined by $y = c\sum_{i=1}^{N} a_i$, with $c > 0$ fixed, and let $Z = Y - y$.

THEOREM 2. *Let us assume that Conditions* I *and* II′ *are satisfied and let $h$ be a solution to the equation $E(Y_h) = 0$. Then, as $\lambda \to \infty$, we have $s \to \infty$. Let us assume moreover that $\mu_4$ exists. Then*

$$P(Y > y) = \frac{1}{\sqrt{2\pi}} \frac{1}{sh} \big(\phi_Z(h) - p_0\big)\big(1 + o(1)\big)$$

*as* $\lambda \to \infty$ *if* $X$ *is absolutely continuous, and*

$$P(Y > y) = \frac{1}{\sqrt{2\pi}} \frac{de^{-hd}}{s(1-e^{-hd})} \phi_Z(h)(1+o(1))$$

*as* $\lambda \to \infty$ *if* $X$ *is a lattice and Condition* IV *is fulfilled.*

Remark 1. If $X$ is absolutely continuous, $Y$ is not absolutely continuous due to a point probability $p_0 = e^{-\lambda}$ at zero. We introduce classically (see [5]) a variable $\tilde{N}$ such that, for $y > 0$,

$$P(Y > y) = P(Y > y \mid N > 0) P(N > 0) = P\Big(\sum_{i=1}^{\tilde{N}} a_i X_i > y\Big)(1-p_0),$$

where $P(\tilde{N} = k) = (1-p_0)^{-1} p_k$ for $k \in N^*$. This ensures that $\tilde{Y} = \sum_{i=1}^{\tilde{N}} a_i X_i$ is absolutely continuous. We obtain easily the equality $E(e^{t\tilde{Y}}) = (1-p_0)^{-1}(\phi_Y(t) - p_0)$.

Remark 2. Propositions 7.2.3 and 7.2.5 in [10] give sufficient conditions for the fulfilment of the conditions of our theorems. A discussion about necessary conditions for the validity of our conditions can also be found in [4] and [9].

Remark 3. When $N = n$ is fixed, we obtain by Theorem 2 the results for weighted sums of i.i.d. random variables presented in [2]. When all $a_i = 1$, we obtain the classical results for Poisson compound sums [5]. However, our results are established under stronger moment conditions.

## 3. PROOFS

This section deals mainly with the proof of Theorem 1 in the lattice case. The proofs of the other results given in this paper follow the same outline with some weaker arguments. The main differences are presented at the end of this section.

Proof of Theorem 1 in the lattice case. In the first three lemmas, we consider that $h$ is such that $\phi_{Y-y}(h) < \infty$. We begin with a very classical result that connects $Y$ and $Y_h$:

LEMMA 1. *We have*

$$P(Y > y) = e^{-hy} \phi_Y(h) I(h),$$

*with* $I(h) = h\int_0^\infty \exp(-hx)[H_h^*(x) - H_h(0)]\,dx$, *where* $H_h^*$ *is defined by*

$$H_h^*(x) = \begin{cases} \frac{1}{2}[H_h(x) + H_h(x-)] & \text{if } x \text{ is on the lattice,} \\ H_h(x) & \text{otherwise.} \end{cases}$$

Proof. This follows from Fubini's theorem. See for example [7]. ∎

Then we rewrite $Y_h$ as a compound sum $Y_h = \sum_{i=1}^{N_h} X_{hi} - y$. The identification of $N_h$ and $X_{hi}$ is performed with the help of the Laplace transform of $Y_h$. Let $\phi_{hi}(t) = E(\exp(tX_{hi}))$ and $q_k = P(N_h = k)$.

LEMMA 2. *We have*

$$q_k = \frac{\prod_{i=1}^{k} \phi(a_i h)}{\sum_{l=0}^{\infty} p_l \prod_{i=1}^{l} \phi(a_i h)} p_k \tag{1}$$

*and*

$$\phi_{hi}(t) = \frac{\phi(a_i(t+h))}{\phi(a_i h)}. \tag{2}$$

Observe that the $X_{hi}$'s are exponentially tilted variables associated with the $a_i X_i$'s.

