# CENTRAL LIMIT THEOREM FOR A GAUSSIAN INCOMPRESSIBLE FLOW WITH ADDITIONAL BROWNIAN NOISE

BY

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Abstract. We generalize the result of Komorowski and Papanicolaou published in [7]. We consider the solution of stochastic differential equation $dX(t) = V(t, X(t)) dt + \sqrt{2\kappa} dB(t)$, where $B(t)$ is a standard $d$-dimensional Brownian motion and $V(t, x)$, $(t, x) \in R \times R^d$, is a $d$-dimensional, incompressible, stationary, random Gaussian field decorrelating in finite time. We prove that the weak limit as $\varepsilon \downarrow 0$ of the family of rescaled processes $X_\varepsilon(t) = \varepsilon X(t/\varepsilon^2)$ exists and may be identified as a certain Brownian motion.

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## 1. INTRODUCTION

Consider the turbulent transport of a tracer particle modeled by the stochastic differential equation

$$(1.1) \qquad dX(t; \omega, \sigma) = V(t, X(t); \omega) dt + \sqrt{2\kappa} dB(t; \sigma),$$

$$X(0; \omega, \sigma) = 0,$$

where $V(t, x; \omega)$, $(t, x) \in R \times R^d$, is a $d$-dimensional random field over a certain probability space $\mathcal{T}_0 = (\Omega, \mathcal{V}, P)$, and $B(t; \sigma)$, $t \geq 0$, is a standard $d$-dimensional Brownian motion over another probability space $\mathcal{T}_1 = (\Sigma, \mathcal{W}, Q)$. The constant $\kappa \geq 0$ stands for a molecular diffusivity of the medium. Let $E$ and $M$ denote the expectations in $\mathcal{T}_0$ and $\mathcal{T}_1$, respectively.

This model is widely used in physics literature to describe the motion in a turbulent flow. We are interested in a long time, large scale behavior of passive tracer over the product probability space. Namely, we consider the macroscopic scaling $x \sim x/\varepsilon$, $t \sim t/\varepsilon^2$. The rescaled process $X_\varepsilon(\cdot)$ satisfies the

following stochastic equation:

$$dX_\varepsilon(t) = \frac{1}{\varepsilon} V\left(\frac{t}{\varepsilon^2}, \frac{X_\varepsilon(t)}{\varepsilon}\right) dt + \sqrt{2\kappa} dB(t).$$

We ask the natural question about the convergence in law of $X_\varepsilon(t)$ as $\varepsilon \downarrow 0$.

This problem has a long history and several results, varying with the assumptions concerning the random velocity field $V$, are well known. Let $V$ be stationary, incompressible and centered. The first group of results deals with the velocity field having a so-called *stream matrix* $H$, i.e.

$$V(t, x) = \overline{V}_x H(t, x).$$

Papanicolaou and Varadhan [10] and also Kozlov [8] proved the convergence of rescaled processes given by a time-independent field $V(x)$, $x \in R^d$, with the stationary stream matrix $H(x)$, under the assumption of boundedness of $V$ and $H$. A similar result was proved in [9] under the same conditions but for the time-dependent velocity field $V(t, x)$. In [4] Fannjiang and Komorowski give the proof of convergence for the random fields $V$ not bounded but with a finite $p$-th moment (for some $p > d+2$).

In the second group we have results concerning the time-dependence assumptions imposed on the velocity field. Here we have for example the convergence for an Ornstein–Uhlenbeck velocity field of finite modes (Carmona and Xu [3]) and for a class of Markovian fields with strong mixing properties (Fannjiang and Komorowski [5]).

In this paper we follow the idea of Komorowski and Papanicolaou published in [7] which contributes to the last group of results. Assume that the field $V$ is Gaussian, incompressible, stationary, centered and that it decorrelates in finite time. The authors considered the equation (1.1) with $\kappa = 0$ (without additional Brownian noise) and proved that the laws of the family of processes $X_\varepsilon(t) = \varepsilon X(t/\varepsilon^2)$ converge as $\varepsilon \downarrow 0$ to that of the Brownian motion with covariance matrix given by

$$D_{ij} = \int_0^\infty E\left[V_i(t, X(t)) V_j(0, 0) + V_j(t, X(t)) V_i(0, 0)\right] dt, \quad i, j = 1, \ldots, d.$$

We show that convergence still holds in the presence of molecular diffusivity $\kappa > 0$. Namely, we prove that the weak limit of the laws of $X_\varepsilon(\cdot)$ over $C\left([0, +\infty); R^d\right)$ is a Brownian motion with covariance matrix

$$(1.2) \qquad D_{ij} = \int_0^\infty M E\left[V_i(t, X(t)) V_j(0, 0) + V_j(t, X(t)) V_i(0, 0)\right] dt + 2\kappa\delta_{ij},$$

$$i, j = 1, \ldots, d.$$

This result confirms the turbulent diffusion hypothesis of G. I. Taylor coming from the early 1920's (see [15]).

Let us describe the main steps of the proof. Thanks to the key assumption of the incompressibility of the field $V$, in the proof we use the result of Port and Stone [11], namely the stationarity of the Lagrangian velocity process $\left(V(t, X(t))\right)_{t \geqslant 0}$. We develop, as in [7], the idea of the "transport operator" $Q$, being a linear operator acting on the space of elements integrable with respect to $P$. The operator $Q$ preserves densities and satisfies

$$ME\left[V_i\left(s, X(s)\right)V_j(0, 0)\right] = ME\left[V_i\left(s-T, X(s-T)\right)Q\left[V_j(0, 0)\right]\right]$$

for $s \geqslant T$, where $T$ is the decorrelation time of the field $V$. Establishing estimates of the rate of convergence of the sequence $\{\|Q^n Y\|_{L^1}\}_{n \in N}$ for any $Y$ measurable with respect to $\{V(t, \cdot), t \leqslant 0\}$ and satisfying $EY = 0$, we prove the convergence of the integrals in (1.2). In addition, we show that for any $L > 0$

$$ME\left[|X_\varepsilon(t) - X_\varepsilon(s)|^2 \, |X_\varepsilon(u) - X_\varepsilon(t)|\right] \leqslant C(u-s)^{1+\nu}$$

for any $0 \leqslant s \leqslant t \leqslant u \leqslant L$ and some constants $C, \nu > 0$. This gives us the tightness of the family $\{X_\varepsilon(\cdot)\}$, $\varepsilon > 0$, in the Skorohod space $D([0, L], R^d)$ for any $L > 0$ and, by the continuity of trajectories, also in $C([0, L], R^d)$. Thanks to Stone [13], it gives tightness in $C([0, +\infty), R^d)$. Finally, we identify the limit as a certain Wiener measure with the help of the Stroock–Varadhan martingale problem.

This idea is strongly based on [7] and we make the references to lemmas presented therein. We skip the proofs which may be generalized to our situation in a straightforward way. However, we present the complete argument for the results which are new or involve some major adjustments.

## 2. NOTATION AND FORMULATION OF THE MAIN RESULT

By $L^p := L^p(\Omega, \mathcal{V}, P)$ we understand the space of $L^p$-integrable random variables over the space $\mathcal{T}_0$ equipped with the standard $\|\cdot\|_p$-norm. Let $E[\cdot|\mathscr{A}]$ denote the conditional expectation with respect to some sub-$\sigma$-algebra $\mathscr{A} \subset \mathcal{V}$.

Let $\tau_{t,x} \colon \Omega \to \Omega$, $(t, x) \in R \times R^d$, be a group of measure-preserving transformations, i.e. such that the map $(t, x, \omega) \mapsto \tau_{t,x}(\omega)$ is jointly $\mathscr{B}(R) \otimes \mathscr{B}(R^d) \otimes \mathcal{V}$ to $\mathcal{V}$ measurable, $\tau_{t,x}\tau_{s,y} = \tau_{t+s,x+y}$, $\tau_{t,x}^{-1}(A) \in \mathcal{V}$ and $P[\tau_{t,x}^{-1}(A)] = P[A]$ for all $(t, x), (s, y) \in R \times R^d$, $A \in \mathcal{V}$. Here $\mathscr{B}(R^d)$ denotes the $\sigma$-algebra of Borel subsets of $R^d$.

Suppose now that $\tilde{V} \colon \Omega \to R^d$ is a $d$-dimensional random vector such that the random field given by $V(t, x; \omega) := \tilde{V}\left(\tau_{t,x}(\omega)\right)$ satisfies the following conditions:

(V1) It is centered, i.e. $E\tilde{V} = 0$.

(V2) It is Gaussian, i.e. all its finite-dimensional distributions are Gaussian random vectors.

(V3) It is divergence-free, i.e. $\operatorname{div} V(t, x) := \sum_{i=1}^{d} \partial_{x_i} V_i(t, x) \equiv 0$.

(V4) Its correlation matrix $R(t, x) := \left[ E\left[ V_i(t, x) V_j(0, 0) \right] \right]_{i,j=1,\ldots,d}$ satisfies

$$|R(0, 0) - R(t, x)| + \sum_{i,j=1}^{d} |\partial_{ij}^2 R(0, 0) - \partial_{ij}^2 R(t, x)| \leqslant \frac{C}{|\ln \sqrt{t^2 + |x|^2}|^{1+\eta}}$$

for some constant $C > 0$ and all $(t, x) \in R \times R^d$. According to [2], Theorem 3.4.1, this guarantees a version of $V$ being jointly continuous in $(t, x)$ and of $C^1$-class in $x$.

