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# ON THE EXISTENCE OF MOMENTS OF STOPPED SUMS IN MARKOV RENEWAL THEORY*

BY

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*Abstract.* Let $(M_n)_{n\geqslant 0}$ be an ergodic Markov chain on a general state space $X$ with stationary distribution $\pi$ and $g: X \to [0, \infty)$ a measurable function. Define $S_0(g) \stackrel{\text{def}}{=} 0$ and $S_n(g) \stackrel{\text{def}}{=} g(M_1)+\ldots+g(M_n)$ for $n \geqslant 1$. Given any stopping time $T$ for $(M_n)_{n\geqslant 0}$ and any initial distribution $\nu$ for $(M_n)_{n\geqslant 0}$, the purpose of this paper is to provide suitable conditions for the finiteness of $E_\nu S_T(g)^p$ for $p > 1$. A typical result states that

$$E_\nu S_T(g)^p \leqslant C_1\left(E_\nu S_T(g^p)+E_\nu T^p\right)+C_2$$

for suitable finite constants $C_1$, $C_2$. Our analysis is based to a large extent on martingale decompositions for $S_n(g)$ and on drift conditions for the function $g$ and the transition kernel $P$ of the chain. Some of the results are stated under the stronger assumption that $(M_n)_{n\geqslant 0}$ is positive Harris recurrent in which case stopping times $T$ which are regeneration epochs for the chain are of particular interest. The important special case where $T = T(t) \stackrel{\text{def}}{=} \inf\{n \geqslant 1: S_n(g) > t\}$ for $t \geqslant 0$ is also treated.

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## 1. INTRODUCTION

Stopped sums of real-valued random variables appear in many areas of probability and statistics. The analysis of such so-called stopped random walks frequently leads to the problem of finding verifiable necessary and/or sufficient conditions for the existence of their moments. Classical random walks $(S_n)_{n\geqslant 0}$ with i.i.d. increments $X_1, X_2, \ldots$ form an extensively studied class of sequences in this respect; a monography by Gut [9] gives a good account of the rele-

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vant results and references. In this paper we will focus on the case where $(S_n)_{n\geqslant 0}$ forms the additive component of a Markov random walk $(M_n, S_n)_{n\geqslant 0}$ driven by an ergodic Markov chain (see Section 2 for definitions) for which much less seems to be known. The main problem we will address can be phrased as follows:

Given $p>1$ and a stopping time $T$ for $(M_n, S_n)_{n\geqslant 0}$, find conditions on $(M_n)_{n\geqslant 0}$, the increments $X_1, X_2, \ldots$ of $(S_n)_{n\geqslant 0}$ and on $T$ such that $E|S_T|^p<\infty$.

Our main motivation for this work arose from the special case where $(M_n)_{n\geqslant 0}$ is a positive Harris chain and $T$ an associated regeneration epoch. To explain, if $(M_n)_{n\geqslant 0}$ is positive Harris recurrent, then it can be decomposed into 1-dependent cycles $(M_k)_{\sigma_n\leqslant k<\sigma_{n+1}}$, $n\geqslant 0$, which are further stationary with finite average length for $n\geqslant 1$. The stopping times $0=\sigma_0<\sigma_1<\ldots$, called *regeneration epochs*, form a renewal process (see Section 2 for further details). This decomposition is of great importance when dealing with various functionals of $(M_n, S_n)_{n\geqslant 0}$ arising in the context of Markov renewal theory, a typical example being the moments of the excess over the boundary at first passage beyond $t\geqslant 0$, i.e. $E(S_{T(t)}-t)^p$ for $p>0$ with $T(t)\overset{\text{def}}{=}\inf\{n\geqslant 1\colon S_n>t\}$. In order to obtain finiteness of $E(S_{T(t)}-t)^p$ or even convergence as $t\to\infty$, the regenerative approach typically imposes suitable moment conditions on the *derived* variable $S_{\sigma_1}$ rather than on the variables $X_1, X_2, \ldots$ given by the model (see, for example, [1]). Hence it does not come by surprise that a lack of verifiable model assumptions for the existence of moments of $S_{\sigma_1}$ is frequently held against the use of regeneration methods despite its mathematical elegance.

Not at least because of their role in renewal theory the afore-mentioned first passage times $T(t)$ have been extensively studied for classical random walks with nonnegative drift ($\mu\overset{\text{def}}{=}EX_1\in[0,\infty]$). A classical result by Gut (see [9]) states that in case $\mu>0$ the equivalences

$$E(X_1^+)^p<\infty\Leftrightarrow ES_{T(t)}^p<\infty, \tag{1.1}$$

$$E(X_1^-)^p<\infty\Leftrightarrow ET(t)^p<\infty \tag{1.2}$$

hold for all $p\geqslant 1$ and $t\geqslant 0$, where $X^+\overset{\text{def}}{=}\max\{X,0\}$ and $X^-\overset{\text{def}}{=}-\min\{X,0\}$ as usual.

Relaxing the independence assumption on $X_1, X_2, \ldots$ only a little leads to random walks with stationary $l$-dependent increments for some $l\geqslant 1$. Janson [10] showed for such $(S_n)_{n\geqslant 0}$ with positive drift that (1.1) as well as "$\Rightarrow$" of (1.2) remain true. But he also gave a simple counterexample ([10], Example 2.1) that proved the converse "$\Leftarrow$" of (1.2) be false in general. In fact, let $Y_0, Y_1, \ldots$ be i.i.d. positive random variables with $P(Y_1<1)>0$, finite mean and $EY_1^p=\infty$ for some $p>1$. Put $X_n=\varphi(Y_{n-1}, Y_n)$ for $n\geqslant 1$, where

$$\varphi(y_1, y_2)=\begin{cases}-y_2 & \text{if } y_1<1<y_2,\\ 1+y_1 & \text{otherwise.}\end{cases}$$

Then $(S_n)_{n\geqslant 0}$ has stationary 1-dependent increments with positive mean and $E(X_1^-)^p = \infty$. On the other hand, $T(t) \leqslant 2(t+1)$ for all $t \geqslant 0$ because $\max(X_n, X_n + X_{n+1}) \geqslant 1$ for all $n \geqslant 1$.

This negative example deserves an explicit mention here for two reasons. First, it belongs to the class of Markov random walks $(M_n, S_n)_{n\geqslant 0}$ driven by an ergodic Markov chain $(M_n)_{n\geqslant 0}$; in fact, $M_n \stackrel{\text{def}}{=} (Y_{n-1}, Y_n)$ for $n \geqslant 1$ and this chain is even uniformly Harris ergodic. Second, it shows that moment results for stopped sums of i.i.d. random variables can already break down under very moderate deviations from the i.i.d. assumption.

As briefly explained at the beginning of Section 3, it suffices to consider sums of the form

$$S_n = S_n(g) \stackrel{\text{def}}{=} \sum_{k=1}^{n} g(M_k)$$

for measurable real-valued, mostly even nonnegative functions $g$. Our conditions for the existence of moments of $S_T(g)$, where attention is also given to the case $T = T(t)$, will be in terms of the Markov chain $(M_n)_{n\geqslant 0}$, notably its initial distribution $\nu$, its stationary distribution $\pi$ and its transition kernel $P$, and of the increment distributions under $P_\nu$ and $P_\pi$. More precisely, the majority of results, stated in Section 3, will provide upper bounds for $E_\nu S_T(g)^p$ in case of nonnegative $g$ which besides finite terms involve $E_\nu T^p$ and $E_\nu S_{T+m}(g^p)$ for some $m \in N_0$. Assuming $E_\nu T^p < \infty$, finiteness of $E_\nu S_T(g)^p$ hence follows if $E_\nu S_{T+m}(g^p) < \infty$. This is a great simplification because finiteness of the latter expectation can be checked by using a Wald-type equation for MRW's recently obtained in [6] and [7]. For instance, if $m = 0$, $T$ is a regeneration epoch and $\nu$ the distribution of $M_T$, then $E_\nu S_T(g^p) = E_\pi g^p(M_0) E_\nu T$ holds (even more directly from the regenerative representation of $\pi$ as a normalized occupation measure, see (2.4) in the next section). The main tools in our proofs will be:

- suitable *martingale representations* of $S_n(g)$, which are therefore presented in Section 3 and by which we will be able to utilize the powerful Burkholder inequality, and
- *drift conditions* which are also of great importance in the analysis of ergodic Markov chains.

More information on this, including a more detailed model description and some basic facts on Harris recurrence, are collected in Section 2. All proofs can be found in Section 5.

## 2. MODEL DESCRIPTION AND PREREQUISITES

Let $(X, \mathfrak{B}(X))$ be a measurable space with countably generated $\sigma$-field (in most applications a Polish space with Borel $\sigma$-field), $\mathfrak{B}(R)$ the Borel $\sigma$-field on $R$ and $\mathsf{P}$: $X \times (\mathfrak{B}(X) \otimes \mathfrak{B}(R)) \to [0, 1]$ a transition kernel. Let further $(M_n, X_n)_{n\geqslant 0}$

be an associated Markov chain, defined on a probability space $(\Omega, \mathfrak{A}, P)$, with state space $X\times \boldsymbol{R}$, i.e.

(2.1) $$P(M_{n+1}\in A,\, X_{n+1}\in B\,|\,M_n,\, X_n) = \mathsf{P}(M_n,\, A\times B) \text{ a.s.}$$

for all $n\geqslant 0$ and $A\in\mathfrak{B}(X)$, $B\in\mathfrak{B}(\boldsymbol{R})$. Thus $(M_{n+1}, X_{n+1})$ depends on the past only through $M_n$. It is easily seen that $(M_n)_{n\geqslant 0}$ forms a Markov chain with state space $X$ and transition kernel $P(x, A) \stackrel{\text{def}}{=} \mathsf{P}(x, A\times \boldsymbol{R})$. Given the driving (or modulating) chain $(M_j)_{j\geqslant 0}$, the $X_n$ are conditionally independent with

(2.2) $$P\big(X_n\in B\,|\,(M_j)_{j\geqslant 0}\big) = Q(M_{n-1},\, M_n,\, B) \text{ a.s.}$$

for all $n\geqslant 1$, $B\in\mathfrak{B}(\boldsymbol{R})$ and a kernel $Q\colon X^2\times\mathfrak{B}(\boldsymbol{R})\to[0, 1]$. The sequence $(M_n, S_n)_{n\geqslant 0}$, where $S_n \stackrel{\text{def}}{=} X_0+\ldots+X_n$, is called a *Markov random walk* (MRW).