Proof. As for $t > 0$

$$\phi_h(t) = E(\exp(tY_h)) = \frac{1}{\phi_{Y-y}(h)} \int_0^{\infty} e^{(t+h)x} dH_0(x) = e^{-ty} \frac{\phi_Y(t+h)}{\phi_Y(h)}$$

and

$$\phi_Y(t) = E(e^{tY}) = E(E(e^{tY} \mid N)) = \sum_{k=0}^{\infty} p_k \prod_{i=1}^{k} \phi(a_i t),$$

where by convention $\prod_{i=1}^{0} \phi(a_i t) = 1$, we obtain for $t > 0$

$$\phi_h(t) = e^{-ty} \frac{\sum_{k=0}^{\infty} p_k \prod_{i=1}^{k} \phi(a_i(t+h))}{\sum_{k=0}^{\infty} p_k \prod_{i=1}^{k} \phi(a_i h)} = e^{-ty} \sum_{k=0}^{\infty} q_k \prod_{i=1}^{k} \phi_{hi}(t),$$

which completes the proof. ∎

Now, we give bounds for $E(N_h)$ and $\mathrm{Var}(N_h)$.

LEMMA 3. *We have*

$$\lambda\phi(\sigma_1 h) \leqslant E(N_h) \leqslant \lambda\phi(\sigma_2 h) \tag{3}$$

*and*

$$\lambda\phi(\sigma_1 h) - \lambda^2(\phi^2(\sigma_2 h) - \phi^2(\sigma_1 h)) \leqslant \mathrm{Var}(N_h) \leqslant \lambda\phi(\sigma_2 h) + \lambda^2(\phi^2(\sigma_2 h) - \phi^2(\sigma_1 h)). \tag{4}$$

Proof. Let us compute $E(N_h)$, using the notation $\psi_k(h) = \prod_{i=1}^{k} \phi(a_i h)$. We obtain

$$E(N_h) = \frac{\sum_{k=0}^{\infty} k e^{-\lambda}(\lambda^k/k!)\psi_k(h)}{\sum_{k=0}^{\infty} e^{-\lambda}(\lambda^k/k!)\psi_k(h)} = \lambda \frac{\sum_{k=0}^{\infty} e^{-\lambda}(\lambda^k/k!)\psi_{k+1}(h)}{\sum_{k=0}^{\infty} e^{-\lambda}(\lambda^k/k!)\psi_k(h)}.$$

Observe that, by the definition of $\sigma_1$ and $\sigma_2$, we have

$$\phi(\sigma_1 h)\psi_k(h) \leqslant \psi_{k+1}(h) \leqslant \psi_k(h)\phi(\sigma_2 h),$$

so that we obtain the following inequalities:

$$\lambda\phi(\sigma_1 h) \leqslant E(N_h) \leqslant \lambda\phi(\sigma_2 h).$$

Very similar computations give the bounds for $\mathrm{Var}(N_h)$. ∎

We now choose a solution $h$ to the equation $E(Y_h) = 0$.

LEMMA 4. *There exists a solution $h$ to the equation $E(Y_h) = 0$ such that $h \to \infty$, $E(N_h) \to \infty$ and $\mathrm{Var}(Y_h) \to \infty$ as $y \to \infty$. Moreover,*

$$(5) \quad Q'(\theta\sigma_2 h)\alpha\theta^2\sigma_2^2 E(N_h) \leqslant \mathrm{Var}(Y_h) \leqslant Q'(\sigma_2 h)\sigma_2^2 E(N_h) + (\sigma_2 Q(\sigma_2 h))^2 \mathrm{Var}(N_h).$$

Proof. Using our representation of $Y_h$ with $N_h$ and the $X_{hi}$'s, and observing that $E(X_{hi}) = a_i Q(a_i h)$ by (2), we obtain

$$E(Y_h) = E\Big(E\big(\sum_{i=1}^{N_h} X_{hi} \mid N_h\big)\Big) - y = \sum_{k=1}^{\infty} q_k \sum_{i=1}^{k} a_i Q(a_i h) - y.$$

Thus, the equation $E(Y_h) = 0$ is equivalent to $\sum_{k=0}^{\infty} q_k \sum_{i=1}^{k} a_i Q(a_i h) = y$. By Condition I, we have

$$(6) \quad \sum_{k=1}^{\infty} q_k \sum_{i=1}^{k} a_i Q(a_i h) \geqslant \alpha\theta\sigma_2 Q(\theta\sigma_2 h) E(N_h).$$

Condition II ensures that $Q$ is positive for sufficiently large $h$'s and that $\lim_{t\to\infty} Q(t) = \infty$, and as $\sigma_1 > 0$, (3) proves that $E(N_h) \to \infty$ as $h \to \infty$. Thus, by (6), the existence of a solution $h$ to the equation $E(Y_h) = 0$ is proved.