(V5) The field $V$ decorrelates in finite time, i.e. there exists $T > 0$ such that for all $|t| \geqslant T$ and $x \in R^d$ we have $R(t, x) = 0$.

Remark. For a Gaussian field $G(t)$, where $t \in \mathcal{T}$ is some abstract parameter, we define a *d-ball* as related to the pseudometric $d(t_1, t_2) = [E|G(t_1) - G(t_2)|^2]^{1/2}$. Let $N(\varepsilon)$ denote the entropy number of the field $V$, i.e. the minimal number of $d$-balls, corresponding to $V$, with radius $\varepsilon > 0$ needed to cover $R \times R^d$. By the condition (V4) this number can be estimated by

$$(2.1) \qquad N(\varepsilon) \leqslant K_1 \exp\left( K_2 (d+1) \varepsilon^{-2/(1+\eta)} \right)$$

for some constants $K_1, K_2 > 0$ independent of $\varepsilon$. According to [2], p. 121, (2.1) will allow us to use some of the Borell–Fernique–Talagrand type of tail estimates later on.

Let $B(\cdot)$ denote the standard $d$-dimensional Brownian motion starting from $0$, considered over a certain probability space $\mathcal{T}_1 = (\Sigma, \mathcal{W}, Q)$, with mathematical expectation corresponding to the probability measure $Q$.

Let us consider a probability space $\mathcal{T}_0 \otimes \mathcal{T}_1 = (\Omega \times \Sigma, \mathcal{V} \otimes \mathcal{W}, P \otimes Q)$ and a stochastic process $X(\cdot)$ over this space, given by the following stochastic differential equation:

$$(2.2) \qquad \begin{aligned} dX(t; \omega, \sigma) &= V\big(t, X(t; \omega, \sigma); \omega\big)\, dt + dB(t; \sigma), \\ X(0; \omega, \sigma) &= 0. \end{aligned}$$

For simplicity we suppose $\sqrt{2\kappa} = 1$ in (1.1) but the proof is still valid for any $\kappa > 0$. For $\varepsilon > 0$ define $X_\varepsilon(t) := \varepsilon X(t/\varepsilon^2)$, $t \geqslant 0$. We will prove the following result:

THEOREM 1. *Suppose that the d-dimensional random field $V$ satisfies the conditions (V1)–(V5) listed above. Then the integrals*

$$(2.3) \qquad D_{ij} = \int_0^\infty ME\left[ V_i\big(t, X(t)\big) V_j(0, 0) + V_j\big(t, X(t)\big) V_i(0, 0) \right] dt + \delta_{i,j},$$

$$i, j = 1, \ldots, d,$$

*converge. The laws of the processes $\{X_\varepsilon(t)\}_{t \geqslant 0}$ induced on the space $C\left([0, +\infty), R^d\right)$*

converge weakly, as $\varepsilon \downarrow 0$, to the law of the Brownian motion with covariance matrix $D = [D_{ij}]$.

Some additional notation: for any $x \in R$, $\text{Ent}(x)$ denotes the biggest integer less than or equal to $x$; for any $x, y \in R$, $x \wedge y$ denotes $\min(x, y)$.

## 3. AUXILIARY LEMMAS

Consider the processes $Y(\cdot)$ over the probability space $\mathscr{T}_0 \otimes \mathscr{T}_1$ given by the equations

$$dY^{0,x}(t; \omega, \sigma) = V\big(t, Y^{0,x}(t; \omega, \sigma); \omega\big) dt + dB(t; \sigma),$$

$$Y^{0,x}(0; \omega, \sigma) = x.$$

By the assumptions (V1)–(V5) we can apply the following result of Port and Stone [11]. First of all, given $x \in R^d$, the equation above determines a unique process $Y^{0,x}$. Next, we have:

LEMMA 1 (cf. [11], Theorem 3, p. 501). *For any $t \geqslant 0$ the random map $x \mapsto Y^{0,x}(t)$ preserves measure on $R^d$ and satisfies*

$$Y^{0,x}\big(t; \tau_y(\omega), \sigma\big) = Y^{0,x+y}(t; \omega, \sigma).$$

*For all $x \in R^d$, $\sigma \in \Sigma$ and $t \geqslant 0$ the random element $\omega \mapsto \tau_{Y^{0,x}(t;\omega,\sigma)}(\omega)$ has the law $P$ on $\Omega$.* ∎

As a corollary we get the following lemma:

LEMMA 2. *Let $\tilde{U}: \Omega \times \Sigma \to R$ be any random variable such that $\tilde{U} \in L^1(\mathscr{T}_0 \otimes \mathscr{T}_1)$ and define*

$$U(t, x; \omega, \sigma) := \tilde{U}\big(\tau_{t,Y^{0,x}(t;\omega,\sigma)}(\omega), \sigma\big).$$

*For any $x \in R^d$ and $\tilde{U} \in L^1(\mathscr{T}_0 \otimes \mathscr{T}_1)$ the process $\{U(t, x)\}_{t \geqslant 0}$ is strictly stationary over $\mathscr{T}_0 \otimes \mathscr{T}_1$.*

Proof. Using the integral form of expressing $Y^{0,x}(t)$ it is easy to see that for any $t, h \geqslant 0$ we have

$$Y^{0,x}(t+h; \omega, \sigma) = Y^{0,x}(h; \omega, \sigma) + Y^{0,0}\big(t; \tau_{h,Y^{0,x}(h;\omega,\sigma)}(\omega), \sigma\big).$$

Moreover, from Lemma 1 it follows that the measure $P \otimes Q$ is preserved by the transformation $\theta_h: (\omega, \sigma) \mapsto \big(\tau_{h,Y^{0,x}(h;\omega,\sigma)}(\omega), \sigma\big)$. We finish our proof essentially with the same argument as used to prove Lemma 1, p. 240, of [7]. ∎

Recall that the random field $V(t, x; \omega) = \big(V_1(t, x; \omega), \dots, V_d(t, x; \omega)\big)$ is Gaussian. Let us introduce some notation. As in [7], let $L^2_{a,b}$ denote the closure in $L^2$-norm of the linear span of $V_i(t, x)$, $t \in [a, b]$, $x \in R^d$, $i = 1, \dots, d$. By $\mathscr{V}_{a,b}$ we denote the $\sigma$-algebra generated by all random vectors from $L^2_{a,b}$. Let $L^{2\perp}_{a,b} = L^2_{-\infty,+\infty} \ominus L^2_{a,b}$ be the orthogonal complement of $L^2_{a,b}$ in $L^2_{-\infty,+\infty}$, and

$\mathscr{V}^{\perp}_{a,b}$ the $\sigma$-algebra similarly generated by elements of $L^{2\perp}_{a,b}$. According to [12], Theorems 10.1 and 10.2, p. 181, the $\sigma$-algebras $\mathscr{V}_{a,b}$ and $\mathscr{V}^{\perp}_{a,b}$ are independent. Let $V_{a,b}(t, x)$ be the orthogonal projection of $V(t, x)$ onto $L^2_{a,b}$, i.e. each component of $V_{a,b}(t, x)$ is the projection of the corresponding component of $V(t, x)$. We define $V^{a,b}(t, x) := V(t, x) - V_{a,b}(t, x)$. Of course, $V_{a,b}(t, x)$ is $\mathscr{V}_{a,b}$-measurable while $V^{a,b}(t, x)$ is $\mathscr{V}^{\perp}_{a,b}$-measurable. We can also see that $V_{a,b}(t, x)$ is jointly continuous and $C^1$-smooth in $x$, $P$-a.s. (see e.g. [1], Theorem 3.4.1). Finally, let $\mathscr{T}_{a,b}$ denote the probability space $(\Omega \times \Omega, \mathscr{V}_{a,b} \otimes \mathscr{V}^{\perp}_{a,b}, P \otimes P)$.

Over the space $\mathscr{T}_0 \otimes \mathscr{T}_0 = (\Omega \times \Omega, \mathscr{V} \otimes \mathscr{V}, P \otimes P)$ we define the random field $\tilde{V}$ by

$$\tilde{V}_{a,b}(t, x; \omega, \omega') = V_{a,b}(t, x; \omega) + V^{a,b}(t, x; \omega').$$

Let $c \in R$ and consider now the process given by the following stochastic differential equation:

$$(3.1) \qquad d\tilde{X}^{b,x}_c(t; \omega, \omega', \sigma) = \tilde{V}_{a,b}(t + c, \tilde{X}^{b,x}_c(t); \omega, \omega') dt + dB(t + c; \sigma),$$
$$\tilde{X}^{b,x}_c(b) = x.$$

If $a = -\infty$ and $b = 0$, we shorten the notation by writing

$$(3.2) \qquad\qquad \tilde{X}(\cdot) = \tilde{X}^{0,0}_0(\cdot).$$

The following lemma concerning the conditional expectation of the process along its trajectory is the compilation of the adapted versions of Lemma 3, p. 241, and Lemma 4, p. 242, of [7] and may be proved with the help of the same argument as presented therein.