Let throughout a standard model be given with probability measures $P_x$, $x\in X$, on $(\Omega, \mathscr{A})$ such that $P_x(M_0 = x, X_0 = 0) = 1$. If $\nu$ denotes any distribution on $X$, put $P_\nu(\cdot) = \int_X P_x(\cdot)\,\nu(dx)$ in which case $(M_0, X_0)$ has initial distribution $\nu\otimes\delta_0$ under $P_\nu$. Expectation under $P_\nu$ is denoted by $E_\nu$. $P$ and $E$ are used for probabilities and expectations, respectively, which are independent of the initial distribution. As usual, $P^m(x, \cdot)$ denotes the $m$-step transition kernel of $(M_n)_{n\geqslant 0}$ and

$$\nu P^m(B) \stackrel{\text{def}}{=} \int_X P^m(x, B)\,\nu(dx).$$

Finally, put

$$\nu(f) \stackrel{\text{def}}{=} \int_X f(x)\,\nu(dx) \quad\text{ and }\quad P^m f(x) \stackrel{\text{def}}{=} \int_X f(y)\,P^m(x, dy)$$

for any function $f\colon X\to\boldsymbol{R}$ for which the respective integrals exist.

A standing assumption throughout this article is that $(M_n)_{n\geqslant 0}$ has a unique stationary distribution $\pi$ and forms an ergodic process under $P_\pi$. Some of our results will also make the stronger assumption of positive Harris recurrence, which means that $\pi$ is also a maximal irreducibility measure for $(M_n)_{n\geqslant 0}$ and that every $\pi$-positive set $A\in\mathfrak{B}(X)$ is (Harris) recurrent, i.e. $P_x(M_n\in A \text{ i.o.}) = 1$ for all $x\in X$, where i.o. means "infinitely often". We note here that the bivariate chain $(M_n, X_n)_{n\geqslant 0}$ automatically inherits these properties from $(M_n)_{n\geqslant 0}$ because of the special transition structure (2.1).

We proceed with a summary of some important facts on positive Harris chains on which we will draw later on. For more detailed information the reader is referred to Meyn and Tweedie's excellent monograph [12], or to [3]. If $(M_n)_{n\geqslant 0}$ is Harris ergodic, i.e. positive Harris recurrent and aperiodic, then

$$\lim_{n\to\infty} \|P_x(M_n\in\cdot)-\pi\| = 0 \quad\text{ for all } x\in X,$$

where $\|\cdot\|$ denotes the total variation norm. Another important property from which the previous one may be derived is that each $\pi$-positive set contains a small set $C$ (called a *regenerative set* in [3]) with the defining properties $\pi(C) > 0$ and

$$P^r(x, \cdot) \geqslant c\phi \tag{2.3}$$

for all $x \in C$ and some $r \geqslant 1$, $c \in (0, 1]$ and a distribution $\phi$ with $\phi(C) = 1$. Each such small set induces an aperiodic renewal process $0 = \sigma_0 < \sigma_1 < \ldots$ by using the Athreya and Ney coin-tossing procedure (see [3], p. 151). Under $P_\phi$, the $\sigma_n$ divide the chain into stationary 1-dependent cycles

$$(M_{\sigma_n}, \ldots, M_{\sigma_{n+1}-1}), \qquad n \geqslant 0,$$

of finite mean length $E_\phi \sigma_1$, and $\pi$ yields as the normalized occupation measure

$$\pi(A) = \frac{1}{E_\phi \sigma_1} E_\phi\Big(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}_A(M_n)\Big), \tag{2.4}$$

where $\mathbf{1}_A$ denotes the indicator function of $A$. If $\nu \neq \phi$, the 1-dependence of the cycles is preserved but stationarity holds only for $n \geqslant 1$. Let us further mention that the cycles are even independent if $r = 1$ in (2.3), which is the so-called strongly aperiodic case.

Sharper conclusions, subsumed as $f$-ergodicity in [12], can be drawn if

$$\pi(f) < \infty \tag{2.5}$$

for some unbounded function $f\colon X \to [1, \infty)$. Define the $f$-norm as

$$\|\nu\|_f \stackrel{\text{def}}{=} \sup_{g: |g| \leqslant f} |\nu(g)|$$

for signed measures $\nu$. Given a Harris ergodic chain $(M_n)_{n \geqslant 0}$, Theorem 14.0.1 in [12] states that (2.5) implies

$$\lim_{n \to \infty} \|P^n(x, \cdot) - \pi\|_f = 0 \tag{2.6}$$

for all $x \in X_f$, where $X_f$ is absorbing and full. Moreover, there exists a function $V\colon X \to [0, \infty]$, finite on $X_f$, a petite set $C$ (see [12], p. 121) and a finite constant $\beta$ such that the drift condition

$$\Delta V(x) \leqslant -f(x) + \beta \mathbf{1}_C(x), \qquad x \in X, \tag{2.7}$$

holds with $\Delta V(x) \stackrel{\text{def}}{=} PV(x) - V(x)$. Up to an additive constant, the minimal function $V$ in (2.7) is given by

$$V_C(f, x) \stackrel{\text{def}}{=} E_x\Big(\sum_{n=0}^{\sigma(C)} f(M_n)\Big), \tag{2.8}$$

where $\sigma(C) \stackrel{\text{def}}{=} \inf\{n \geqslant 0\colon M_n \in C\}$. This means that $V - V_C(f, \cdot)$ is bounded from below for any other function $V \geqslant 0$ for which (2.7) holds (see [12], Theorem 14.2.3). Finally, if $\pi(V) < \infty$, then

$$(2.9) \qquad \sum_{n \geqslant 0} \|P^n(x, \cdot) - \pi\|_f \leqslant K_f(V(x)+1)$$

for all $x \in X$ and a suitable finite constant $K_f$.

Geometric ergodicity of the chain yields under the following geometric drift condition:

For a function $V: X \to [1, \infty]$, finite on an absorbing and full set $X_V$, a petite set $C$ and constants $\alpha \in (0, 1)$, $\beta \in [0, \infty)$

$$(2.10) \qquad \Delta V(x) \leqslant -\alpha V(x) + \beta \mathbf{1}_C(x), \qquad x \in X.$$

As stated in Theorem 15.0.1 in [12], (2.10) implies

$$(2.11) \qquad \sum_{n \geqslant 0} r^n \|P^n(x, \cdot) - \pi\|_V \leqslant K_V V(x)$$

for all $x \in X_V$ and some constants $r \in (0, 1)$, $K_V < \infty$.

Drift conditions of a similar type as (2.7) and (2.10) play an important role in some of our main results (Theorems 3.1 and 3.2) and are perhaps among the most amenable ones when trying to prove finiteness of moments of stopped Markov random walks.

## 3. RESULTS

Turning to the statement of our results, we begin with a simple observation. If $(M_n, S_n)_{n \geqslant 0}$ is an MRW, the same holds true for $(M_n^*, S_n)_{n \geqslant 0}$, where $M_n^* \stackrel{\text{def}}{=} (M_n, X_n)$ for $n \geqslant 0$. Moreover, all previously discussed properties of the driving chain $(M_n)_{n \geqslant 0}$ automatically carry over to $(M_n^*)_{n \geqslant 0}$. But replacing $(M_n)_{n \geqslant 0}$ with $(M_n^*)_{n \geqslant 0}$, we further see that $X_n = g(M_n^*)$ for $g(x, y) \stackrel{\text{def}}{=} y$. Hence we may confine ourselves hereafter to MRW's with sums $S_n$ of the form

$$S_n(g) \stackrel{\text{def}}{=} \sum_{k=1}^{n} g(M_k)$$

for functions $g: X \to \mathbf{R}$, in other words, to additive functionals of the Markov chain $(M_n)_{n \geqslant 0}$.

We recall our standing assumption that $(M_n)_{n \geqslant 0}$ forms a Markov chain on $X$ with unique stationary distribution $\pi$ under which it is ergodic. However, the positive Harris recurrence of $(M_n)_{n \geqslant 0}$ will only be assumed where explicitly stated. Given any distribution $\nu$ on $X$, we put

$$\nu^* \stackrel{\text{def}}{=} \sum_{n \geqslant 0} 2^{-n-1} P_\nu(M_n \in \cdot) = \sum_{n \geqslant 0} 2^{-n-1} \nu P^n$$

and note that $\pi^* = \pi$. Let $(\mathscr{F}_n)_{n \geqslant 0}$ be any filtration to which $(M_n)_{n \geqslant 0}$ is Markov adapted.

THEOREM 3.1. *Given $p > 1$ and any distribution $\nu$ on $X$, let $g: X \to [0, \infty)$ satisfy*

$$P^m g \leqslant \alpha g + \beta \quad \nu^*\text{-a.s.} \tag{3.1}$$

*for some $m \in N$, $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Then*

$$E_\nu S_T(g)^p \leqslant C\big(E_\nu S_{T+m-1}(g^p) + \nu^*(g^p) + E_\nu T^p\big) \tag{3.2}$$

*for every stopping time $T$ with respect to $(\mathcal{F}_n)_{n \geqslant 0}$ and some finite positive constant $C$ which depends only on occurring parameters.*

If $(M_n)_{n \geqslant 0}$ is geometrically ergodic and satisfies the drift condition (2.10) for some function $V$, then $V$ also satisfies (3.1) with $m = 1$ and for every distribution $\nu$ on $X$. Hence

$$E_\nu S_T(V)^p \leqslant C\big(E_\nu S_T(V^p) + \nu^*(V^p) + E_\nu T^p\big) \tag{3.3}$$

for every stopping time $T$ with respect to $(\mathcal{F}_n)_{n \geqslant 0}$ and some finite positive constant $C$ which only depends on $p$.