We now prove that $h \to \infty$ as $y \to \infty$. Similarly to (6) we have

$$0 = E(Y_h) \leqslant \sum_{k=0}^{\infty} q_k k\sigma_2 Q(\sigma_2 h) - y.$$

It follows that

$$(7) \quad Q(\sigma_2 h) = \frac{\phi'(\sigma_2 h)}{\phi(\sigma_2 h)} \geqslant \frac{y}{\sigma_2 E(N_h)}.$$

Then, using (3) and (7), we obtain

$$\frac{y}{\sigma_2}\frac{\phi(\sigma_2 h)}{\phi'(\sigma_2 h)} \leqslant E(N_h) \leqslant \lambda\phi(\sigma_2 h),$$

which yields to

$$\phi'(\sigma_2 h) \geqslant y/(\sigma_2\lambda).$$

This proves that, as $y \to \infty$, $\phi'(\sigma_2 h) \to \infty$, and so $h \to \infty$. Observe that in view of (3) we have obviously $E(N_h) \to \infty$ as $y \to \infty$.

We now study $\operatorname{Var}(Y_h)$. Since

$$\operatorname{Var}(Y_h) = E\big[\sum_{i=1}^{N_h} \operatorname{Var}(X_{hi})\big] + \operatorname{Var}\big[\sum_{i=1}^{N_h} E(X_{hi})\big],$$

and $E(X_{hi}) = a_i Q(a_i h)$, $\operatorname{Var}(X_{hi}) = a_i^2 Q'(a_i h)$, we obtain

$$\operatorname{Var}(Y_h) = \sum_{k=0}^{\infty} q_k \sum_{i=1}^{k} a_i^2 Q'(a_i h) + \operatorname{Var}\big[\sum_{i=1}^{N_h} a_i Q(a_i h)\big].$$

By Condition I, we have for all solutions $h$ to the equation $E(Y_h) = 0$

$$Q'(\theta\sigma_2 h)\alpha\theta^2 \sigma_2^2 E(N_h) \leqslant \operatorname{Var}(Y_h) \leqslant Q'(\sigma_2 h)\sigma_2^2 E(N_h) + (\sigma_2 Q(\sigma_2 h))^2 \operatorname{Var}(N_h),$$

and we have proved that, as $y \to \infty$, $\operatorname{Var}(Y_h) \to \infty$. ■

From now on, we always consider that $h$ is a solution to the equation $E(Y_h) = 0$. Let $d_h$ be the span of $Y_h/s$. We approximate $H_h^*(sx)$ by means of $H_h^{\#}(sx)$ convolution of $H_h$ by the triangular distribution on $[-d_h/2, d_h/2]$. Then we use a local central limit theorem for $H_h^{\#}(sx)$ that we now state. Let us define

$$G(x) = \mathfrak{N}(x) + \frac{\mu_3}{6s^3}(1-x^2)\mathfrak{n}(x) + \frac{\mu_3^2}{76s^6}(-15x+10x^3-x^5)\mathfrak{n}(x)$$
$$+\frac{\mu_4-3s^4}{24s^4}(3x-x^3)\mathfrak{n}(x),$$

where $\mathfrak{N}(x)$ is the distribution function and $\mathfrak{n}(x)$ the density of a standard Gaussian variable.