LEMMA 3. *Let* $f \in L^2(\Omega, \mathscr{V}_{-\infty, +\infty}, P)$ *and* $-\infty \leqslant a \leqslant b \leqslant +\infty$. *Then there exists* $\tilde{f} \in L^2(\mathscr{T}_{a,b})$ *such that* $f(\omega) = \tilde{f}(\omega, \omega)$ *and both* $f$ *and* $\tilde{f}$ *have the same probability distributions. Let* $c \in R$. *Then for any* $C \in \mathscr{V}^{\perp}_{a,b}$ *we have:*

$$ME\left[ f\left(\tau_{0, X^b, 0(t; \tau_c, 0(\omega), \sigma)}(\omega)\right) \mathbf{1}_C(\omega) \middle| \mathscr{V}_{a,b} \right]$$
$$= ME_{\omega'}\left[ \tilde{f}\left(\tau_{0, \tilde{X}^b_c, 0(t; \omega, \omega', \sigma)}(\omega), \tau_{0, \tilde{X}^b_c, 0(t; \omega, \omega', \sigma)}(\omega')\right) \mathbf{1}_C(\omega') \right],$$

*where* $E_{\omega'}$ *denotes the expectation applied to the variable* $\omega'$ *only.* ∎

## 4. TRANSPORT OPERATOR $Q$ AND ITS PROPERTIES

Recall that $\tilde{X}$ is given by (3.2). For a given $(\omega, \sigma)$ define a transformation $Z^t_{\omega, \sigma}: \Omega \to \Omega$ by

$$Z^t_{\omega, \sigma}(\omega') = \tau_{0, \tilde{X}(t; \omega, \omega', \sigma)}(\omega').$$

Let $J^t(\cdot; \omega, \sigma)$ be the probability measure on $(\Omega, \mathscr{V}^{\perp}_{-\infty, 0})$ given by

$$J^t(A; \omega, \sigma) = P\left[(Z^t_{\omega, \sigma})^{-1}(A)\right].$$

Denote by $\mathscr{D}$ the family of densities $f \in L^1(\Omega, \mathscr{V}^{\perp}_{-\infty,0}, P)$, $f \geqslant 0$, $\int_{\Omega} f dP = 1$. For any $f \in \mathscr{D}$ we define a measure on $(\Omega, \mathscr{V}^{\perp}_{-\infty,0})$ by

$$(4.1) \qquad [Qf][A] = M \int_{\Omega} J^T(A; \omega, \sigma) f(\omega) P(d\omega).$$

Let $\mathscr{V}^s_{0,T}$ denote the $\sigma$-algebra generated by the random vectors $V^{-\infty,0}(t, x)$, $0 \leqslant t \leqslant T$, $x \in \mathbf{R}^d$, and let $\mathscr{V}^0_{-\infty,0} = \tau_{T,0}(\mathscr{V}^s_{0,T}) \subseteq \mathscr{V}_{-\infty,0}$. As in [7] we conclude that for any $f \in \mathscr{D}$, $[Qf]$ is absolutely continuous with respect to $P$ and that its Radon–Nikodym derivative is $\mathscr{V}^s_{0,T}$-measurable. Moreover, we have $[Q1] = P$.

Now we define the transport operator acting on $f \in \mathscr{D}$:

$$(4.2) \qquad Qf = \frac{d[Qf]}{dP} \circ \tau_{-T,0}.$$

Similarly to Lemma 7, p. 247, of [7] we infer that $Q: L^1(\Omega, \mathscr{V}_{-\infty,0}, P) \to L^1(\Omega, \mathscr{V}_{-\infty,0}, P)$ is a positive linear operator, preserving densities, $Q1 = 1$ and it extends to a contraction in every space $L^p(\Omega, \mathscr{V}_{-\infty,0}, P)$, $1 \leqslant p \leqslant \infty$. Now we will prove the following lemma, which displays the usefulness of the transport operator $Q$.

LEMMA 4. *Let* $f, g \in L^1(\Omega)$ *be random variables* $\mathscr{V}^{-\infty,0}$- *and* $\mathscr{V}_{-\infty,0}$-*measurable, respectively. Let* $t \geqslant T > 0$. *Then we have*

$$ME\left[f\left(\tau_{t,X(t;\omega,\sigma)}(\omega)\right)g(\omega)\right] = ME\left[f\left(\tau_{t-T,X(t-T;\omega,\sigma)}(\omega)\right)Qg(\omega)\right].$$

Proof. To shorten the notation, from now on $dP$, $dQ$, $dP'$ and $dQ'$ will stand for $P(d\omega)$, $Q(d\sigma)$, $P(d\omega')$, $Q(d\sigma')$, respectively, and we will omit the argument $\sigma$. If not indicated otherwise, all the integrals should be understood as the integrals over the whole space $\Omega \times \Sigma$. Using Lemma 3 we may write

$$(4.3) \quad ME\left[f\left(\tau_{t,X(t;\omega)}(\omega)\right)g(\omega)\right] = ME\left[ME\left[f\left(\tau_{t,X(t;\omega)}(\omega)\right) \mid \mathscr{V}_{-\infty,0}\right]g(\omega)\right]$$

$$= \iint f\left(\tau_{t,\tilde{X}(t;\omega,\omega')}(\omega')\right)g(\omega) dP'dQ'dPdQ$$

$$= \iint f\left(\tau_{t,\tilde{X}^{T,0}(t;\tau_{0,\tilde{X}(T)}(\omega),\tau_{0,\tilde{X}(T)}(\omega'))}\left(\tau_{0,\tilde{X}(T)}(\omega')\right)\right)g(\omega) dP'dQ'dPdQ.$$

The last equality above comes from the fact that

$$\tilde{X}(t; \omega, \omega') = \tilde{X}(T; \omega, \omega') + \tilde{X}^{T,0}\left(t; \tau_{0,\tilde{X}(T;\omega,\omega')}(\omega), \tau_{0,\tilde{X}(T;\omega,\omega')}(\omega')\right).$$

Let $p^{\omega,\omega'}(s, x, t, y)$ denote the density of probability distribution of the process $\tilde{X}^{s,x}(t; \omega, \omega')$, $t \geqslant s$. If we introduce the notation $y := \tilde{X}(T; \omega, \omega')$, we may rewrite (4.3) in the form

$$(4.4) \quad \iint_{\mathbf{R}^d}\int f\left(\tau_{t,\tilde{X}^{T,0}(t;\tau_{0,y}(\omega),\tau_{0,y}(\omega'))}\left(\tau_{0,y}(\omega')\right)\right)p^{\omega,\omega'}(0, 0, T, y)g(\omega) dydP'dQ'dPdQ$$

for $t \geqslant T$. Since $t \geqslant T$, where $T$ is the decorrelation time of the field $V$, from the definition of $\tilde{X}$ we infer that, in fact, $\tilde{X}^{T,0}\left(t; \tau_{0,y}(\omega), \tau_{0,y}(\omega')\right) = X^{T,0}\left(t; \tau_{0,y}(\omega')\right)$.

Hence (4.4) is equal to

$$\iint\limits_{\mathbf{R}^d}\int f\left(\tau_{t,X^{T,0}(t;\omega')}(\omega')\right)p^{\omega,\tau_0,-y(\omega')}(0,\,0,\,T,\,y)\,g\left(\omega\right)dy dP' dQ' dP dQ$$

$$=\iint\limits_{\mathbf{R}^d}\int f\left(\tau_{t-T,X^{T,0}(t;\tau-T,0(\omega'))}(\omega')\right)p^{\tau-T,-y(\omega),\tau-T,-y(\omega')}(0,\,0,\,T,\,y)\,g\left(\tau_{-T,-y}(\omega)\right)$$

$$\times dy dP' dQ' dP dQ$$

$$=\iint\limits_{\mathbf{R}^d}\int f\left(\tau_{t-T,X(t-T;\omega')}(\omega')\right)p^{\omega,\omega'}(-T,\,-y,\,0,\,0)\,g\left(\tau_{-T,-y}(\omega)\right)dy dP' dQ' dP dQ.$$

Define

$$(4.5)\qquad Q'g\left(\omega\right)=\iint\limits_{\mathbf{R}^d}p^{\omega,\omega'}(-T,\,-y,\,0,\,0)\,g\left(\tau_{-T,-y}(\omega)\right)dy\,dP' dQ'$$

for any $\mathscr{V}_{-\infty,0}$-measurable $g$. It is easy to see that it is an equivalent way of expressing the transport operator $Q$ introduced before. We get the assertion of our lemma. ∎

As a corollary we have

LEMMA 5. *Let* $p \geqslant 0$, $N$, $k \in N$ *and* $s_{k+2} \geqslant s_{k+1} \geqslant \ldots \geqslant s_1 \geqslant NT$, $i = 1, \ldots, d$. *Assume that* $Y \in L^1(\Omega, \mathscr{V}_{-\infty,0}, P)$. *Then*

$$ME\Big[\Big|\int\limits_{s_{k+1}}^{s_{k+2}}V\left(\varrho, X(\varrho)\right)d\varrho\Big|^p V\left(s_1, X(s_1)\right)\ldots V\left(s_k, X(s_k)\right)Y\Big]$$

$$=ME\Big[\Big|\int\limits_{s_{k+1}-NT}^{s_{k+2}-NT}V\left(\varrho, X(\varrho)\right)d\varrho\Big|^p$$

$$\times V\left(s_1-NT, X(s_1-NT)\right)\ldots V\left(s_k-NT, X(s_k-NT)\right)Q^N Y\Big].\ \blacksquare$$

## 5. RATE OF CONVERGENCE OF $\{Q^n Y\}_{n\geqslant 0}$

In this section we will prove the following lemma which will constitute the main tool in establishing diverse estimates later on.