A particular application of Theorem 3.1 is to stopped sums of stationary $l$-dependent increments which were studied in some detail in [2] and [10]. To see this, let $(X_n)_{n \in \mathbf{Z}}$ be a doubly infinite sequence of stationary, $l$-dependent and nonnegative random variables. Put $M_n \stackrel{\text{def}}{=} (X_k)_{k \leqslant n}$ and $S_n \stackrel{\text{def}}{=} \sum_{k=1}^n X_k$ for $n \geqslant 0$. It is readily verified that $(M_n)_{n \geqslant 0}$ forms an ergodic Markov chain and $(M_n, S_n)_{n \geqslant 0}$ an MRW. Moreover, $S_n = S_n(g)$ with $g\big((x_k)_{k \leqslant 0}\big) \stackrel{\text{def}}{=} x_0$. Provided that the $X_k$ have finite mean $\mu$, condition (3.1) holds with $m = l+1$ because

$$P^{l+1} g(x) = E(X_{l+1} \mid M_0 = x) = EX_{l+1} = \mu \text{ a.s.}$$

Hence Theorem 3.1 yields the conclusion

$$ES_T^p \leqslant C\big(ES_{T+l}(g^p) + EX_1^p + ET^p\big) \tag{3.4}$$

for each stopping time $T$ with respect to $(M_n)_{n \geqslant 0}$. Furthermore

$$ES_{T+l}(g^p) = E\Big(\sum_{i=1}^{T+l} X_i^p\Big) = EX_1^p\, E(T+l) \tag{3.5}$$

by Janson's identity (see [10], Theorem 1.1) which is the pendant to Wald's first identity for sums of stationary $l$-dependent random variables. Combining (3.4) and (3.5), we finally conclude Theorem 1.3 (ii) in [10], namely

$$ES_T^p \leqslant CEX_1^p\, ET^p \tag{3.6}$$

for suitable constant $C$ only depending on $p$ and $l$.

Besides constituting a well-behaving particular example to which Theorem 3.1 applies, random walks with $l$-dependent increments are interesting for yet another reason. As already mentioned in the Introduction and in Section 2, if $(M_n)_{n \geqslant 0}$ is a positive Harris chain and $(\sigma_n)_{n \geqslant 0}$ a sequence of regeneration epochs associated with a small set $C$ (see Section 2), then, under every $P_\nu$, $(S_{\sigma_n}(g))_{n\geq 1}$ has 1-dependent increments which are further stationary for $n \geq 2$. Moreover,

$$\hat{M}_0 \overset{\text{def}}{=} M_{\sigma_0} = M_0 \quad \text{and} \quad \hat{M}_n \overset{\text{def}}{=} (M_{\sigma_{n-1}+1}, \ldots, M_{\sigma_n}) \quad \text{for } n \geq 1$$

forms a Markov chain with state space $\hat{X} \overset{\text{def}}{=} \bigcup_{k\geq 1} X^k$ and transition kernel $\hat{P}$, say. Defining

$$\hat{S}_n(h) \overset{\text{def}}{=} \sum_{k=1}^{n} h(\hat{M}_k)$$

for any function $h\colon \hat{X} \to \mathbf{R}$ and

$$\hat{g}(x_1, \ldots, x_k) \overset{\text{def}}{=} \sum_{i=1}^{k} g(x_i) \quad \text{for } (x_1, \ldots, x_k) \in X^k \text{ and } k \geq 1,$$

we infer that $S_{\sigma_n}(g) = \hat{S}_n(\hat{g})$ for each $n \geq 0$ and that $(\hat{M}_n, \hat{S}_n(\hat{g}))_{n\geq 0}$ constitutes an MRW. Given a distribution $\nu$ on $X$, the regenerative construction implies

$$\nu\hat{P}^n = P_\nu(\hat{M}_n \in \cdot) = \hat{\phi} \overset{\text{def}}{=} P_\phi(\hat{M}_1 \in \cdot) \quad \text{and} \quad \hat{P}^n g = E_{\hat{\phi}}\hat{g}(\hat{M}_n) = E_\phi S_{\sigma_1}$$

for all $n \geq 2$. Hence $\hat{\nu}^* \overset{\text{def}}{=} \sum_{n\geq 0} 2^{-n-1}\nu\hat{P}^n$ takes the simple form $\hat{\nu}^* = (\nu/2) + (\nu\hat{P} + \hat{\phi})/4$, and

$$\hat{\nu}^*(\hat{g}^p) = \tfrac{1}{2}\nu(g) + \tfrac{1}{4}E_\nu S_{\sigma_1}(g)^p + \tfrac{1}{4}E_\phi S_{\sigma_1}(g)^p \leq E_\zeta S_{\sigma_1}(g)^p,$$

where $\zeta \overset{\text{def}}{=} (\nu + \phi)/2$. Furthermore, condition (3.1) is satisfied with $\hat{P}$, $\hat{g}$ instead of $P$, $g$, and with $m = 2$.

Now consider any stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geq 0}$ with $E_\nu T^p < \infty$. Put $\tau \overset{\text{def}}{=} \inf\{n \geq 0\colon \sigma_n \geq T\}$ and $\hat{T} \overset{\text{def}}{=} \sigma_\tau$. Since $(\sigma_n)_{n\geq 0}$ constitutes a delayed renewal process with finite drift $E_\phi \sigma_1$ (by positive recurrence), a trivial extension of Theorem I.5.2 in [9] to the delayed case implies (even if $E_\nu T^p = \infty$)

$$E_\nu \hat{T}^p \leq C(E_\nu \sigma_1^p + E_\phi \sigma_1^p E_\nu \tau^p)$$

for some constant $C$ only depending on $p$. Note that $E_\nu \tau^p < \infty$ follows from $\tau \leq T$, whence $E_\nu \hat{T}^p < \infty$ holds if $E_\zeta \sigma_1^p < \infty$ for $\zeta$ as defined above.

After these considerations an application of Theorem 3.1 to the MRW $(\hat{M}_n, \hat{S}_n(\hat{g}))_{n\geq 0}$ yields for any $p > 1$ and nonnegative $g$

$$\begin{aligned}(3.7)\qquad E_\nu S_{\hat{T}}(g)^p = E_\nu \hat{S}_\tau(\hat{g})^p &\leq C\big(E_\nu \hat{S}_{\tau+1}(\hat{g}^p) + E_\zeta S_{\sigma_1}(g)^p + E_\nu \hat{T}^p\big)\\ &\leq C\big(E_\nu \hat{S}_{\tau+1}(\hat{g}^p) + E_\zeta S_{\sigma_1}(g)^p + E_\nu \sigma_1^p + E_\phi \sigma_1^p E_\nu \tau^p\big)\\ &\leq C\big(E_\nu \hat{S}_{\tau+1}(\hat{g}^p) + E_\zeta S_{\sigma_1}(g)^p + E_\zeta \sigma_1^p E_\nu \tau^p\big)\end{aligned}$$

for some constant $C$ depending only on $p$, but which may differ from line to line. Moreover, by Wald's identity for MRW's with $l$-dependent increments (the extension of Janson's identity to the delayed case), which has been given in [2], identity (3.2), we obtain ($l=1$)

$$(3.8) \qquad E_\nu \hat{S}_{\tau+1}(\hat{g}^p) = E_\nu \hat{S}_1(\hat{g}^p) + E_\phi \hat{S}_1(\hat{g}^p) E_\nu \tau$$
$$= E_\nu S_{\sigma_1}(g)^p + E_\phi S_{\sigma_1}(g)^p E_\nu \tau \leqslant 2E_\zeta S_{\sigma_1}(g)^p E_\nu \tau.$$

Clearly, $E_\nu S_T(g)^p \leqslant E_\nu S_{\hat{T}}(g)^p$ follows from $T \leqslant \hat{T}$ and $g \geqslant 0$. Therefore the conclusion of the previous considerations is the following: Given the positive Harris recurrence of $(M_n)_{n\geqslant 0}$, any first regeneration epoch $\sigma_1$ associated with a small set and a stopping time $T$ with finite $p$-th moment, $E_\nu S_T(g)^p < \infty$ follows whenever $E_\zeta \sigma_1^p < \infty$ and $E_\zeta S_{\sigma_1}(g)^p < \infty$. This means that the problem of finding conditions on $(M_n, S_n)_{n\geqslant 0}$ for the finiteness of $E_\nu S_T(g)^p$ reduces to the very same problem for the special case when $T = \sigma_1$. We summarize the result in the following corollary:

COROLLARY 3.2. *Let $(M_n)_{n\geqslant 0}$ be a positive Harris chain, $\sigma_1$ the first regeneration epoch associated with a small set $C$ and $\phi$, $r$ be defined by* (2.3). *Given $p > 1$ and any distribution $\nu$ on $X$, let $\zeta = (\nu+\phi)/2$ and $\tau$ be as defined above. Then it follows for every function $g\colon X \to [0, \infty)$ that*

$$(3.9) \qquad E_\nu S_T(g)^p \leqslant C\big(E_\zeta S_{\sigma_1}(g)^p E_\nu \tau + E_\zeta \sigma_1^p E_\nu \tau^p\big)$$

*for every stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geqslant 0}$ and some finite positive constant $C$ which depends only on occurring parameters.*

Of course, (3.9) holds trivially true if $E_\zeta S_{\sigma_1}(g)^p = \infty$. A combination of Theorem 3.1 and Corollary 3.2 further leads to:

COROLLARY 3.3. *Let $(M_n)_{n\geqslant 0}$ be a positive Harris chain, $\sigma_1$ the first regeneration epoch associated with a small set $C$ and $\phi$ be defined by* (2.3). *Given $p > 1$ and any distribution $\nu$ on $X$, let $g\colon X \to [0, \infty)$ satisfy* (3.1) *for some $m \in N$, $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Then*

$$(3.10) \qquad E_\nu S_T(g)^p \leqslant C\Big(\big(E_\nu S_{\sigma_1}(g^p) + \pi(g^p) E_\phi \sigma_1 + \nu^*(g^p)\big) E_\nu \tau + E_\zeta \sigma_1^p E_\nu \tau^p\Big)$$

*for every stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geqslant 0}$ and some finite positive constant $C$ which depends only on occurring parameters.*

We now turn to our second result which is actually a generalization of Theorem 3.1 as will be discussed further below.