THEOREM 3. *If $Y_h$ is such that $(\mu_4-3s^4)/s^4 = O(1/s^2)$ when $y \to \infty$, then*

$$H_h^{\#}(sx) = G(x) + s^{-2} r_s(x),$$

*where $r_s(x) \to 0$ as $y \to \infty$ uniformly on $[0, +\infty]$.*

Proof. Let us denote by $G^{\#}$ the convolution of $G$ by the triangular distribution on $[-d_h/2, d_h/2]$:

$$G^{\#}(x) = \frac{2}{d_h}\int_{-d_h/2}^{d_h/2}\left(1-\frac{2|y|}{d_h}\right)G(x-y)\,dy.$$

As $|G'(x)| \to 0$ uniformly with respect to $x$ as $y \to \infty$, it follows from the two--term Taylor expansion of $G$ at the point $x$ that

$$|G^{\#}(x) - G(x)| < d_h^2\, o(1).$$

As $d_h$ is of order $1/s$, to prove the theorem it suffices to show that, uniformly with respect to $x$,

$$|H_h^{\#}(sx) - G^{\#}(x)| = o(1/s^2).$$

We rewrite $\omega_h(\zeta) = \phi_h(i\zeta)$ as $e^{v(\zeta)}$. Let us note that $\omega_h(0) = 1$, $\omega_h'(0) = E(Y_h) = 0$, $\omega_h''(O) = i^2 s^2$, $\omega_h^{(3)}(0) = i^3 \mu_3$ and $\omega_h^{(4)}(0) = i^4 \mu_4$. Successive derivations show that we have $v(0) = 0$, $v'(0) = 0$, $v''(0) = i^2 s^2$, $v'''(0) = i^3 \mu_3$ and $v^{(4)}(0) = i^4(\mu_4 - 3s^4)$. Let

$$\beta = \frac{v'''(0)}{6s^3} i^3 \zeta^3 + \frac{v^{(4)}(0)}{24 s^4} i^4 \zeta^4.$$

We let $C^{\#} = H_h^{\#} - D^{\#}$, where $D^{\#}$ has the Fourier transform $\exp(-\frac{1}{2}\zeta^2) + \exp(-\frac{1}{2}\zeta^2)(\beta + \beta^2/2)$.

Assume that $\varepsilon > 0$ is fixed. As $(\mu_4 - 3s^4)/s^4 = O(1/s^2)$, we have by Lemma 7.2.1 of [10] that $|G'(x)| \to 0$ as $y \to \infty$, and we can choose a constant $a$ so large that $|G'(x)| < \varepsilon a$ for all $x$ and $y$. By a smoothing theorem ([7], p. 538) we have

(8) $$|C^{\#}(x)| \leqslant \int_{-as^2}^{as^2} \left| \frac{\exp(v(\zeta/s)) - \exp(-\frac{1}{2}\zeta^2) - \exp(-\frac{1}{2}\zeta^2)(\beta + \beta^2/2)}{\zeta} \right| |v(\zeta)|\, d\zeta + \frac{24\varepsilon}{\pi s^2},$$

where

$$v(\zeta) = \frac{\sin^2(\frac{1}{2} d_h \zeta)}{(\frac{1}{2} d_h \zeta)^2}$$

is the characteristic function of the triangular distribution. We partition the interval of integration into two parts. The first one is defined by $\delta s \leqslant |\zeta| \leqslant as^2$, and the latter one by $|\zeta| < \delta s$, where $\delta$ is a fixed positive real number. On the domain $\delta s \leqslant |\zeta| \leqslant as^2$, we have

$$\int_{\delta s}^{as^2} \frac{|\exp(v(\zeta/s))\, v(\zeta)|}{\zeta} d\zeta = \frac{4}{(d_h s)^2} \int_{\delta}^{as} \frac{|e^{v(y)} \sin^2((d_h s y)/2)|}{y^3} dy.$$

As $e^{v(y)}$ and $\sin^2((d_h s y)/2)$ have period $2\pi/(d_h s)$, it is sufficient to prove that

(9) $$\int_0^{\pi/(d_h s)} \frac{|E_{N_h}(\phi_{h1}(iy) \ldots \phi_{hN_h}(iy))|}{y^3} dy = o(1/s^2),$$

which is true because within a neighbourhood of the origin

$$|E_{N_h}(\phi_{h1}(iy) \ldots \phi_{hN_h}(iy))| < \exp(-y^2/4)$$

(see [7], p. 516), and outside of this neighbourhood $|E_{N_h}(\phi_{h1}(iy) \ldots \phi_{hN_h}(iy))|$ is bounded away from 0, and hence the integrand in (9) decreases faster than any power of $n$. Thus, the contribution of this domain to the integral (8) is less than

$$o(1/s^2)+\int_{\delta s\leqslant|\zeta|\leqslant as}\frac{\exp(-\frac{1}{2}\zeta^2)}{\zeta}(1+|\beta+\beta^2/2|)\,d\zeta,$$

and this is an $o(1/s^2)$.