LEMMA 6. *Let* $Y \in L^2(\Omega, \mathscr{V}_{-\infty,0}, P)$ *be a random variable such that* $EY = 0$. *Then for any* $s > 0$ *there exists a constant* $C$ *depending only on* $s$ *and* $\|Y\|_{L^2}$ *such that*

$$\|Q^n Y\|_{L^1} \leqslant C/n^s \quad \text{for all } n \in N.$$

Proof. This proof is a modification of the proof of Lemma 10, p. 249, of [7]. According to [6], the part of $J^T(d\omega'; \omega, \sigma)$, which is absolutely continuous with respect to $P$, has a density given by the formula

$$(5.1)\qquad\qquad\qquad \int\limits_{\mathbf{R}^d}\frac{v_0^T(dx; \omega, \omega', \sigma)}{G_T(0; \omega, \tau_{0,x}(\omega'), \sigma)},$$

where $v_0^T(U; \omega, \omega', \sigma)$ stands for the cardinality of those $y \in U$ for which $\psi_t(y; \omega, \omega', \sigma) = 0$ and

$$G_T(x; \omega, \omega', \sigma) = \det V_x \psi_t(x; \omega, \omega', \sigma),$$

where

(5.2) $$\psi_t(x; \omega, \omega', \sigma) = x + \tilde{X}(t; \omega, \tau_{0,x}(\omega'), \sigma).$$

As in [7], using the integral form of expressing $\tilde{X}$ (coming from (3.1)), we get

$$\frac{d}{dt} V_x \psi_t(0; \omega, \omega', \sigma) = V_x V_{-\infty,0}\big(t, \tilde{X}(t; \omega, \omega', \sigma); \omega\big)\big[V\psi_t(0; \omega, \omega^t, \sigma) - I\big]$$

$$+ V_x V^{-\infty,0}\big(t, \tilde{X}(t; \omega, \omega', \sigma); \omega'\big) V\psi_t(0; \omega, \omega', \sigma)$$

and

$$V_x \psi_0(0; \omega, \omega', \sigma) = I.$$

Let $\alpha(\varrho)$ be a smooth function, increasing on $\varrho \geqslant 0$, satisfying $\alpha(-\varrho) = \alpha(\varrho)$, $\alpha(0) = 0$, and $\alpha(\varrho) = \sqrt{\varrho}$ for $\varrho \geqslant 1$. Let $\varphi(x) = \alpha(|x|)$ for $x \in \mathbf{R}^d$. Fix $\gamma \in (\frac{1}{2}, 1)$ and for any $\lambda > 0$ let us introduce a set

$$K_n(\lambda) = \Big[\omega \in \Omega: \sup_{0 \leqslant t \leqslant T} \big[|V_{-\infty,0}(t, x)| + |V_x V_{-\infty,0}(t, x)|\big] \leqslant \lambda\big(\varphi(x) + \log^\gamma n\big)\Big].$$

According to the Remark of Section 2 and Theorem 5.4, p. 121, of [1], we may find $\Lambda > 0$ such that for $\lambda \geqslant \Lambda$ we have

$$P[\Omega \setminus K_n(\lambda)] = P[K_n(\lambda)^c] \leqslant C \exp(C_1 \lambda^{2/(2+\eta)}) \exp\left(-\frac{\lambda^2}{4\sigma_n^2}\right) \leqslant C \exp\left(-\frac{\lambda^2}{8\sigma_n^2}\right),$$

where

$$\sigma_n^2 = \sup_{0 \leqslant t \leqslant T, x \in \mathbf{R}^d} E\left[\frac{|V_{-\infty,0}(t, x)|^2 + |V_x V_{-\infty,0}(t, x)|^2}{\big(\varphi(x) + \log^\gamma n\big)^2}\right]$$

$$= \frac{1}{\log^{2\gamma} n} E\left[|V_{-\infty,0}(t, 0)|^2 + |V_x V_{-\infty,0}(t, 0)|^2\right] = \frac{C}{\log^{2\gamma} n}.$$

Hence

$$P[K_n^c] = P[\Omega \setminus K_n(\lambda)] \leqslant C_{01} \exp\{-C_{02} \lambda^2 \log^{2\gamma} n\}$$

for some constants $C_{01}, C_{02} > 0$. Take $v \in (0, 1)$ such that $[\Lambda/(\Lambda + 1)] e^{v(\Lambda + 1)T} < 1$ and $v\Lambda < 1$. Define also two families of sets by

$$L_m = \Big[\omega \in \Omega: \sup_{0 \leqslant t \leqslant T} \big[|V^{-\infty,0}(t, x)| + |V_x V^{-\infty,0}(t, x)|\big] \leqslant v|x| + \log^\gamma m\Big],$$

$$S_p = \Big[\sigma \in \Sigma: \sup_{0 \leqslant t \leqslant T} |B(t; \sigma)| \leqslant \log^{2\gamma} p\Big].$$

Since the field $V(t, x)/(\sqrt{t^2+|x|^2+1})$ is $P$-a.s. bounded, we get $\lim_{m \to +\infty} P[L_m] =$
$= 1$. From the properties of Brownian motion we obtain

$$Q[S_p^c] = Q[\Sigma \setminus S_p] = Q\left[\sup_{0 \leqslant t \leqslant T} |B(t; \sigma)| \geqslant \log^{2\gamma} p\right] \leqslant \frac{C}{\log^{4\gamma} p}$$

for any $p \geqslant 2$ and some constant $C > 0$.

The properties of $\alpha$ imply that there exists a constant $C_\nu$ such that $\varphi(x) \leqslant \nu|x| + C_\nu$. For $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$ we get

$$\left|\frac{d(\tilde{X}(t)-B(t))}{dt}\right| = |V_{-\infty,0}(t, x; \omega) + V^{-\infty,0}(t, x; \omega')|$$

$$\leqslant \nu(\Lambda+1)|\tilde{X}(t)| + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m$$

$$\leqslant \nu(\Lambda+1)|\tilde{X}(t)-B(t)| + \nu(\Lambda+1)\log^{2\gamma} p + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m.$$

Hence, by the Gronwall inequality

$$\sup_{0 \leqslant t \leqslant T} |\tilde{X}(t)-B(t)| \leqslant \frac{e^{\nu(\Lambda+1)T}-1}{\nu(\Lambda+1)}[\nu(\Lambda+1)\log^{2\gamma} p + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m]$$

and since $\sigma \in S_p$, we get

$$(5.3)\qquad \sup_{0 \leqslant t \leqslant T} |\tilde{X}(t)| \leqslant \frac{e^{\nu(\Lambda+1)T}-1}{\nu(\Lambda+1)}$$

$$\times[\nu(\Lambda+1)\log^{2\gamma} p + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m] + \log^{2\gamma} p.$$

Using the estimate from above we write

$$\frac{d}{dt}|\nabla_x \psi_t(0)| \leqslant \left|\frac{d}{dt}\nabla_x \psi_t(0)\right|$$

$$\leqslant |\nabla_x V_{-\infty,0}(t, \tilde{X}(t); \omega)|[|\nabla\psi_t(0)|+1] + |\nabla_x V^{-\infty,0}(t, \tilde{X}(t); \omega')||\nabla_x \psi_t(0)|$$

$$\leqslant \sup_{0 \leqslant t \leqslant T} |\nabla_x V_{-\infty,0}(t, \tilde{X}(t); \omega)|[|\nabla_x \psi_t(0)|+1]$$

$$+ \sup_{0 \leqslant t \leqslant T} |\nabla_x V^{-\infty,0}(t, \tilde{X}(t); \omega')||\nabla_x \psi_t(0)|$$

$$\leqslant [\nu(\Lambda+1)\sup_{0 \leqslant t \leqslant T}|\tilde{X}(t)| + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m]|\nabla_x \psi_t(0)|$$

$$+ [\nu(\Lambda+1)\sup_{0 \leqslant t \leqslant T}|\tilde{X}(t)| + \Lambda C_\nu + \Lambda \log^\gamma n] \leqslant a|\nabla_x \psi_t(0)| + b,$$

where $a$ and $b$ come from the estimate (5.3) of $\sup_{0 \leqslant t \leqslant T} |\tilde{X}(t)|$. Again from the Gronwall inequality we get

$$\sup_{0 \leqslant t \leqslant T} |\nabla_x \psi_t(\mathbf{0})| \leqslant \frac{(a+b)e^{aT}-b}{a} \leqslant \frac{(a+b)e^{aT}}{a} \leqslant 2e^{aT}$$

$$= 2\exp\left[T\left[\exp\left(v(\Lambda+1)T\right)\right]\left(v(\Lambda+1)\log^{2\gamma}p + \Lambda C_v + \Lambda\log^\gamma n + \log^\gamma m\right)\right]$$

$$\leqslant C_{21}\exp\left(C_{22}\log^\gamma n\right)\exp\left(C_{23}\log^\gamma m\right)\exp\left(C_{24}\log^{2\gamma}p\right)$$

for $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$.

Let $B(\mathbf{0}, R)$ denote the ball of radius $R$ centered in $\mathbf{0} \in \mathbf{R}^d$. We will now investigate the size of the set $[\mathbf{x}: \psi_T(\mathbf{x}) = \mathbf{0}]$ for $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$. We have the following lemma which may be proved in the same way as Lemma 11, p. 252, of [7].