THEOREM 3.4. *Given $p > 1$, $\gamma \in [1, 2)$ and any distribution $\nu$ on $X$, put $s \overset{\text{def}}{=} \lceil \log_2 p \rceil$ (i.e. $2^{s-1} < p \leqslant 2^s$), $v \overset{\text{def}}{=} (2\gamma)^{s-1}$, and let $g\colon X \to [0, \infty)$ satisfy*

$$(3.11) \qquad P^m g^{v\gamma} \leqslant g^{v\gamma} - \alpha_v g^v + \beta_v \quad \nu^*\text{-a.s.}$$

*for some $m \in N$, $\alpha_v \in (0, \infty)$ and $\beta_v \in [0, \infty)$. Then*

$$(3.12) \qquad E_\nu S_T(g)^p \leqslant C\big(E_\nu S_{T+m-1}(g^{p\gamma^s}) + \nu^*(g^{p\gamma^s}) + E_\nu T^p\big)$$

*for every stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geqslant 0}$ and a finite positive constant $C$ which depends only on occurring parameters.*

We will show in Lemma 5.2 that (3.11) implies the same condition for each $q\in(0, v]$, i.e.

$$P^m g^{q\gamma} \leqslant g^{q\gamma} - \alpha_q g^q + \beta_q \quad v^*\text{-a.s.} \tag{3.13}$$

for suitable constants $\alpha_q > 0$ and $\beta_q \geqslant 0$. The choice $q = \gamma = 1$ now shows that Theorem 3.1 is a special case of Theorem 3.4.

If $(M_n)_{n\geqslant 0}$ is $f$-ergodic, and thus satisfies the drift condition (2.7) for suitable functions $f, V$: $X \to [0, \infty)$, $f \geqslant 1$, and if $V \geqslant af^\gamma - b$ $v^*$-a.s. for some $a > 0$, $b \geqslant 0$ and $\gamma\in[1, 2)$, then (3.11) holds with $m = 1$ and $g = V^{1/v\gamma}$, as one can easily verify. Notice that $g^{p\gamma^s} = V^{p/2^{s-1}}$. Hence, by Theorem 3.4,

$$E_v S_T(V^{1/v\gamma})^p \leqslant C\big(E_v S_T(V^{p/2^{s-1}}) + v^*(V^{p/2^{s-1}}) + E_v T^p\big) \tag{3.14}$$

for every stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geqslant 0}$ and some finite positive constant $C$.

If $(M_n)_{n\geqslant 0}$ is positive Harris recurrent and $\sigma_1$ the first regeneration epoch associated with a small set, then a combination of Theorem 3.4 with Corollary 3.2 yields the following counterpart of Corollary 3.3.

COROLLARY 3.5. *Let $(M_n)_{n\geqslant 0}$ be a positive Harris chain, $\sigma_1$ the first regeneration epoch associated with a small set $C$ and $\phi$ be defined by* (2.3). *Given $p > 1$ and any distribution $v$ on $X$, let $g$: $X \to [0, \infty)$ satisfy* (3.11) *for some $m\in N$, $\gamma\in[1, 2)$, $\alpha_p\in(0, \infty)$, $\beta_p\in[0, \infty)$ and $s = \lceil\log_2 p\rceil$. Then*

$$E_v S_T(g)^p \leqslant C\big(\big(E_v S_{\sigma_1}(g^{p\gamma^s}) + \pi(g^{p\gamma^s}) E_\phi \sigma_1 + v^*(g^{p\gamma^s})\big) E_v \tau + E_\zeta \sigma_1^p E_v \tau^p\big) \tag{3.15}$$

*for every stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geqslant 0}$ and some finite positive constant $C$ which depends only on occurring parameters.*

For our third theorem we continue to assume that $(M_n)_{n\geqslant 0}$ be positive Harris recurrent. We then call a function $g$: $X \to [0, \infty)$ *l-regular*, $l\in N_0$, if the set $\{P^l g > t\}$ is small for some $t \geqslant 0$. The following result shows that for regular functions $g$ we obtain the same inequality for $E_v S_T(g)^p$ as in Corollary 3.3. This means that regularity provides a substitute for condition (3.1).

THEOREM 3.6. *Given $p > 1$, the positive Harris recurrence of $(M_n)_{n\geqslant 0}$ and any distribution $v$ on $X$, let $g$: $X \to [0, \infty)$ be such that $g^p$ is $l$-regular for some $l\in N_0$ and satisfies $v^*(g^p) < \infty$. Let $\sigma_1$ be the first regeneration epoch associated with the small set $\{P^l g > t\}$, $\phi$ as given in* (2.3) *for this set, and $\tau$, $\zeta$ be as defined earlier. Then* (3.10) *holds true for every stopping time $T$ with respect to $(\mathscr{F}_n)_{n\geqslant 0}$ and some finite positive constant $C$ depending only on occurring parameters.*

We finally turn to a discussion of the first passage times

$$T(t) = \inf\{n \geqslant 1\colon\ S_n(g) > t\}, \qquad t \geqslant 0,$$

where $g: \boldsymbol{X} \to \boldsymbol{R}$ is now any function such that $\pi(g) > 0$. The latter implies $n^{-1} S_n(g) \to \pi(g)$ $\boldsymbol{P}_\pi$-a.s. by Birkhoff's ergodic theorem, and thus $T(t) < \infty$ $\boldsymbol{P}_\pi$-a.s. for all $t \geqslant 0$. Our goal is to provide conditions for the finiteness of $\boldsymbol{E}_\nu S^p_{T(t)}$ for $p \geqslant 1$ and all $t \geqslant 0$. We make the basic assumption that $(M_n)_{n\geqslant 0}$ is a positive Harris chain and that $(\sigma_n)_{n\geqslant 0}$ means the sequence of regeneration epochs associated with a small set $\boldsymbol{C}$ ($\sigma_0 \stackrel{\text{def}}{=} 0$). Let also $\phi$ and $m \in \boldsymbol{N}$ be given by (2.3).

It is obviously true that $g(M_{T(t)}) = g^+(M_{T(t)}) \geqslant 0$ and

$$g^+(M_1) \leqslant S_{T(t)}(g) \leqslant t + g^+(M_{T(t)}). \tag{3.16}$$

For i.i.d. increments $g(M_1), g(M_2), \ldots$, a combination of (3.16) with Wald's equation implies for each $p > 1$ and $t \geqslant 0$ that

$$Eg^+(M_1)^p \leqslant ES_{T(t)}(g)^p \leqslant 2^p\big(t^p + Eg^+(M_1)^p\, ET(t)\big),$$

whence $Eg^+(M_1)^p < \infty$ forms a necessary and sufficient condition for $ES_{T(t)}(g)^p < \infty$ for one (and then all) $t \geqslant 0$. The result extends to the case of $l$-dependent stationary increments [10] by almost the same argument in which Janson's equation (see (3.5)) replaces Wald's equation. However, a further generalization to the situation described above and to general initial distributions $\nu$ requires greater care and will be dealt with in the next theorem for which we define

$$L_n(g) \stackrel{\text{def}}{=} \max_{0\leqslant k\leqslant n} g(M_n) \quad \text{for } n \geqslant 1.$$

THEOREM 3.7. *Given the general situation described before and $p \geqslant 1$, let $g: \boldsymbol{X} \to \boldsymbol{R}$ be such that $\pi(g) > 0$ and $\nu$ be any initial distribution with $\boldsymbol{E}_\nu \sigma_1 < \infty$ and $\boldsymbol{E}_\nu S_{\sigma_1}(g^-) < \infty$. Then $\boldsymbol{E}_{\nu P^k} T(t) < \infty$ for all $k \geqslant 0$ and $t \geqslant 0$. Provided that additionally $\pi((g^+)^p) < \infty$ and $\boldsymbol{E}_\nu L_{\sigma_1}(g^+)^p < \infty$, the following assertions are equivalent*:

(a) $\nu P^k((g^+)^p) = \boldsymbol{E}_\nu g^+(M_k)^p < \infty$ *for all* $k \geqslant 0$;
(b) $\boldsymbol{E}_{\nu P^k} S_{T(t)}(g)^p < \infty$ *for all* $k \geqslant 0$ *and* $t \geqslant 0$;
(c) $\boldsymbol{E}_{\nu P^k} S_{T(0)}(g)^p < \infty$ *for all* $k \geqslant 0$.

If $\nu$ equals the stationary distribution $\pi$, the previous result simplifies because $\pi P^k = \pi$ for each $k \geqslant 0$:

COROLLARY 3.8. *Given the general situation described before and $p \geqslant 1$, let $g: \boldsymbol{X} \to \boldsymbol{R}$ be any function such that $\pi(g) > 0$. Then $\boldsymbol{E}_\pi \sigma_1 < \infty$ and $\boldsymbol{E}_\pi S_{\sigma_1}(g^-) < \infty$ imply $\boldsymbol{E}_\pi T(t) < \infty$ for all $t \geqslant 0$. If furthermore $\boldsymbol{E}_\pi L_{\sigma_1}(g^+)^p < \infty$, then the following assertions are equivalent*:

(a) $\pi((g^+)^p) < \infty$;
(b) $\boldsymbol{E}_\pi S_{T(t)}(g)^p < \infty$ *for all* $k \geqslant 0$ *and* $t \geqslant 0$;
(c) $\boldsymbol{E}_\pi S_{T(0)}(g)^p < \infty$ *for all* $k \geqslant 0$.

We note, though trivial, that in the previous corollary $\pi((g^+)^p) < \infty$ also implies assertions (b) and (c) when $\pi$ is replaced with any $\nu \leqslant c\pi$ for some $c > 0$. The relevance of this remark lies in the fact that, by (2.4), this condition holds for all $\phi P^k$, where $k \geqslant 0$ and $\phi$ is a minorizing distribution associated (via (2.3)) with any small set $C$. On the other hand, even more can be concluded in this situation. If $\nu = \phi$, then $E_\phi \sigma_1 < \infty$ and $E_\phi S_{\sigma_1}(g^-) = \pi(g^-) E_\phi \sigma_1 < \infty$ automatically hold by the positive Harris recurrence of $(M_n)_{n\geqslant 0}$ and the assumption $\pi(g) > 0$. Moreover, $\pi((g^+)^p) < \infty$ implies $\phi((g^+)^p) < \infty$ and

$$E_\phi L_{\sigma_1}(g^+)^p \leqslant E_\phi\Big(\sum_{n=0}^{\sigma_1} g^+(M_n)^p\Big) = \phi((g^+)^p) + \pi((g^+)^p) E_\phi \sigma_1 < \infty.$$

Hence the following corollary is an immediate consequence of Theorem 3.7.