On the domain $|\zeta|<\delta s$, we let

$$\psi(\zeta)=v(\zeta)+\tfrac{1}{2}s^2\zeta^2$$

and the integrand in (8) can be rewritten in the form

$$\frac{\exp(-\frac{1}{2}\zeta^2)}{\zeta}\left|\exp(\psi(\zeta/s))-1-\beta-\beta^2/2\right|\,|\nu(\zeta)|\,d\zeta$$

and estimated by using the following inequality from [7]:

$$|e^{\alpha}-1-\beta-\beta^2/2|\leqslant(|\alpha-\beta|+\tfrac{1}{6}|\beta|^3)\,e^{\gamma}, \tag{10}$$

where $\gamma\geqslant\max(|\alpha|,|\beta|)$ for any $\alpha$ and $\beta$, real or complex. We use a four-term Taylor expansion for $\psi$. As $v^{(4)}$ is continuous and $v^{(4)}(0)/s^4=O(1/s^2)$, we deduce that there exists a $\delta$ such that

$$\left|\psi\left(\frac{\zeta}{s}\right)-\frac{v'''(0)}{6s^3}i^3\zeta^3-\frac{v^{(4)}(0)}{24s^4}i^4\zeta^4\right|<\varepsilon s^2\left|\frac{\zeta}{s^4}\right|^4\qquad\text{for }|\zeta|<\delta s.$$

We choose $\delta$ so small that

$$\left|\psi\left(\frac{\zeta}{s}\right)\right|<\frac{1}{4}\zeta^2,\qquad\left|\frac{v'''(0)}{6s^3}\zeta^3-\frac{v^{(4)}(0)}{24s^4}\zeta^4\right|\leqslant\frac{1}{4}\zeta^2\qquad\text{for }|\zeta|<\delta s.$$

With this choice of $\delta$ we majorize the integral (8) on the domain $|\zeta|<\delta s$ using (10):

$$\int_{|\zeta|<\delta s}\exp(-\tfrac{1}{4}\zeta^2)\left|\frac{\varepsilon}{s^2}|\zeta|^3+\frac{1}{6\zeta}\left|i^3\frac{\mu_3}{6s^3}\zeta^3+i^4\frac{\mu_4-3s^4}{24s^4}\zeta^4\right|^3\right|\,|\nu(\zeta)|\,d\zeta. \tag{11}$$

We choose $y$ so large that the integral (11) is less than $(1000\varepsilon)/s^2$, and we have proved that for all $x$

$$|C^{\#}(x)|\leqslant\frac{24\varepsilon}{\pi s^2}+\frac{\varepsilon}{s^2}+\frac{1000\varepsilon}{s^2}+o\left(\frac{1}{s^2}\right),$$

and as $\varepsilon$ is arbitrary, we conclude that $C^{\#}(x)=o(1/s^2)$ uniformly with respect to $x$. We get the desired expansion by dropping in $D^{\#}$ the terms of the polynomial involving powers $1/s^k$, with $k\geqslant 2$, to obtain $G^{\#}$. ∎

We now return to the proof of the large deviation theorem. Let us recall that we have to estimate $I(h)$:

$$I(h) = h \int_0^\infty e^{-hx} [H_h^\#(x) - H_h(0)]\, dy.$$

Since $Y$ is a lattice with span $d$, this integral can be rewritten as

$$I(h) = \sum_{k=0}^{\infty} h \int_{kd}^{(k+1)d} e^{-hx} [H_h^*((k+1/2)\,d) - H_h(0)]\, dx.$$

