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GENERATING FUNCTIONS OF ORTHOGONAL POLYNOMIALS AND SZEGÖ–JACOBI PARAMETERS

BY

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Abstract. In this paper, we present a more direct way to compute the Szegö–Jacobi parameters from a generating function than that in [5] and [6]. Our study is motivated by the notions of one-mode interacting Fock spaces defined in [1] and Segal–Bargmann transform associated with non-Gaussian probability measures introduced in [2]. Moreover, we examine the relationships between the representations of orthogonal polynomials in terms of differential or difference operators and our generating functions. The connections provide practical criteria to determine when functions of a certain form are orthogonal polynomials.

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1. INTRODUCTION

The theory of orthogonal polynomials [9], [23] has a long history with a wide range of applications, for example, to stochastic analysis [13], [16] and mathematical physics [10]. Its connection with the notion of one-mode interacting Fock space has been recently examined by Accardi and Bożejko [1]. In the papers by the first-named author [2], [3] and the authors [4], important aspects related to Accardi–Bożejko's work have been recently studied from the viewpoint of the Segal–Bargmann transforms associated with non-Gaussian measures. The results in this paper will be useful for relating the analysis of the Segal–Bargmann transform to quantum probability theory [19], [20].

Let μ be a probability measure on $\boldsymbol{R}$ with finite moments of all orders such that the linear span of the monomials x^n, $n \geqslant 0$, is dense in $L^2(\mu)$. Then we have a unique complete orthogonal system $\{P_n\}_{n=0}^{\infty}$ such that $P_n(x)$ is a polynomial of degree n with leading coefficient 1, which is called a *monic polynomial.* It is well known [9], [23] that $\{P_n\}_{n=0}^{\infty}$ satisfies the following recursion formula:

$$(x-\alpha_n)P_n(x) = P_{n+1}(x)+\omega_n P_{n-1}(x), \qquad n \geqslant 0, \tag{1.1}$$

where $\alpha_n \in \mathbf{R}$, $\omega_n \geqslant 0$ for $n \geqslant 0$ and, by convention, $\omega_0 = 1$, $P_{-1} = 0$. The numbers α_n and ω_n are called the *Szegö–Jacobi parameters* of μ. Define a sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ by

$$\lambda_n = \omega_0\,\omega_1 \ldots \omega_n, \qquad n \geqslant 0.$$

For such a sequence λ, we can define the associated Hilbert space Γ_λ as the l^2-space with weight λ. The annihilation operator A and creation operator A^* acting on Γ_λ are defined from ω_n. We point out that in general the pair $\{A, A^*\}$ is different from the pair $\{b, b^*\}$ of the bosonic annihilation and creation operators. The neutral operator α_N is defined from $\{\alpha_n\}$ and the number operator N. The Hilbert space Γ_λ equipped with $\{A, A^*, \alpha_N\}$ is called the *one-mode interacting Fock space.* See [1], [2], [4] for explicit definitions of Γ_λ, A, A^*, N, and α_N.

In [1], it is proved that there exists a unitary isomorphism $U\colon \Gamma_\lambda \to L^2(\mu)$ such that the following intertwining formulas hold:

(1) $U\Phi_0 = 1$,
(2) $UA^*U^*P_n = P_{n+1}$,
(3) $U(A+A^*+\alpha_N)U^* = X$,

where Φ_0 is the vacuum vector and X is the multiplication operator by x in $L^2(\mu)$. This unitary isomorphism U is canonical in the sense of condition (3), namely, under U, the classical random variable x on $L^2(\mu)$ considered as the multiplication operator X can be decomposed into the sum of the operators A, A^*, and α_N on the interacting Fock space Γ_λ. In this sense, U provides a natural starting point to develop probability theory from the noncommutative algebraic point of view. In fact, central limit theorems and random walks on noncommutative algebras are examined as an application of one-mode interacting Fock spaces in [12] and references cited therein.