COROLLARY 3.9. *Given the general situation described before and $p \geqslant 1$, let $g: X \to R$ be any function such that $\pi(g) > 0$ and $\phi$ be any minorizing distribution associated with a small set. Then $E_{\phi P^k} T(t) < \infty$ for all $k \geqslant 0$ and $t \geqslant 0$. If furthermore $\pi((g^+)^p) < \infty$, then the following assertions are equivalent:*

(a) $\phi P^k((g^+)^p) = E_\phi g^+(M_k)^p < \infty$ *for all* $k \geqslant 0$;

(b) $E_{\phi P^k} S_{T(t)}(g)^p < \infty$ *for all* $k \geqslant 0$ *and* $t \geqslant 0$;

(c) $E_{\phi P^k} S_{T(0)}(g)^p < \infty$ *for all* $k \geqslant 0$.

## 4. MARTINGALE DECOMPOSITIONS

In the following we consider an MRW $(M_n, S_n(g))_{n\geqslant 0}$, where $g: X \to R$ is supposed to be $\pi$-integrable, i.e. $\pi(g) < \infty$. Given a stopping time $T$, the first step towards finding bounds for the moments of $S_T(g)$ is to decompose it into a martingale $W_T(g)$ and a remainder $R_T(g)$. Put $\bar{g} \stackrel{\text{def}}{=} g - \pi(g)$ and note that $S_n(\bar{g}) = S_n(g) - n\pi(g)$ for $n \geqslant 0$.

There are various ways of extracting a martingale $(W_n(g))_{n\geqslant 0}$ from $(S_n(g))_{n\geqslant 0}$ such that $(M_n, W_n(g))_{n\geqslant 0}$ is also an MRW. The first one is based on the common method of *conditional centering*. Suppose that all $g(M_n)$ are integrable under a given $P_\nu$, which is particularly true if $\nu^*(g) < \infty$. Put $R_n(g) \stackrel{\text{def}}{=} g(M_0) - g(M_n)$. Then $(g(M_n) - Pg(M_{n-1}))_{n\geqslant 1}$ forms a sequence of martingale differences under $P_\nu$ and we have

$$S_n(g) = W_n(g) + S_n(Pg) + R_n(Pg) \tag{4.1}$$

with

$$W_n(g) \stackrel{\text{def}}{=} \sum_{k=1}^{n} (g(M_k) - Pg(M_{k-1})) = S_n(g) - S_n(Pg) + Pg(M_n) - Pg(M_0) \tag{4.2}$$

being a $P_\nu$-martingale. Since the sum $S_n(Pg)$ is of the same type as $S_n(g)$ but with $Pg$ instead of $g$, an iteration leads to

$$S_n(g) = \sum_{k=0}^{m-1} W_n(P^k g) + S_n(P^m g) + \sum_{k=1}^{m} R_n(P^k g) \tag{4.3}$$

for each $m \geqslant 1$, where $P^0 g \stackrel{\text{def}}{=} g$. Noting that $E_\nu |P^k g(M_n)| \leqslant E_\nu |g(M_{n+k})| < \infty$ for all $n, k \geqslant 1$, each $(W_n(P^k g))_{n\geqslant 0}$ forms a $P_\nu$-martingale with respect to $(M_n)_{n\geqslant 0}$, and so does

$$W_n^{(m)}(g) \stackrel{\text{def}}{=} \sum_{k=0}^{m} W_n(P^k g), \qquad n \geqslant 0 \text{ for each } m \geqslant 0.$$

A second way of getting a martingale decomposition under a given $P_\nu$ works in the case when there is a $\nu$-integrable solution $\hat{g}$ to the *Poisson equation*

$$\hat{g} - P\hat{g} = \bar{g}$$

for $g$. If $(M_n)_{n\geqslant 0}$ is Harris ergodic and satisfies the drift condition (2.7) with $f = |g| \vee 1$ and some $\nu$-integrable $V$, then $\hat{g} \stackrel{\text{def}}{=} \sum_{n\geqslant 0} P^n \bar{g} = \sum_{n\geqslant 0} (P^n g - \pi(g))$ forms indeed such a solution because, by (2.9),

$$|\hat{g}(x)| \leqslant \sum_{n\geqslant 0} |P^n \bar{g}(x)| \leqslant \sum_{n\geqslant 0} \|P^n(x, \cdot) - \pi\|_f \leqslant K_f(V(x)+1) < \infty \quad \nu\text{-a.s.}$$

and

$$\nu(\hat{g}) \leqslant K_f(\nu(V)+1) < \infty;$$

see also [12], Theorem 17.4.2. Notice that the previous conclusions are true as well for $P^k \hat{g} = \sum_{n\geqslant k} P^n \bar{g}$ for every $k \geqslant 1$. In particular, $\hat{g}(x) < \infty$ $\nu P^k$-a.s. for all $k \geqslant 0$, and thus $\pi$-a.s. because $\sum_{k\geqslant 0} 2^{-k-1} \nu P^k$ and $\pi$ are equivalent distributions by the Harris ergodicity. Now $\bar{g} = \hat{g} - P\hat{g}$ gives the decomposition

$$S_n(g) - n\pi(g) = S_n(\hat{g}) - S_n(P\hat{g}) = W_n(\hat{g}) + R_n(P\hat{g}) \tag{4.4}$$

into the $P_\nu$-martingale $(W_n(\hat{g}))_{n\geqslant 0}$ and the $P_\nu$-integrable remainders $(R_n(P\hat{g}))_{n\geqslant 0}$; see also [12], Theorem 17.4.3. Notice that $W_n(P^k \bar{g}) = W_n(P^k g)$ and $R_n(P^k \bar{g}) = R_n(P^k g)$ for all $k, n \geqslant 0$. Hence

$$R_n(\hat{g}) = \sum_{k\geqslant 1} R_n(P^k g) \tag{4.5}$$

as well as

$$W_n(\hat{g}) = \sum_{k\geqslant 0} W_n(P^k g) \tag{4.6}$$

for $n \geqslant 0$. Since furthermore $\lim_{m\to\infty} S_n(P^m g) = n\pi(g)$ $P_\nu$-a.s. for each $n \geqslant 1$ under (2.7) with $f = |g| \vee 1$, we see that, under appropriate conditions, decomposition (4.4) can also be obtained as the limit of (4.3) as $m \to \infty$.

In the case where $\nu = \pi$ and $W_n(\hat{g})$, $R_n(P\hat{g})$ are square-integrable, decomposition (4.4) can be used to show asymptotic normality of $S_n(g)/n^{1/2}$; see [4], [5], [8], [12], [13]. For the same purpose, Maxwell and Woodroofe [11] give yet another decomposition of $S_n(g)$ obtained by solving the *perturbed Poisson equation*

$$(1+\varepsilon)\hat{g}_\varepsilon - P\hat{g}_\varepsilon = \bar{g}$$

for $\varepsilon > 0$. The solution is

$$\hat{g}_\varepsilon = \sum_{k\geq 1} \frac{P^{k-1}\bar{g}}{(1+\varepsilon)^k} = \varepsilon \sum_{k\geq 1} \frac{\Sigma_k \bar{g}}{(1+\varepsilon)^{k+1}}, \tag{4.7}$$

where

$$\Sigma_n \bar{g} \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} P^k \bar{g}, \qquad n \geq 1,$$

and all $\hat{g}_\varepsilon(M_n)$ are $P_\nu$-integrable if $\sup_{n\geq 0}(1+\varepsilon)^{-n} E_\nu g(M_n) < \infty$. Now

$$S_n(g) - n\pi(g) = W_n(\hat{g}_\varepsilon) + \varepsilon S_n(\hat{g}_\varepsilon) + R_n(P\hat{g}_\varepsilon); \tag{4.8}$$

see [11], equation (6). Notice that

$$S_n(\hat{g}_\varepsilon) = \sum_{k\geq 1} \frac{S_n(P^{k-1}g)}{(1+\varepsilon)^k} = \frac{1}{1+\varepsilon}\left(S_n(g) + \sum_{k\geq 1} \frac{S_n(P^k g)}{(1+\varepsilon)^k}\right) = \frac{S_n(g) + S_n(P\hat{g}_\varepsilon)}{1+\varepsilon}$$

for all $n \geq 1$, whence (4.8) may be rewritten as

$$S_n(g) - n\pi(g) = (1+\varepsilon) W_n(\hat{g}_\varepsilon) + \varepsilon S_n(P\hat{g}_\varepsilon) + (1+\varepsilon) R_n(\hat{g}_\varepsilon). \tag{4.9}$$

The martingale decomposition (4.4) based on solving the Poisson equation for $g$ can be used to obtain Wald's equation for MRW's. This has been done by Fuh and Zhang [7], see also [6] for another approach for uniformly ergodic $(M_n)_{n\geq 0}$. We state the result for the case of bounded stopping times $T$ in which it follows immediately by an appeal to the Optional Sampling Theorem (see [7], Corollary 1).

LEMMA 4.1. *Given an arbitrary distribution $\nu$ on $X$, let $g: X \to \mathbf{R}$ satisfy $\pi(|g|) < \infty$, $\nu P^k(|g|) < \infty$ and $\nu P^k(|h|) < \infty$ for all $k \geq 0$, where*

$$h \stackrel{\text{def}}{=} \sum_{n\geq 0} P^n(g - \pi(g)).$$

*Then*

$$E_\nu S_T(g) = \pi(g) E_\nu T + E_\nu h(M_T) - E_\nu h(M_0) \tag{4.10}$$

*for every bounded stopping time $T$ for $(M_n)_{n\geq 0}$.*

Notice that if $g$ is nonnegative as will be the case for most of our results in Section 5, then, by monotone convergence, (4.10) extends to unbounded stopping times $T$ in the following way: Providing additionally $E_\nu T < \infty$,

$$E_\nu S_T(g) = \pi(g) E_\nu T + \lim_{n\to\infty} E_\nu h(M_{T\wedge n}) - E_\nu h(M_0) \tag{4.11}$$

and the right-hand side is finite iff the limit of $E_\nu h(M_{T\wedge n})$ is finite. Sufficient conditions for this to hold may again be found in [7]. An extension of Wald's second identity to MRW's is also proved there by using a suitable martingale decomposition of $W_n(h)^2$. We refrain from a statement here because we will not make use of this identity.