LEMMA 7. *For* $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ *and* $\sigma \in S_p$ *the random set* $[\mathbf{x}: \psi_T(\mathbf{x}) = \mathbf{0}]$ *is nonempty and there exists a constant* $C_{31} > 0$ *independent of* $n, m, p$ *such that*

$$[\mathbf{x}: \psi_T(\mathbf{x}) = \mathbf{0}] \subseteq B\left(\mathbf{0}, C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p)\right). \;\blacksquare$$

For $\omega' \in L_m$ and $\mathbf{x}_1 \in B\left(\mathbf{0}, C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p)\right)$ we have

$$\left|V^{-\infty,0}\left(t, \mathbf{x}; \tau_{0,\mathbf{x}_1}(\omega')\right)\right| = \left|V^{-\infty,0}\left(t, \mathbf{x}+\mathbf{x}_1; \omega'\right)\right|$$

$$\leqslant v|\mathbf{x}+\mathbf{x}_1| + \log^\gamma m \leqslant v|\mathbf{x}| + vC_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p) + \log^\gamma m$$

$$\leqslant v|\mathbf{x}| + (1+vC_{31})(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p)$$

$$\leqslant v|\mathbf{x}| + C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p),$$

provided $C_{31}$ is chosen sufficiently large, since $v \in (0, 1)$. Hence $\tau_{0,\mathbf{x}_1}(L_m) \subset L_{\mathrm{Ent}(M)+1}$, where $\log^\gamma M = C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p)$, and

$$\bigcup_{|\mathbf{x}| \leqslant C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma}p)} \tau_{0,\mathbf{x}}(L_m) \subseteq L_{\mathrm{Ent}(M)+1}.$$

Using all these arguments we have

$$\int_{\mathbf{R}^d} \frac{v_0^T(d\mathbf{x}; \omega, \omega'\sigma)}{G_T(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega'), \sigma)} \geqslant C_{41} \int_{\mathbf{R}^d} \frac{v_0^T(d\mathbf{x}; \omega, \omega'\sigma)}{\left|\nabla_x \psi_T(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega'), \sigma)\right|^d}$$

$$\geqslant C_{42} \frac{1}{\exp\left(C_{23}\log^\gamma(\mathrm{Ent}(M)+1)\right)} \frac{1}{\exp\left(C_{22}\log^\gamma n\right)} \frac{1}{\exp\left(C_{24}\log^{2\gamma}p\right)}$$

$$\geqslant C_{43} \frac{1}{\exp\left(C_{44}\log^\gamma n\right)} \frac{1}{\exp\left(C_{45}\log^\gamma m\right)} \frac{1}{\exp\left(C_{46}\log^{2\gamma}p\right)}$$

for $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$. Hence

$$[Qf][A] = \int\limits_\Omega \int\limits_\Sigma J^T(A; \omega, \sigma) f(\omega) P(d\omega) Q(d\sigma)$$

$$\geqslant \int\limits_A \Gamma(\omega') P(d\omega') \int\limits_\Omega \int\limits_\Sigma f(\omega) \Delta(\omega, \sigma) P(d\omega) Q(d\sigma),$$

where

$$\Gamma(\omega') = C_{51} \frac{1}{\exp(C_{45} \log^\gamma m)} \quad \text{for } \omega' \in L_{m+1} \backslash L_m,$$

$$\Delta(\omega, \sigma) = \frac{1}{\exp(C_{44} \log^\gamma n)} \frac{1}{\exp(C_{46} \log^{2\gamma} p)} \quad \text{for } \omega \in K_{n+1} \backslash K_n, \ \sigma \in S_{p+1} \backslash S_p.$$

Finally we obtain the formula equivalent to the formula (50) of [7]:

$$Qf(\omega') \geqslant \Gamma(\tau_{-T,0}(\omega')) \int\limits_\Omega \int\limits_\Sigma f(\omega) \Delta(\omega, \sigma) dP dQ.$$

Now let us put $Y_n = Q^n Y$. Choose the minimum of $\int_\Sigma \int_\Omega Y_n^+ \Delta dP dQ$ and $\int_\Sigma \int_\Omega Y_n^- \Delta dP dQ$; say it is the first one. Then we have

$$\|Y_{n+1}\|_{L^1} \leqslant \|QY_n^+\|_{L^1} - \|QY_n^-\|_{L^1} - \int\limits_\Sigma \int\limits_\Omega Y_n^+ \Delta dP dQ \int\limits_\Omega \Gamma dP$$

$$\leqslant \|Y_n\|_{L^1} - \int\limits_{S_n} \int\limits_{K_n} Y_n^+ \Delta dP dQ \int\limits_\Omega \Gamma dP$$

$$\leqslant \|Y_n\|_{L^1} - \exp(-C_{44} \log^\gamma n) \exp(-C_{46} \log^{2\gamma} n) \int\limits_{S_n} \int\limits_{K_n} Y_n^+ dP dQ \int\limits_\Omega \Gamma dP$$

$$= \|Y_n\|_{L^1} + \exp(-C_{44} \log^\gamma n) \exp(-C_{46} \log^{2\gamma} n) \int\limits_\Omega \Gamma dP$$

$$\times \left( \int\limits_{S_n} \int\limits_{K_n^c} Y_n^+ dP dQ + \int\limits_{S_n^c} \int\limits_{K_n^c} Y_n^+ dP dQ + \int\limits_{S_n^c} \int\limits_{K_n} Y_n^+ dP \, dQ - \int\limits_\Sigma \int\limits_\Omega Y_n^+ dP dQ \right).$$

Since $\int_\Omega Y_n dP = 0$, we get $\int_\Omega Y_n^+ dP = \frac{1}{2}\|Y_n\|_{L^1}$. From this and the Schwarz inequality we have

$$\int\limits_{S_n} \int\limits_{K_n^c} Y_n^+ dP dQ + \int\limits_{S_n^c} \int\limits_{K_n^c} Y_n^+ dP dQ = \int\limits_{K_n^c} Y_n^+ dP \leqslant \sqrt{P[K_n^c]} \|Y_n^+\|_{L^2}$$

$$\leqslant C_{61} \exp(-C_{62} \log^{2\gamma} n) \|Y\|_{L^2},$$

$$\int\limits_{S_n^c} \int\limits_{K_n} Y_n^+ dP dQ = Q[S_n^c] \int\limits_{K_n} Y_n^+ dP \leqslant \frac{C}{\log^{4\gamma} n} \sqrt{P[K_n]} \|Y_n\|_{L^2} \leqslant \frac{C}{\log^{4\gamma} n} \|Y\|_{L^2}$$

for some positive constants. We conclude with

$$\|Y_{n+1}\|_{L^1} \leqslant \|Y_n\|_{L^1} \left(1 - C_{70} \exp(-C_{72} \log^\gamma n)\right) + C_{71} \exp(-C_{73} \log^{2\gamma} n),$$

where the constants depend only on $\|Y\|_{L^2}$. Following the same argument we get

$$\|Y_n\|_{L^1} \leqslant C_{71} \sum_{k=1}^{n} \exp(-C_{73} \log^{2\gamma} k) \prod_{p=k+1}^{n} \left(1 - C_{70} \exp(-C_{72} \log^{\gamma} p)\right)$$

$$+ \|Y\|_{L^1} \prod_{p=1}^{n} \left(1 - C_{70} \exp(-C_{72} \log^{\gamma} p)\right).$$

Since $1 - x \leqslant e^{-x}$, the $k$-th term in the sum above can be estimated by

$$C_{71} \exp(-C_{73} \log^{2\gamma} k) \exp\left(-C_{70} \sum_{p=k+1}^{n} \exp(-C_{72} \log^{\gamma} p)\right).$$

We will estimate the last expression in two separate cases. For $k > [n/2]$ we may estimate it by

$$\frac{C_{81}}{n^{C_{82} \log^{2\gamma-1}(n/2)}}.$$

For $k < [n/2]$ we have

$$\sum_{p=k+1}^{n} \exp(-C_{72} \log^{\gamma} p) = \sum_{p=k+1}^{n} \frac{1}{p^{C_{72}/\log^{1-\gamma} p}} \geqslant n^{1-r},$$

where $r \in (0, 1)$ and $n$ is sufficiently large. Hence the $k$-th term of the sum is estimated by $C_{91} \exp(-C_{71} n^{1-r})$, which completes the proof of our lemma. ∎

R e m a r k. One can see that the proof above remains valid for the sequence of $L^p$-norms of the iterates $Q^n Y$ for any $p \in (1, 2)$.