As $H_h^*$ and $H_h^\#$ are equal at mid points of the lattice, we can apply Theorem 3. We note that $H_h(0) = H_h^\#(d/2)$ so that we have

$$\begin{aligned} I(h) &= \sum_{k=0}^{\infty} h \int_{kd}^{(k+1)d} e^{-hx} \left[ G\left( (k+1/2)\frac{d}{s} \right) - G\left( \frac{d}{2s} \right) \right] dx + o(1/s^2) \\ &= \sum_{k=0}^{\infty} (e^{-hkd}\, e^{-h(k+1)d}) \left[ G\left( (k+1/2)\frac{d}{s} \right) - G\left( \frac{d}{2s} \right) \right] + o(1/s^2) \\ &= \sum_{k=0}^{\infty} (e^{-hkd}\, e^{-h(k+1)d}) \int_{d/(2s)}^{((k+1/2)d)/s} G'(x)\, dx + o(1/s^2). \end{aligned}$$

Observe that

$$\begin{aligned} G'(x) &= \mathrm{n}(x) + \frac{\mu_3}{6s^3}(x^3 - 3x)\,\mathrm{n}(x) + \frac{\mu_3^2}{76s^6}(x^6 - 15x^4 + 45x^2 - 15)\,\mathrm{n}(x) \\ &+ \frac{\mu_4 - 3s^4}{24s^4}(x^4 - 6x^2 + 3)\,\mathrm{n}(x) = \mathrm{n}(x) + G_1(x). \end{aligned}$$

First we compute the contribution of $\mathrm{n}(x)$:

$$J(h) = \sum_{k=0}^{\infty} (e^{-hkd}\, e^{-h(k+1)d}) \frac{1}{\sqrt{2\pi}} \int_{d/(2s)}^{((k+1/2)d)/s} \exp\left( -\frac{x^2}{2} \right) dx = J_1(h) + J_2(h),$$

where $J_1(h)$ means the sum from 0 to $[s^{1/4}]$, and $J_2(h)$ the sum from $[s^{1/4}]+1$ to $+\infty$, and $[x]$ denotes the greatest integer contained in $x$. For $J_1(h)$ we use the expansion $\exp(-x^2/2) = 1 - x^2/2 + o(x^2)$, and so we obtain

$$J_1(h) = \sum_{k=0}^{[s^{1/4}]} (e^{-hkd} - e^{-h(k+1)d}) \frac{kd}{\sqrt{2\pi}\, s} + o(1/s^2).$$

An upper bound for $J_2(h)$ is

$$J_2(h) < 1/2 \sum_{k=[s^{1/4}]+1}^{\infty} (e^{-hkd} - e^{-h(k+1)d})$$

and is an $o(1/s^2)$. Since

$$\sum_{k=0}^{[s^{1/4}]} ke^{-hkd} = \frac{e^{-hd}}{(1-e^{-hd})^2} + o(1/s^2),$$

we obtain

$$J(h) = \frac{e^{-hd}}{1-e^{-hd}} + o(1/s^2).$$

Since $(\mu_4 - 3s^4)/s^4 = O(1/s^2)$, the contribution of $G_1(x)$ to the integral is an $O(1/s^2)$, and thus, as $h/s \to 0$ when $y \to \infty$, we obtain

$$I(h) = \frac{1}{\sqrt{2\pi}} \frac{e^{-hd} d}{s(1-e^{-hd})}(1+o(1)),$$

which completes the proof of the theorem. ∎

Elements of proof of Theorem 1 in the absolutely continuous case. In the absolutely continuous case, we work with $\tilde{Y}$ that is absolutely continuous, as shown in Remark 1. As $Y_h$ is absolutely continuous, the smoothing of $H_{ch}$ is not necessary and we use a version of Theorem 3 with $H_{ch}$ instead of $H_{ch}^{\#}$. The remaining of the proof is similar, with a direct computation of $I(h)$. ∎

Elements of proof of Theorem 2. The asymptotic $\lambda \to \infty$ is easier to handle because a solution $h$ of the equation $E(Y_h) = 0$ is uniformly bounded. The outline of the proof is similar to the one of Theorem 1, but we need only a weaker version of Theorem 3 (an expansion in $o(1/s)$ is sufficient because $1/hs = O(1/s)$). ∎

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