## 5. PROOFS

Throughout the whole section $C$ denotes a generic positive constant which may differ from line to line, but which does not depend on occurring random variables or their distributions.

In order to prove Theorem 3.1 we first give a lemma concerning the drift condition (3.1).

LEMMA 5.1. *If* (3.1) *holds, then*

$$(P^m g)^q \leqslant \alpha_q g^q + \beta_q \quad \nu^*\text{-a.s.} \tag{5.1}$$

*for each* $q > 0$ *and suitable constants* $\alpha_q \in [0, 1)$ *and* $\beta_q \in [0, \infty)$.

Proof. The assertion follows because (3.1) implies for each $q > 0$

$$(P^m g)^q \leqslant (\alpha g + \beta)^q \leqslant (\alpha g + \beta)^q \mathbb{1}_{\{g > 2\beta/(1-\alpha)\}} + \beta^q \left(\frac{2}{1-\alpha} + 1\right)^q$$
$$\leqslant \left(\alpha + \frac{1-\alpha}{2}\right)^q g^q + \beta^q \left(\frac{2}{1-\alpha} + 1\right)^q \quad \nu^*\text{-a.s.} \ \blacksquare$$

Proof of Theorem 3.1. Without loss of generality let $\nu^*(g^p) < \infty$ and $E_\nu T^p < \infty$ because there is nothing to prove otherwise. Suppose first $m = 1$ in (3.1) and consider decomposition (4.1). Then a combination of (3.1) and $R_n(Pg) \leqslant Pg(M_0)$ leads to the inequality

$$S_n(g) \leqslant (1-\alpha)^{-1}\big(W_n(g) + R_n(Pg) + \beta n\big) \leqslant (1-\alpha)^{-1}\big(W_n(g) + Pg(M_0) + \beta n\big) \tag{5.2}$$

for all $n \geqslant 1$. We consider the martingale transform

$$W_{T\wedge n}(g) = \sum_{k=1}^{n} \mathbb{1}_{\{T \geqslant k\}}\big(g(M_k) - Pg(M_{k-1})\big), \qquad n \geqslant 0.$$

An application of Fatou's lemma, Burkholder's inequality and (5.1) give

$$E_\nu |W_T(g)|^p \leqslant \liminf_{n\to\infty} E_\nu |W_{T\wedge n}(g)|^p \leqslant C E_\nu \Big(\sum_{n=1}^{T} \big(g(M_n) - Pg(M_{n-1})\big)^2\Big)^{p/2} \tag{5.3}$$
$$\leqslant C E_\nu \Big(\sum_{n=1}^{T} \big(g^2(M_n) + (Pg)^2(M_{n-1})\big)\Big)^{p/2}$$
$$\leqslant C E_\nu \Big(\sum_{n=1}^{T} \big(g^2(M_n) - \alpha_2 g^2(M_{n-1}) + \beta_2\big)\Big)^{p/2} \leqslant$$

$$\leqslant CE_\nu\big((1+\alpha_2)S_T(g^2)+\alpha_2 g^2(M_0)+\beta_2 T\big)^{p/2}$$
$$\leqslant C\big(E_\nu S_T(g^2)^{p/2}+E_\nu g^p(M_0)+E_\nu T^{p/2}\big),$$

so that with (5.2) we get

$$E_\nu S_T(g)^p \leqslant 3^p(1-\alpha)^{-p}\big(E_\nu|W_T(g)|^p+E_\nu(Pg)^p(M_0)+E_\nu T^p\big)$$
$$\leqslant C\big(E_\nu S_T(g^2)^{p/2}+E_\nu g^p(M_0)+E_{\nu P} g^p(M_0)+E_\nu T^p\big)$$
$$\leqslant C\big(E_\nu S_T(g^2)^{p/2}+\nu^*(g^p)+E_\nu T^p\big).$$

If $1<p\leqslant 2$, $E_\nu S_T(g^2)^{p/2}\leqslant E_\nu S_T(g^p)$, and thus

(5.4) $$E_\nu S_T(g)^p \leqslant C\big(E_\nu S_T(g^p)+\nu^*(g^p)+E_\nu T^p\big),$$

which is (3.2) for $m=1$.

If $2^k<p\leqslant 2^{k+1}$ for $k\in N$, use the previous estimation $k$ further times (successively for $E_\nu S_T(g^2)^{p/2},\ldots, E_\nu S_T(g^{2k})^{p/2^k}$, which is possible by Lemma 5.1), to obtain

$$E_\nu S_T(g)^p \leqslant C\big(E_\nu S_T(g^{2^{k+1}})^{p/2^{k+1}}+\nu^*(g^p)+E_\nu T^p\big),$$

and thus again (5.4) because $0<p/2^{k+1}\leqslant 1$.

If $m\geqslant 2$ in (3.1), then put

$$S_{r,n}(g)\overset{\text{def}}{=}\sum_{k=0}^{n-1} g(M_{km+1}) \quad \text{for } 1\leqslant r\leqslant m,\ n\geqslant 1$$

and

$$\tau_r \overset{\text{def}}{=} \inf\{n\geqslant 1\colon\ nm+r\geqslant T\}.$$

Notice that $\tau_r$ is a stopping time with respect to $(\mathscr{F}_{mn+r})_{n\geqslant 0}$ because

$$\{\tau_r=n\}=\{m(n-1)+r<T\leqslant mn+r\}\in\mathscr{F}_{mn+r}$$

for all $n\geqslant 1$ and $1\leqslant r\leqslant m$. Moreover, $(M_{mn+r})_{n\geqslant 0}$ is Markov adapted to $(\mathscr{F}_{mn+r})_{n\geqslant 0}$ with transition kernel $P^m$ and $\sum_{n\geqslant 0}2^{-n-1}P_\nu(M_{mn+r}\in\cdot)\leqslant C\nu^*$. Consequently, (5.4) applies to each $S_{r,\tau_r}(g)$, $1\leqslant r\leqslant m$, giving

$$E_\nu S_{r,\tau_r}(g)^p\leqslant C\big(E_\nu S_{r,\tau_r}(g)^p+\nu^*(g^p)+E_\nu \tau_r^p\big).$$

Finally, use $m\tau_r+r\leqslant T+m-1$, and thus $\sum_{r=1}^m S_{r,\tau_r}(g^p)\leqslant S_{T+m-1}(g^p)$ to conclude

$$E_\nu S_T(g)^p\leqslant C\sum_{r=1}^m E_\nu S_{r,\tau_r}(g)^p\leqslant C\Big(\sum_{r=1}^m \big(E_\nu S_{r,\tau_r}(g^p)+m\nu^*(g^p)+mE_\nu\tau_r^p\big)\Big)$$
$$\leqslant C\big(E_\nu S_{T+m-1}(g^p)+\nu^*(g^p)+E_\nu T^p\big),$$

which is (3.2). ∎

Proof of Corollary 3.3. In view of (3.9) it suffices to examine

$$E_\zeta S_{\sigma_1}(g)^p = \big(E_\nu S_{\sigma_1}(g)^p + E_\phi S_{\sigma_1}(g)^p\big)/2.$$

But $E_\phi S_{\sigma_1}(g)^p = \pi(g^p) E_\phi \sigma_1$ (use (2.4)) and Theorem 3.1 imply

$$E_\nu S_{\sigma_1}(g)^p \leqslant C\big(E_\nu S_{\sigma_1+m-1}(g^p) + \nu^*(g^p) + E_\nu \sigma_1^p\big)$$

for a suitable constant $C$. Furthermore,

$$E_\nu\big(S_{\sigma_1+m-1}(g^p) - S_{\sigma_1}(g^p)\big) = \phi P(g^p) + \ldots + \phi P^{m-1}(g^p) \leqslant (m-1)\pi(g^p).$$

Condition (3.10) now easily follows (of course, in general with a different $C$). ■

For the proof of Theorem 3.4 we again need a lemma that extends the drift condition (3.11) to arbitrary $q \leqslant p$.

LEMMA 5.2. *If* (3.11) *holds, then*

$$P^m g^{q\gamma} \leqslant g^{q\gamma} - \alpha_q g^q + \beta_q \quad \nu^*\text{-a.s.} \tag{5.5}$$

*for all* $q \in (0, v]$ *and suitable* $\alpha_q \in (0, \infty)$, $\beta_q \in [0, \infty)$.

Proof. Fix an arbitrary $q \in (0, v)$. The following estimation uses twice the subadditivity of $x \mapsto x^{q/v}$ for $x \geqslant 0$, which particularly implies $(x-y)^{q/v} \leqslant x^{q/2} - y^{q/v}$ for $0 \leqslant y < x$. It follows on the event $\{g^{v\gamma} - \alpha_v g^v > 0\}$ that

$$P^m g^{q\gamma} \leqslant (P^m g^{v\gamma})^{q/v} \leqslant (g^{v\gamma} - \alpha_v g^v + \beta_v)^{q/v} = g^{q\gamma} - \alpha_v^{q/v} g^q + \beta_v^{q/v} \quad \nu^*\text{-a.s.},$$

while Jensen's inequality and $K_q \stackrel{\text{def}}{=} \sup_{x\in\mathbf{X}}(\alpha_v^{q/v} g^q - g^{q\gamma}) < \infty$ imply on $\{g^{v\gamma} - \alpha_v g^v \leqslant 0\}$

$$0 \leqslant (P^m g^{q\gamma})^{v/q} \leqslant P^m g^{v\gamma} \leqslant \beta_v \leqslant (g^{q\gamma} - qp^{-1}\alpha_v g^q + K_q + \beta_v^\delta)^{v/q} \quad \nu^*\text{-a.s.}$$