## 6. TIGHTNESS

To establish the tightness of the family $\{X_\varepsilon(t)\}_{t \geqslant 0}$, $\varepsilon > 0$, we will use the following lemma (see [7], Lemma 12, p. 259):

LEMMA 8. *Suppose that* $\{Y_\varepsilon(t)\}_{t \geqslant 0}$, $\varepsilon > 0$, *is a family of processes with trajectories in* $C[0, \infty)$, *such that* $Y_\varepsilon(0) = 0$ *for* $\varepsilon > 0$. *Suppose further that for any* $L > 0$ *there exist constants* $p, C, v > 0$ *such that for any* $0 \leqslant s \leqslant t \leqslant u \leqslant L$

(6.1) $$ME\left[|Y_\varepsilon(t) - Y_\varepsilon(s)|^2 |Y_\varepsilon(u) - Y_\varepsilon(t)|^p\right] \leqslant C(u-s)^{1+v}.$$

*Then the family* $\{Y_\varepsilon(t)\}_{t \geqslant 0}$, $\varepsilon > 0$, *is tight.* ∎

We will show that the process $\{X_\varepsilon(t)\}$ satisfies the condition (6.1) with $p = 1$ and $v = \frac{1}{2}$. Let us introduce the notation

$$V(\varrho) := V\left(\varrho, X(\varrho; \omega, \sigma); \omega\right)$$

and similarly for each component $V_i$, $i = 1, \ldots, d$, of the field $V$. To begin with we show that

LEMMA 9. *For any $L > 0$ there exists a constant $C > 0$ such that for any $0 \leqslant s \leqslant t \leqslant u \leqslant L$ and $\varepsilon > 0$ we have*

$$(6.2) \qquad ME\left[|X_\varepsilon(t) - X_\varepsilon(s)|^2\right] \leqslant C(t-s).$$

Proof. The left-hand side of the inequality above may be rewritten in the form

$$(6.3) \quad ME\left[|\varepsilon \int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho)\,d\varrho + \varepsilon\left[B(t/\varepsilon^2;\sigma) - B(s/\varepsilon^2;\sigma)\right]|^2\right]$$

$$\leqslant 2ME\left[\varepsilon^2 |\int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho)\,d\varrho|^2\right] + 2ME\left[|\varepsilon\left[B(t/\varepsilon^2;\sigma) - B(s/\varepsilon^2;\sigma)\right]|^2\right].$$

For any $\varepsilon > 0$ the process $\{\varepsilon B(t/\varepsilon^2)\}_{t \geqslant 0}$ has the law of the standard Brownian motion in $\mathbf{R}^d$, so

$$ME\left[|\varepsilon\left[B(t/\varepsilon^2;\sigma) - B(s/\varepsilon^2;\sigma)\right]|^2\right]$$

$$= \sum_{i=1}^d ME\left[\left[\varepsilon\left[B_i(t/\varepsilon^2;\sigma) - B_i(s/\varepsilon^2;\sigma)\right]\right]^2\right] = d(t-s).$$

The first term of the right-hand side of (6.3) may be estimated as follows. Using the stationarity of $\{V(t)\}_{t \geqslant 0}$ (see Lemma 2) we may write

$$(6.4) \quad ME\left[\varepsilon^2 |\int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho)\,d\varrho|^2\right] = \varepsilon^2 \sum_{i=1}^d ME\left[2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{\eta} V_i(\varrho)\,V_i(\eta)\,d\varrho\,d\eta\right]$$

$$= 2\varepsilon^2 \sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} d\eta \left\{\sum_{k=1}^{\mathrm{Ent}(\eta/T)} \int_{\eta-kT}^{\eta-(k-1)T} ME\left[V_i(\eta-\varrho)\,V_i(0)\right]\,d\varrho\right\}$$

$$+ 2\varepsilon^2 \sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} d\eta \int_0^{\eta-\mathrm{Ent}(\eta/T)T} ME\left[V_i(\eta-\varrho)\,V_i(0)\right]\,d\varrho.$$

Then from the Schwarz inequality for the last term above we have:

$$(6.5) \quad 2\varepsilon^2 \sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} d\eta \int_0^{\eta-\mathrm{Ent}(\eta/T)T} ME\left[V_i(\eta-\varrho)\,V_i(0)\right]\,d\varrho$$

$$\leqslant 2\varepsilon^2 \sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} T\|V_i\|_{L^2}^2\,d\eta \leqslant 2\varepsilon^2 \sum_{i=1}^d T\|V_i\|_{L^2}^2 \frac{t-s}{\varepsilon^2} = C(t-s)$$

for a constant $C > 0$. Now we will estimate the first sum of the right-hand side of (6.4). Notice that for $\varrho \in [\eta-kT, \eta-(k-1)T]$ we have $\eta-\varrho \in [(k-1)T, kT]$. From Lemma 5 we get

$$ME\left[V_i(\eta-\varrho)\,V_i(0)\right] = ME\left[V_i(\eta-\varrho-(k-1)T)\,Q^{k-1}\,V_i(0)\right].$$

Using the stationarity of $\{V(t)\}_{t \geqslant 0}$ and the estimation coming from Lemma 6 we may majorize the $k$-th term of the sum under consideration by $C/k^2$ for some

constant $C > 0$ depending only on $\|V\|_2^2$. Hence the whole sum will be bounded from above by $C'(t-s)$. This completes the proof of (6.2). ∎

Since the sum $\sum_{k=1}^{\infty} 1/k^2$ is convergent, from the argument above we can deduce the following corollary:

COROLLARY 1. *The integrals*

$$\int_0^{\infty} ME\left[V_i\left(\varrho, X(\varrho)\right) V_j(0, \mathbf{0})\right] d\varrho$$

*for* $i, j = 1, 2, \ldots, d$ *are convergent.* ∎

To make use of Lemma 8 we will now establish the estimates of the following:

$$(6.6) \quad ME\left[|X_\varepsilon(u) - X_\varepsilon(t)| \, |X_\varepsilon(t) - X_\varepsilon(s)|^2\right]$$

$$\leqslant 2ME\left[\left|\varepsilon \int_{t/\varepsilon^2}^{u/\varepsilon^2} V(\varrho) \, d\varrho\right| \left|\varepsilon \int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho) \, d\varrho\right|^2\right]$$

$$+ 2ME\left[\left|\varepsilon \int_{t/\varepsilon^2}^{u/\varepsilon^2} V(\varrho) \, d\varrho\right| \left|\varepsilon \left[B(t/\varepsilon^2) - B(s/\varepsilon^2)\right]\right|^2\right]$$

$$+ 2ME\left[\left|\varepsilon \left[B(u/\varepsilon^2) - B(t/\varepsilon^2)\right]\right| \left|\varepsilon \int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho) \, d\varrho\right|^2\right]$$

$$+ 2ME\left[\left|\varepsilon \left[B(u/\varepsilon^2) - B(t/\varepsilon^2)\right]\right| \left|\varepsilon \left[B(t/\varepsilon^2) - B(s/\varepsilon^2)\right]\right|^2\right] = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

We will estimate each of the four terms of the utmost right-hand side of (6.6) separately. From (6.2) and the Schwarz inequality we infer that both II and IV can be estimated from above by $C(u-s)^{3/2}$. Now turn to the term I of (6.6). Define

$$\Gamma_{1,k} := \left|\varepsilon \int_{(t-s)/\varepsilon^2 - \varrho - (k-1)T}^{(u-s)/\varepsilon^2 - \varrho - (k-1)T} V(\tau) \, d\tau\right|, \quad \Gamma_{2,k} := V_i\left(\eta - \varrho - (k-1)T\right).$$

We transform I in a similar way as we did in (6.4) to get

$$I \leqslant 4\varepsilon^2 \sum_{i=1}^{d} \int_0^{(t-s)/\varepsilon^2} d\eta \sum_{k=1}^{\mathrm{Ent}(\eta/T)} \int_{\eta - kT}^{\eta - (k-1)T} ME\left[\Gamma_{1,k} \Gamma_{2,k} Q^{k-1}\left(V_i(0)\right)\right] d\varrho$$

$$+ 4\varepsilon^2 \sum_{i=1}^{d} \int_0^{(t-s)/\varepsilon^2} d\eta \int_0^{\eta - \mathrm{Ent}(\eta/T)T} ME\left[\Gamma_{1,1} \Gamma_{2,1} V_i(0)\right] d\varrho.$$

We will now estimate $ME\left[\Gamma_{1,k} \Gamma_{2,k} Q^{k-1}\left(V_i(0)\right)\right]$. We have

$$(6.7) \quad ME\left[\Gamma_{1,k} \Gamma_{2,k} Q^{k-1}\left(V_i(0)\right)\right]$$

$$= \int_{\Gamma_{1,k} \leqslant k^{\nu}(u-t)^{1/2}} \Gamma_{1,k} \Gamma_{2,k} Q^{k-1}\left(V_i(0)\right) + \int_{\Gamma_{1,k} > k^{\nu}(u-t)^{1/2}} \Gamma_{1,k} \Gamma_{2,k} Q^{k-1}\left(V_i(0)\right),$$

where the constant $v > 0$ will be determined later. The first integral above will be less than or equal to $Ck^{-2}(u-t)^{1/2}$. Since by (6.2) we have

$$P \otimes Q\left[\Gamma_{1,k} > k^v(u-t)^{1/2}\right] \leqslant \frac{1}{k^{2v}(u-t)}\|\Gamma_{1,k}\|_{L^2}^2 \leqslant \frac{C}{k^{2v}},$$

for the second term of (6.7) we may write

$$\left|\int_{\Gamma_{1,k} > k^v(u-t)^{1/2}} \Gamma_{1,k}\Gamma_{2,k}Q^{k-1}\left(V_i(0)\right)\right|$$

$$\leqslant \left(P \otimes Q\left[\Gamma_{1,k} > k^v(u-t)^{1/2}\right]\right)^{\beta_1}\left(ME\Gamma_{1,k}^2\right)^{1/2}\left(ME\Gamma_{2,k}^{1/\beta_2}\right)^{\beta_2}\left\|Q^{k-\overline{1}}\left(V_i(0)\right)\right\|_{L^{1/\beta_3}}$$

$$\leqslant \frac{C(u-t)^{1/2}}{k^2}$$

for some $\beta_1, \beta_2, \beta_3 > 0$, $\beta_1 + \beta_2 + \beta_3 + \frac{1}{2} = 1$, $v\beta_1 > 0$. From the same argument it follows that

$$\int_0^{\eta - \text{Ent}(\eta/T)T} ME\left[\Gamma_{1,1}\Gamma_{2,1}V_i(0)\right] \leqslant \int_0^T C(u-t)^{1/2} \leqslant C'(u-t)^{1/2}.$$