Both inequalities combined prove (5.5). ■

Proof of Theorem 3.4. Without loss of generality let $\nu^*(g^{p\gamma^s}) < \infty$ and $E_\nu T^p < \infty$ because there is nothing to prove otherwise. We only consider the case $m = 1$ in (3.11) (and thus (5.5)) because the extension to general $m$ follows by almost the same argument as in the proof of Theorem 3.1, and can thus be omitted. A combination of (5.5) and (4.1) shows that for all $q \in (0, v]$

$$\begin{aligned} S_n(g^q) &\leqslant \alpha_q^{-1}\big(S_n(g^{q\gamma}) - S_n(Pg^{q\gamma}) + \beta_q n\big) = \alpha_q^{-1}\big(W_n(g^{q\gamma}) + R_n(Pg^{q\gamma}) + \beta_q n\big) \\ &\leqslant \alpha_q^{-1}\big(W_n(g^{q\gamma}) + Pg^{q\gamma}(M_0) + \beta_q n\big), \end{aligned}$$

which in turn implies

$$E_\nu S_T(g^q)^r \leqslant C\big(E_\nu |W_T(g^{q\gamma})|^r + \nu^*(g^{q\gamma r}) + E_\nu T^r\big) \tag{5.6}$$

for $r > 0$; in particular,

$$E_\nu S_T(g)^p \leqslant C\big(E_\nu |W_T(g^\gamma)|^p + \nu^*(g^{p\gamma}) + E_\nu T^p\big). \tag{5.7}$$

Using $Pg^{q\gamma} \leqslant g^{q\gamma} + \beta_q$ for each $q \in (0, q]$ and Burkholder's inequality, we obtain by a similar estimation to that in (5.3)

$$(5.8) \qquad \begin{aligned} E_\nu |W_T(g^\gamma)|^p &\leqslant C\big(E_\nu S_T(g^{2\gamma})^{p/2} + E_\nu S_T(Pg^{2\gamma})^{p/2} + E_{\nu P}\, g(M_0)^{p\gamma}\big) \\ &\leqslant C\big(E_\nu S_T(g^{2\gamma})^{p/2} + E_{\nu P}\, g(M_0)^{p\gamma} + E_\nu T^p\big) \\ &\leqslant C\big(E_\nu S_T(g^{2\gamma})^{p/2} + \nu^*(g^{p\gamma}) + E_\nu T^p\big). \end{aligned}$$

If $p \in (1, 2]$, then (3.12) follows because $E_\nu S_T(g^{2\gamma})^{p/2} \leqslant E_\nu S_T(g^{p\gamma})$. Otherwise, a repeated use of (5.6) and Burkholder's inequality lead to

$$\begin{aligned} E_\nu S_T(g^{2\gamma})^{p/2} &\leqslant C\big(E_\nu |W_T(g^{2\gamma^2})|^{p/2} + \nu^*(g^{p\gamma^2}) + E_\nu T^{p/2}\big) \\ &\leqslant C\big(E_\nu S_T(g^{(2\gamma)^2})^{p/4} + E_\nu S_T(Pg^{(2\gamma)^2})^{p/4} + \nu^*(g^{p\gamma^2}) + E_\nu T^p\big) \\ &\leqslant \ldots \leqslant C\big(E_\nu |W_T(g^{2^{s-1}\gamma^s})|^{p/2^{s-1}} + \nu^*(g^{p\gamma^{s-1}}) + E_\nu T^p\big) \\ &\leqslant C\big(E_\nu S_T(g^{(2\gamma)^s})^{p/2^s} + E_\nu S_T(Pg^{(2\gamma)^s})^{p/2^s} + \nu^*(g^{p\gamma^s}) + E_\nu T^p\big) \\ &\leqslant C\big(E_\nu S_T(g^{p\gamma^s})\, E_\nu S_T(Pg^{p\gamma^s}) + \nu^*(g^{p\gamma^s}) + E_\nu T^p\big) \\ &\leqslant C\big(E_\nu S_T(g^{p\gamma^s}) + \nu^*(g^{p\gamma^s}) + E_\nu T^p\big), \end{aligned}$$

which together with (5.7) and (5.8) yields the asserted inequality (3.12). ∎

Proof of Theorem 3.6. Let $\phi, r, c$ be given by (2.3) for the small set $C \stackrel{\text{def}}{=} \{P^l g^p > t\}$. With the help of Corollary 3.2 it suffices to prove that

$$(5.9) \qquad E_\nu S_{\sigma_1}(g)^p \leqslant C\big(E_\nu S_{\sigma_1}(g^p) + \pi(g^p)\, E_\phi\, \sigma_1 + \nu^*(g^p)\big)$$

for a suitable $C$ depending only on occurring parameters. We will do so by induction over $l$. Once again, without loss of generality we assume $\nu^*(g^p) < \infty$ and, for simplicity, also $r = 1$ (strongly aperiodic case). The extension to the case $r > 1$ requires no extra argument but is more technical because of a more unpleasant definition of the regeneration epoch $\sigma_1$.

Assuming $l = 0$, let $\chi_n$ be the successive visits to $C$ and $(Z_n)_{n \geqslant 1}$ be the sequence of i.i.d. Bernoulli($c$) variables describing the outcomes of the coin tosses each time the set $C$ is hit and thus determining the first regeneration epoch $\sigma_1$. Namely, $\sigma_1 = \chi_\varrho$, where $\varrho \stackrel{\text{def}}{=} \inf\{k \geqslant 1\colon Z_k = 1\}$ (see [3], p. 151). Observe that $g^p(M_n) \leqslant t$ for all $n$ outside the random set $\{\chi_1, \chi_2, \ldots\}$. Hence

$$E_\nu S_{\sigma_1}(g)^p \leqslant t E_\nu \sigma_1^p + E_\nu \Big(\sum_{k=1}^{\varrho} g(M_{\chi_k})\Big)^p,$$

which leaves us with a further estimation of the final sum on the right-hand side. To this end, by using the (infinite) Minkowski inequality, the independence of $\mathbf{1}_{\{\varrho \geqslant n\}}$ and $g(M_{\chi_n})$ for each $n \geqslant 1$, and $g^p(M_{\chi_n}) \leqslant S_{\sigma_1}(g^p)$ for $n = 1, \ldots, \varrho$, we obtain

$$\begin{aligned}\Big(E_\nu\big(\sum_{k=1}^{\varrho} g(M_{\chi_k})\big)^p\Big)^{1/p} &= \Big(E_\nu\big(\sum_{n\geqslant 1} \mathbf{1}_{\{\varrho \geqslant n\}}\, g(M_{\chi_n})\big)^p\Big)^{1/p} \\ &\leqslant \sum_{n\geqslant 1} \big(E_\nu \mathbf{1}_{\{\varrho \geqslant n\}}\, g^p(M_{\chi_n})\big)^{1/p} \leqslant \sum_{n\geqslant 1} \big(P_\nu(\varrho \geqslant n) E_\nu g^p(M_{\chi_n})\big)^{1/p} \\ &\leqslant C \sum_{n\geqslant 1} (1-c)^{(n-1)/p} \big(E_\nu S_{\sigma_1}(g^p)\big)^{1/p} \leqslant C\big(E_\nu S_{\sigma_1}(g^p)\big)^{1/p},\end{aligned}$$

and thus

$$E_\nu\big(\sum_{k=1}^{\varrho} g(M_{\chi_k})\big)^p \leqslant C E_\nu S_{\sigma_1}(g^p),$$

where $C$ is a constant which depends on $p$, $c$ and $t$ and which as before may vary from line to line.

For the inductive step $l-1 \to l$, we first observe that if $\{P^l g^p > t\}$ is small, then $Pg^p$ satisfies the same condition with $l-1$. Moreover, for $q \in [1, p]$, $(Pg^q)^{1/q} \leqslant (Pg^p)^{1/p}$ a.s. implies that

$$\{Pg^p > t^{q/p}\} \subset \{Pg^p > t\} \text{ a.s.},$$

and therefore that $Pg^q$ is $(l-1)$-regular for every $q \in [1, p]$. Again, using the martingale decomposition (4.1), we obtain

$$E_\nu S_{\sigma_1}(g)^p \leqslant C\big(E_\nu |W_{\sigma_1}(g)|^p + E_\nu S_{\sigma_1}(Pg)^p + E_{\nu P}\, g(M_0)^p\big). \tag{5.10}$$

We apply the inductive hypothesis to $E_\nu S_{\sigma_1}(Pg)^p$ (i.e. (5.9) with $Pg$ instead of $g$) which, together with the inequality $(Pg)^p \leqslant Pg^p$ and $\pi P = \pi$, leads to

$$\begin{aligned}E_\nu S_{\sigma_1}(Pg)^p &\leqslant C\big(E_\nu S_{\sigma_1}((Pg)^p) + \pi((Pg)^p) E_\phi \sigma_1 + \nu^*((Pg)^p)\big) \\ &\leqslant C\big(E_\nu S_{\sigma_1}(Pg^p) + \pi(g^p) E_\phi \sigma_1 + \nu^*(g^p)\big).\end{aligned} \tag{5.11}$$

We also have

$$\begin{aligned}E_\nu S_{\sigma_1}(Pg^p) &= E_\nu\big(\sum_{i\geqslant 1} \mathbf{1}_{\{\sigma_1 \geqslant i\}} Pg^p(M_i)\big) = E_\nu\big(\sum_{i\geqslant 1} \mathbf{1}_{\{\sigma_1 \geqslant i\}} g^p(M_{i+1})\big) \\ &\leqslant E_\nu S_{\sigma_1+1}(g^p) \leqslant E_\nu S_{\sigma_1}(g^p) + \pi(g^p),\end{aligned} \tag{5.12}$$

which, when combined with (5.11) and $E_{\nu P}\, g(M_0)^p \leqslant 2\nu^*(g^p)$, finally shows in (5.10) that

$$E_\nu S_{\sigma_1}(g)^p \leqslant C\big(E_\nu |W_{\sigma_1}(g)|^p + E_\nu S_{\sigma_1}(g)^p + \pi(g^p) E_\phi \sigma_1 + \nu^*(g^p)\big) \tag{5.13}$$

for a suitable constant $C$.