Finally we have

$$I \leqslant 4\varepsilon^2 \sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} d\eta \left(\sum_{k=1}^{\text{Ent}(\eta/T)} \frac{C(u-t)^{1/2}}{k^2} + C'(u-t)^{1/2}\right)$$

$$\leqslant C''(t-s)(u-t)^{1/2} \leqslant C''(u-s)^{3/2}.$$

Now, with the help of Hölder's inequality, the proof of the estimation of the third term in (6.6) may be reduced to the problem of showing that

$$\left[ME\left|\varepsilon\int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho)\,d\varrho\right|^3\right]^{2/3} \leqslant C(t-s).$$

Since

$$ME\left|\varepsilon\int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho)\,d\varrho\right|^3 \leqslant \varepsilon^2 ME\left[\left|\varepsilon\int_{s/\varepsilon^2}^{t/\varepsilon^2} V(\varrho_1)\,d\varrho_1\right|\int_{s/\varepsilon^2}^{t/\varepsilon^2}\int_{s/\varepsilon^2}^{t/\varepsilon^2} V_i(\varrho_2)\,V_i(\varrho_3)\,d\varrho_2\,d\varrho_3\right],$$

it suffices to show the inequality in the case of $\varrho_1 \geqslant \varrho_2 \geqslant \varrho_3$ (other cases are also covered by symmetry of the term under the last integral above). But now we can use the same method as presented in the proof of the estimate of I and we finally get

$$\text{III} \leqslant C(u-s)^{3/2}.$$

This completes the proof of the inequality

$$ME\left[|X_\varepsilon(u) - X_\varepsilon(t)|\,|X_\varepsilon(t) - X_\varepsilon(s)|^2\right] \leqslant C(u-s)^{3/2}$$

and by Lemma 8 the family of the laws of processes $\{X_\varepsilon(t)\}_{t \geqslant 0}$, $\varepsilon > 0$, is tight on $C([0, +\infty), \mathbf{R}^d)$.

## 7. LIMIT IDENTIFICATION

To complete the proof of our theorem we have to show that there exists only one process that can be the weak limit of the family $\{X_\varepsilon(t)\}_{t \geqslant 0}$ as $\varepsilon \downarrow 0$. We prove this with the help of the Stroock–Varadhan martingale identification theorem. We start with the following lemma:

LEMMA 10 (cf. [7], Lemma 13, p. 260). *For any* $\gamma \in (0, 1)$, $M \in N$, $\psi: (R^d)^M \to R_+$ *continuous and bounded,* $0 \leqslant s_1 \leqslant \ldots \leqslant s_M \leqslant s$ *and* $i = 1, \ldots, d$, *there exists a constant* $C > 0$ *such that for any* $\varepsilon > 0$ *and* $0 \leqslant s \leqslant t \leqslant L$

$$\left| ME\left[ \left( X_\varepsilon^i(t + \varepsilon^\gamma) - Y_\varepsilon^i(t) \right) \psi \left( X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M) \right) \right] \right| \leqslant C\varepsilon.$$

Proof. We have

$$
\begin{aligned}
(7.1) \quad & \left| ME\left[ \left( X_\varepsilon^i(t + \varepsilon^\gamma) - X_\varepsilon^i(t) \right) \psi \left( X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M) \right) \right] \right| \\
&= \left| ME\left[ \left( \varepsilon \int_{t/\varepsilon^2}^{t/\varepsilon^2 + \varepsilon^{\gamma-2}} V_i(\varrho)\, d\varrho \right) \psi \left( X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M) \right) \right] \right| \\
&\quad + \left| ME\left[ \varepsilon \left( B_i\left( \frac{t + \varepsilon^\gamma}{\varepsilon^2} \right) - B_i\left( \frac{t}{\varepsilon^2} \right) \right) \psi \left( X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M) \right) \right] \right|.
\end{aligned}
$$

The second term above is equal to

$$\left| ME\left[ \varepsilon \left( B_i\left( \frac{t + \varepsilon^\gamma}{\varepsilon^2} \right) - B_i\left( \frac{t}{\varepsilon^2} \right) \right) \right] ME\left[ \psi \left( X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M) \right) \right] \right| = 0.$$

The first term on the right-hand side of (7.1) can be rewritten in the form

$$ME\left[ \varepsilon \int_0^{\varepsilon^{\gamma-2}} V_i(\varrho)\, \Psi d\varrho \right],$$

where $\Psi$ is a $\mathscr{V}_{-\infty,0}$-measurable random variable for any $\sigma \in \Sigma$. We get

$$ME\left[ \varepsilon \int_0^{\varepsilon^{\gamma-2}} V_i(\varrho)\, \Psi d\varrho \right] \leqslant \sum_{k=1}^{\mathrm{Ent}(\varepsilon^{\gamma-2}/T)} \varepsilon \int_{kT}^{(k+1)T} ME\left[ V_i(\varrho)\, \Psi \right] d\varrho + \varepsilon \int_0^T ME\left[ V_i(\varrho)\, \Psi \right] d\varrho.$$

The last integral above is of order $\varepsilon$. Since $ME\left[ V_i(\varrho) \right] = 0$, we may estimate as we have already done in the previous section:

$$
\begin{aligned}
\sum_{k=1}^{\mathrm{Ent}(\varepsilon^{\gamma-2}/T)} \varepsilon \int_{kT}^{(k+1)T} ME\left[ V_i(\varrho)\, \Psi \right] d\varrho &= \sum_{k=1}^{\mathrm{Ent}(\varepsilon^{\gamma-2}/T)} \varepsilon \int_{kT}^{(k+1)T} ME\left[ V_i(\varrho)(\Psi - ME\Psi) \right] d\varrho \\
&= \sum_{k=1}^{\mathrm{Ent}(\varepsilon^{\gamma-2}/T)} \varepsilon \int_{kT}^{(k+1)T} ME\left[ V_i\left(\varrho - (k-1)\, T\right) Q^{k-1}(\Psi - ME\Psi) \right] d\varrho \\
&\leqslant \sum_{k=1}^{\mathrm{Ent}(\varepsilon^{\gamma-2}/T)} \varepsilon \int_{kT}^{(k+1)T} \frac{C}{k^2} d\varrho \leqslant C'\varepsilon. \quad \blacksquare
\end{aligned}
$$

Now we have

LEMMA 11. *Under the assumptions of Lemma 10, for any $\varepsilon > 0$ we have*

$$\left| ME\left[\left[\left(X_\varepsilon^i(t+\varepsilon^\gamma) - X_\varepsilon^i(t)\right)\left(X_\varepsilon^j(t+\varepsilon^\gamma) - X_\varepsilon^j(t)\right) - \varepsilon^\gamma D_{ij}\right] \psi\left(X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M)\right)\right]\right| = o(\varepsilon^\gamma)$$

*for $i, j = 1, \ldots, d$, where the matrix $[D_{ij}]$ is given by (2.3), i.e.*

$$D_{ij} = \int_0^\infty ME\left[V_i(t) V_j(0) + V_j(t) V_i(0)\right] dt + \delta_{ij}.$$

Proof. Notice that

$$(7.2) \quad ME\left[\left(X_\varepsilon^i(t+\varepsilon^\gamma) - X_\varepsilon^i(t)\right)\left(X_\varepsilon^j(t+\varepsilon^\gamma) - X_\varepsilon^j(t)\right) \psi\left(X_\varepsilon(s_1), \ldots, X_\varepsilon(s_M)\right)\right]$$

$$= ME\left[\varepsilon^2 \int_{t/\varepsilon^2}^{(t+\varepsilon^\gamma)/\varepsilon^2} \int_{t/\varepsilon^2}^{(t+\varepsilon^\gamma)/\varepsilon^2} V_i(\varrho) V_j(\varrho') \, d\varrho \, d\varrho' \psi\right]$$

$$+ ME\left[\varepsilon^2 \left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_i\left(\frac{t}{\varepsilon^2}\right)\right)\left(B_j\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_j\left(\frac{t}{\varepsilon^2}\right)\right) \psi\right]$$

$$+ ME\left[\varepsilon^2 \int_{t/\varepsilon^2}^{(t+\varepsilon^\gamma)/\varepsilon^2} V_i(\varrho)\left(B_j\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_j\left(\frac{t}{\varepsilon^2}\right)\right) d\varrho \psi\right]$$

$$+ ME\left[\varepsilon^2 \int_{t/\varepsilon^2}^{(t+\varepsilon^\gamma)/\varepsilon^2} V_j(\varrho)\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_i\left(\frac{t}{\varepsilon^2}\right)\right) d\varrho \psi\right]$$

$$= I + II + III + IV.$$

It is easy to see that $|II - \varepsilon^\gamma \delta_{ij} ME[\psi]| = 0$. Following [7] we also get the estimate:

$$(7.3) \quad \left| I - \varepsilon^\gamma \int_0^{+\infty} \left[V_i(t) V_j(0) + V_j(t) V_i(0)\right] dt \, ME[\psi]\right| = o(\varepsilon^\gamma).$$

To complete the proof of the lemma we are left with the proof of the fact that the terms III and IV in (7.2) are of order $o(\varepsilon^\gamma)$. Once again we may find a $\mathscr{V}_{-\infty,0}$-measurable random variable $\Psi$ such that III can be rewritten in the form

$$\left|\varepsilon^2 \int_0^{\varepsilon^{\gamma-2}} ME\left[V_i(\varrho) B_j(\varepsilon^{\gamma-2}) \Psi\right] d\varrho\right|.$$