Thus it remains to derive a bound for the martingale term $E_\nu |W_{\sigma_1}(g)|^p$. If $2^{s-1} < p \leqslant 2^s$ for $s \in N$, a similar estimation as in the proof of Theorem 3.1 (see (5.3) and the subsequent argument) shows that

$$\begin{aligned}
E_\nu |W_{\sigma_1}(g)|^p &\leqslant CE_\nu\Big(\sum_{n=1}^{\sigma_1}\big(g^2(M_n)+(Pg)^2(M_{n-1})\big)\Big)^{p/2}\\
&\leqslant C\big(E_\nu S_{\sigma_1}(g^2)^{p/2}+E_\nu S_{\sigma_1}\big((Pg)^2\big)^{p/2}+E_{\nu P}\, g^p(M_0)\big)\\
&\leqslant C\big(E_\nu |W_{\sigma_1}(g^2)|^{p/2}+E_\nu S_{\sigma_1}(Pg^2)^{p/2}+\nu^*(g^p)\big)\\
&\leqslant \ldots \leqslant C\Big(E_\nu |W_{\sigma_1}(g^{2^{s-1}})|^{p/2^{s-1}}+\sum_{i=0}^{s-1} E_\nu S_{\sigma_1}(Pg^{2^i})^{p/2^i}+\nu^*(g^p)\Big).
\end{aligned}$$

Since $0 < p/2^s \leqslant 1$, we have

$$\begin{aligned}
E_\nu |W_{\sigma_1}(g^{2^{s-1}})|^{p/2^{s-1}} &\leqslant C\Big(E_\nu S_{\sigma_1}(g^{2^s})^{p/2^s}+E_\nu S_{\sigma_1}\big((Pg^{2^{s-1}})^2\big)^{p/2^s}+E_{\nu P}\, g^p(M_0)\Big)\\
&\leqslant C\Big(E_\nu S_{\sigma_1}(g^p)+E_\nu S_{\sigma_1}\big((Pg^{2^{s-1}})^{p/2^{s-1}}\big)+\nu^*(g^p)\Big)\\
&\leqslant C\big(E_\nu S_{\sigma_1}(g^p)+E_\nu S_{\sigma_1}(Pg^p)+\nu^*(g^p)\big),
\end{aligned}$$

which gives

$$E_\nu |W_{\sigma_1}(g)|^p \leqslant C\big(E_\nu S_{\sigma_1}(g^p)+sE_\nu S_{\sigma_1}(Pg^p)+\nu^*(g^p)\big).$$

Putting this into (5.13) and using once again (5.12), we arrive at the asserted result. ∎

Proof of Theorem 3.7. Let us define $S_{k,n}(g) \stackrel{\text{def}}{=} S_n(g)-S_k(g)$ for $0 \leqslant k \leqslant n$. In order to show that $E_\nu T(t) < \infty$ put further $\tau(t) \stackrel{\text{def}}{=} \inf\{n \geqslant 1:\ S_{\sigma_n}(g) > t\}$ and observe that $T(t) \leqslant \sigma_{\tau(t)}$. Since $(S_{\sigma_n}(g))_{n\geqslant 0}$ has 1-dependent stationary increments under $P_\phi$, we have $E_\phi \tau(t) \leqslant C(t+1)$ for all $t \geqslant 0$ and some $C > 0$; see [10], Theorem 2.2. Moreover, $(\sigma_n)_{n\geqslant 0}$ has i.i.d. increments under $P_\phi$, whence we infer with Wald's identity that

$$\begin{aligned}
(5.14)\qquad E_\nu T(t) &\leqslant E_\nu \sigma_{\tau(t)} = E_\nu \sigma_1 + \int_{\{\tau(t)>1\}} \sigma_{\tau(t)}\, dP_\nu\\
&= E_\nu \sigma_1 + \int_{(-\infty,t]} E_\phi \sigma_{\tau(t-x)}\, P_\nu\big(S_{\sigma_1}(g)\in dx\big)\\
&= E_\nu \sigma_1 + E_\phi \sigma_1 \int_{(-\infty,t]} E_\phi \tau(t-x)\, P_\nu\big(S_{\sigma_1}(g)\in dx\big)\\
&\leqslant E_\nu \sigma_1 + CE_\phi \sigma_1 \int_{(-\infty,t]} (t-x+1)\, P_\nu\big(S_{\sigma_1}(g)\in dx\big)\\
&\leqslant E_\nu \sigma_1 + CE_\phi \sigma_1\big(t+1+E_\nu S_{\sigma_1}(g^-)\big) < \infty
\end{aligned}$$

for all $t \geqslant 0$.

In order to get the same result under $P_{\nu P^k}$ for $k \geqslant 1$, consider the MRW $(M_n, S_{k,n}(g))_{n \geqslant k}$ under $P_\nu$, where $M_k$ has distribution $\nu P^k$ under $P_\nu$. Define

$$T_k(t) \stackrel{\text{def}}{=} \inf\{n > k:\ S_{k,n}(g) > t\} \qquad \text{and} \qquad \tau_k(t) \stackrel{\text{def}}{=} \inf\{n > k:\ S_{k,\sigma_n}(g) > t\}$$

for $t \geqslant 0$. Then $E_{\nu P^k} T(t) = E_\nu T_k(t)$ and $T_k(t) \leqslant \sigma_{\tau_k(t)}$ for all $t \geqslant 0$. Moreover, we again have $E_\phi \tau_k(t) \leqslant C(t+1)$ for all $t \geqslant 0$ and some $C > 0$, and furthermore

$$E_\nu(\sigma_{k+1} - k) \leqslant E_\nu \sigma_{k+1} = E_\nu \sigma_1 + kE_\phi \sigma_1 < \infty$$

as well as

$$\begin{aligned} E_\nu\big(S_{\sigma_{k+1}}(g^-) - S_k(g^-)\big) &\leqslant E_\nu S_{\sigma_{k+1}}(g^-) \\ &\leqslant E_\nu S_{\sigma_1}(g^-) + kE_\phi S_{\sigma_1}(g^-) = E_\nu S_{\sigma_1}(g^-) + k\pi(g^-) E_\phi \sigma_1 < \infty. \end{aligned}$$

Consequently, an analogous estimation as in (5.14) leads to

$$E_{\nu P^k} T(t) = E_\nu T_k(t) \leqslant E_\nu \sigma_{k+1} + CE_\phi \sigma_1 \big(t + 1 + E_\nu S_{\sigma_{k+1}}(g)^-\big) < \infty.$$

From now on assume that additionally $\pi((g^+)^p) < \infty$ and $E_\nu L_{\sigma_1}(g^+)^p < \infty$ hold true.

(a) $\Rightarrow$ (b). In the following estimation we will make use of the independence of $\{T(t) > \sigma_n - r\}$ and $S_{\sigma_n,\sigma_{n+1}}((g^+)^p)$ for each $n \geqslant 1$, which is a consequence of the construction of the regeneration scheme (see [3], p. 151). We first prove (b) for $k = 0$, i.e. $E_\nu S_{T(t)}(g)^p < \infty$. By (3.16), it suffices to prove $E_\nu g^+(M_{T(t)})^p < \infty$ which follows from

$$\begin{aligned} (5.15) \quad E_\nu g^+(M_{T(t)})^p &= \sum_{n \geqslant 0} E_\nu g^+(M_{T(t)})^p \mathbf{1}_{\{\sigma_n < T(t) \leqslant \sigma_{n+1}\}} \\ &\leqslant E_\nu L_{\sigma_1}(g^+)^p + \sum_{n \geqslant 1} E_\nu S_{\sigma_n,\sigma_{n+1}}\big((g^+)^p\big) \mathbf{1}_{\{T(t) > \sigma_n - r\}} \\ &= E_\nu L_{\sigma_1}(g^+)^p + E_\phi S_{\sigma_1}\big((g^+)^p\big) \sum_{n \geqslant 1} P_\nu\big(T(t) > \sigma_n - r\big) \\ &\leqslant E_\nu L_{\sigma_1}(g^+)^p + E_\phi S_{\sigma_1}\big((g^+)^p\big) E_\nu\big(T(t) + r\big) \\ &= E_\nu L_{\sigma_1}(g^+)^p + \pi\big((g^+)^p\big) E_\phi \sigma_1 E_\nu\big(T(t) + r\big) < \infty. \end{aligned}$$

Proceeding with a proof of $E_{\nu P^k} g^+(M_{T(t)})^p < \infty$ for $k \geqslant 1$ we consider again the MRW $(M_n, S_{k,n}(g))_{n \geqslant k}$ under $P_\nu$ and utilize the fact that $E_{\nu P^k} S_{T(t)}(g)^p = E_\nu S_{k,T_k(t)}(g)^p$ for all $t \geqslant 0$. Since

$$\begin{aligned} E_\nu L_{\sigma_{k+1}}(g^+)^p &\leqslant E_\nu L_{\sigma_1}(g^+)^p + E_\nu S_{\sigma_1,\sigma_{k+1}}\big((g^+)^p\big) \\ &\leqslant E_\nu L_{\sigma_1}(g^+)^p + k\pi\big((g^+)^p\big) E_\phi \sigma_1 < \infty, \end{aligned}$$

a similar estimation to that in (5.15) now yields

$$(5.16)\qquad \begin{aligned} E_{\nu P^k} S_{T(t)}(g)^p &= E_\nu S_{k,T_k(t)}(g)^p \leqslant E_{\nu P^k} g^+ (M_{T_k(t)})^p \\ &\leqslant E_\nu L_{\sigma_{k+1}}(g^+)^p + \pi\big((g^+)^p\big) E_\phi \sigma_1 E_\nu\big(T_k(t)+r\big) < \infty. \end{aligned}$$

Since (b) $\Rightarrow$ (c) is trivial, we turn to the implication (c) $\Rightarrow$ (a). Notice that $\nu\big((g^+)^p\big) \leqslant E_\nu L_{\sigma_1}(g^+)^p < \infty$, while for $k \geqslant 1$ we infer from the left-hand side inequality in (3.16) that

$$\nu P^k\big((g^+)^p\big) = E_{\nu P^{k-1}} g^+(M_1) \leqslant E_{\nu P^{k-1}} S_{T(0)}(g)^p < \infty. \blacksquare$$

Proof of Corollary 3.9. Here it suffices to note that if $\nu = \phi$ in Theorem 3.7, then (5.15) simplifies to

$$\begin{aligned} E_\phi g^+(M_{T(t)})^p &\leqslant \sum_{n \geqslant 0} E_\phi S_{\sigma_n, \sigma_{n+1}}\big((g^+)^p\big) \mathbf{1}_{\{T(t) > \sigma_n - r\}} \\ &\leqslant \pi\big((g^+)^p\big) E_\phi \sigma_1 E_\phi\big(T(t)+r\big) < \infty. \blacksquare \end{aligned}$$

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