Notice that up to a term of order $o(\varepsilon^\gamma)$ we have

$$\varepsilon^2 \int_0^{\varepsilon^{\gamma-2}} \left|ME\left[V_i(\varrho) B_j(\varepsilon^{\gamma-2}) \Psi\right]\right| d\varrho$$

$$= \varepsilon^\gamma \frac{1}{\text{Ent}(\varepsilon^{\gamma-2}/T) T} \int_0^{\text{Ent}(\varepsilon^{\gamma-2}/T) T} \left|ME\left[V_i(\varrho) B_j(\varepsilon^{\gamma-2}) \Psi\right]\right| d\varrho.$$

Since $\varepsilon^{\gamma-2} \to +\infty$ as $\varepsilon \downarrow 0$, it suffices to show that

$$\lim_{N \uparrow +\infty} \frac{1}{N} \int_0^{NT} |ME[V_i(\varrho) B_j(NT) \Psi]| \, d\varrho = 0.$$

We may write

$$(7.4) \quad \frac{1}{N} \int_0^{NT} |ME[V_i(\varrho) B_j(NT) \Psi]| \, d\varrho$$

$$= \frac{1}{N} \int_0^T |ME[V_i(\varrho) B_j(NT) \Psi]| \, d\varrho + \frac{1}{N} \int_T^{NT} |ME[V_i(\varrho) B_j(NT) \Psi]| \, d\varrho$$

$$= \frac{1}{N} \int_0^T |ME[V_i(\varrho) B_j(NT) \Psi]| \, d\varrho + \frac{1}{N} \int_0^{(N-1)T} |ME[V_i(\varrho) B_j((N-1)T) Q\Psi]| \, d\varrho.$$

Continue this way and we see that (7.4) is equal to

$$(7.5) \quad \frac{1}{N} \sum_{k=0}^{N-1} \int_0^T |ME[V_i(\varrho) B_j((N-k)T) Q^k \Psi]| \, d\varrho.$$

Now let $\tilde{\Psi} = \Psi - ME\Psi$. Since $Q1 = 1$, (7.5) can be rewritten in the form

$$\frac{1}{N} \sum_{k=0}^{N-1} \int_0^T |ME[V_i(\varrho) B_j((N-k)T) Q^k \tilde{\Psi}]| \, d\varrho$$

$$+ \frac{1}{N} \sum_{k=0}^{N-1} \int_0^T |ME[V_i(\varrho) B_j((N-k)T)] ME[Q^k \tilde{\Psi}]| \, d\varrho.$$

But

$$ME[V_i(\varrho) B_j((N-k)T)] = M[E[V_i(\varrho)] B_j((N-k)T)] = 0,$$

since $E[V_i(\varrho)] = 0$. The first term above may be estimated with the help of Lemma 6. We have $\|Q^k \tilde{\Psi}\|_{L^q} \leqslant C/k^2$ for any $q \in [1, 2)$, $k \in N$, the constant $C > 0$ depending only on $q$ and $\|\tilde{\Psi}\|_{L^q}$. And we estimate (7.5) from above by

$$\frac{1}{N} \sum_{k=0}^{N-1} T \|V_i\|_{L^{4q/(3q-4)}} \|B_j((N-k)T)\|_{L^4} \|Q^k \tilde{\Psi}\|_{L^q}$$

$$\leqslant \frac{C'}{N} \sum_{k=0}^{N-1} \frac{\sqrt{N-k}}{k^2} \leqslant \frac{C''}{\sqrt{N}} \to 0 \quad \text{as } N \uparrow +\infty.$$

This completes the proof of Lemma 11. ∎

LEMMA 12. *For any $\gamma \in (0, 1)$ and $0 < \gamma' < \gamma$ there exists a constant $C > 0$ such that*

$$ME[|X_\varepsilon(t+\varepsilon^\gamma) - X_\varepsilon(t)|^4] \leqslant C\varepsilon^{2\gamma'}.$$

Proof. We have

$$X_\varepsilon(t+\varepsilon^\gamma)-X_\varepsilon(t) = \varepsilon \int_{t/\varepsilon^2}^{t/\varepsilon^2+\varepsilon^{\gamma-2}} V_i(\varrho)\,d\varrho + \varepsilon\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right)-B_i\left(\frac{t}{\varepsilon^2}\right)\right).$$

Define

$$A = \varepsilon \int_{t/\varepsilon^2}^{t/\varepsilon^2+\varepsilon^{\gamma-2}} V_i(\varrho)\,d\varrho, \qquad B = \varepsilon\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right)-B_i\left(\frac{t}{\varepsilon^2}\right)\right).$$

It suffices to find the estimates of the following type:

   (i) $ME[|A^4|] \leqslant C_1\varepsilon^{2\gamma'}$,
   (ii) $ME[|A^3 B|] \leqslant C_2\varepsilon^{2\gamma'}$,
   (iii) $ME[|A^2 B^2|] \leqslant C_3\varepsilon^{2\gamma'}$,
   (iv) $ME[|AB^3|] \leqslant C_4\varepsilon^{2\gamma'}$,
   (v) $ME[|B^4|] \leqslant C_5\varepsilon^{2\gamma'}$

for some constants $C_1, \ldots, C_5 > 0$. It is easy to see that once we have established (i) and (v) we get (ii)–(iv) by the Schwarz inequality. But (v) is obvious. To prove (i) we have to find an appropriate estimate from above of the term

(7.6) $$\varepsilon^4 \int_0^{\varepsilon^{\gamma-2}} d\varrho_1 \int_0^{\varrho_1} d\varrho_2 \int_0^{\varrho_2} d\varrho_3 \int_0^{\varrho_3} ME[V_i(\varrho_2)\,V_i(\varrho_3)\,V_i(\varrho_4)\,V_i(0)]\,d\varrho_4,$$

and this is exactly the result of Lemma 15 of [7]. ∎

Now let $f \in C_0^\infty(\mathbf{R}^d)$. Let $\psi$ be as in Lemma 11 and let $t_m = s+m\varepsilon^\gamma$. From the Taylor expansion formula we have

$$ME\left[\left(f(X_\varepsilon(t))-f(X_\varepsilon(s))\right)\Psi\right]$$

$$= \sum_{m:s<t_m<t} ME\left\{\left\langle[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)], (\nabla f)(X_\varepsilon(t_m))\right\rangle\Psi\right\}$$

$$+\frac{1}{2}\sum_{m:s<t_m<t} ME\left\{\left\langle[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]\otimes[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)], (\nabla\otimes\nabla f)(X_\varepsilon(t_m))\right\rangle\Psi\right\}$$

$$+\frac{1}{6}\sum_{m:s<t_m<t} ME\left\{\left\langle[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]\otimes[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]\otimes[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)],\right.\right.$$

$$\left.\left.(\nabla\otimes\nabla\otimes\nabla f)(X_\varepsilon(\theta_m))\right\rangle\Psi\right\},$$

where $\theta_m$ is a point in the segment $[X_\varepsilon(t_m), X_\varepsilon(t_{m+1})]$. The first term of the sum above is estimated with the help of Lemma 10 by the term of order $o(\varepsilon^\gamma)$. The third one, by the Hölder inequality and Lemma 12, may be estimated from above by $C\varepsilon^{3\gamma/2}$. Choosing carefully $\gamma \in (0, 1)$ and $\gamma' \in (0, \gamma)$ we get again the term of order $o(\varepsilon^\gamma)$. The second term can be rewritten in the form

$$\frac{1}{2}\sum_{m:s<t_m<t} ME\left\{\left\langle [X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]\otimes[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]-D^\varepsilon,\right.\right.$$

$$\left.\left.(\nabla\otimes\nabla f)\left(X_\varepsilon(t_m)\right)\right\rangle\Psi\right\}$$

$$+\frac{1}{2}\sum_{m:s<t_m<t} ME\left\{\left\langle D^\varepsilon, (\nabla\otimes\nabla f)\left(X_\varepsilon(t_m)\right)\right\rangle\Psi\right\},$$

where

$$D^\varepsilon = ME\left[[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]\otimes[X_\varepsilon(t_{m+1})-X_\varepsilon(t_m)]\right].$$

And this, by Lemma 11, up to the terms of order $o(\varepsilon^\gamma)$ is equal to

$$(7.7)\qquad \frac{1}{2}\sum_{m:s<t_m<t}\left[\sum_{i,j=1}^{d}\varepsilon^\gamma D_{ij}ME\left[\partial_{ij}^2 f\left(X_\varepsilon(t_m)\right)\Psi\right]+o(\varepsilon^\gamma)\right].$$

Thus the limit of the family of processes $\{X_\varepsilon(t)\}_{t\geqslant 0}$ as $\varepsilon\downarrow 0$ must have the law $\mu$ on the space $C\left([0,+\infty), R^d\right)$ such that for any $f\in C_0^\infty(R^d)$

$$f\left(x(t)\right)-\frac{1}{2}\sum_{i,j=1}^{d} D_{ij}\int_0^t \partial_{ij}^2 f\left(x(\varrho)\right)d\varrho, \quad t\geqslant 0,$$

is a martingale under $\mu$. By the theorem of Stroock and Varadhan (see e.g. [14]) it may be identified as a diffusion with covariance matrix $D = [D_{ij}]$. This completes the proof of our theorem. ■

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