The transform associated with the Gaussian measure was introduced originally in [7], [21], [22]. The generating function for the Hermite polynomials plays important roles as the integral kernel function. This integral transform, so-called the *Segal–Bargmann transform,* has been extended by many authors within Gaussian analysis. Consult the paper by Gross and Malliavin [11] and references cited therein. Motivated by Accardi and Bożejko [1], the first-named author has extended the Segal–Bargmann transform to the one associated with non-Gaussian measures to study intertwining property, as (1)–(3) above, among classical random variables on $L^2(\mu)$ and creation, annihilation and neutral operators on the L^2-space of holomorphic functions [2]. As one of the interesting applications, the classical Gaussian and Poisson random variables can be represented as the sum of b and b^*, and that of b, b^* and N, respectively. This point of view shares the same spirit as Hudson and Parthasarathy [15]. In addition, if "appropriate" integral kernel functions are chosen, the bosonic expressions of the classical Gaussian and Poisson random variables can be realized on the common L^2-space of holomorphic functions with respect to the Gaussian measure $\tilde{\mu}$ on C, $\mathscr{H}L^2(C, \tilde{\mu})$. This case study has been considered in [5]. The q-Gaussian [18], q-Poisson [3], and free Gaussian ($q = 0$) [8] cases have been also studied.

The notion of *generating function* has several different definitions, and hence many of classical formulas in [9], [10], [23] should be modified in case by case for applications. In particular, to obtain "appropriate" generating function as an integral kernel for the associated Segal–Bargmann transform with a given probability measure and noncommutative realizations of classical random variables, we need alternative methods to remove such inconvenience in a unified manner. Moreover, in order to keep good contacts with the theory of interacting Fock space, it is a crucial point to derive the Szegö–Jacobi parameters from the generating function [5], [6].

In this paper, we will adopt notions of pre-generating functions, generating functions, the multiplicative renormalization, and related theorem from our previous papers [5], [6].

By a *pre-generating function* for μ we mean a function $\varphi(t, x)$ having a power series expansion in t near $t = 0$

$$\varphi(t, x) = \sum_{n=0}^{\infty} g_n(x) t^n, \tag{1.2}$$

where $g_n(x)$ is a polynomial of degree n for each $n \geqslant 0$ satisfying

$$\limsup_{n \to \infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty. \tag{1.3}$$

The *multiplicative renormalization* of a pre-generating function $\varphi(t, x)$ is defined to be the function

$$\psi(t, x) = \frac{\varphi(t, x)}{E_\mu[\varphi(t, \cdot)]},$$

where E_μ denotes the expectation in the x-variable with distribution μ. Then $\psi(t, x)$ is also a pre-generating function. A *generating function* for μ means a function $\psi(t, x)$ given by

$$\psi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n,$$

with some nonnegative coefficients $\{a_n\}$. Here $\{P_n\}$ is given by equation (1.1).

Inspired by Hida's multiplicative renormalization in the theory of generalized Brownian functionals in [13], [14], called *white noise theory* nowadays

[17], we have given a systematic way to find generating functions for given probability measures in [5], [6]. Then the following question arises naturally:

QUESTION. How can we obtain the Szegö–Jacobi parameters directly from generating functions without deriving the P_n's?

One of the answers can be found in the following theorem proved in [6].

THEOREM 1.1 (Theorem 2.6 in [6]). *Let* $\psi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n$ *be a generating function for* μ. *Then*

$$\lim_{t\to 0} \psi(t, x/t) = \sum_{n=0}^{\infty} a_n x^n, \qquad E_\mu[\psi(t, \cdot)^2] = \sum_{n=0}^{\infty} a_n^2 \lambda_n t^{2n},$$

$$E_\mu[x\psi(t, \cdot)^2] = \sum_{n=0}^{\infty} (a_n^2 \lambda_n \alpha_n t^{2n} + 2a_n a_{n-1} \lambda_n t^{2n-1}), \tag{1.4}$$

where $a_{-1} = 0$ *by convention.*

The basic idea of this theorem is the following. Once we have a generating function $\psi(t, x)$ for μ, we can find the power series of $E_\mu[\psi(t, \cdot)^2]$ and $E_\mu[x\psi(t, \cdot)^2]$. Then by Theorem 1.1 we can find a_n and the Szegö–Jacobi parameters α_n and ω_n. We remark that in [6] the Gram–Schmidt process is not used to get P_n's.

However, there are still difficulties in applying Theorem 1.1 to certain examples. That is, if the integrals $E_\mu[\psi(t, \cdot)^2]$ and $E_\mu[x\psi(t, \cdot)^2]$ are very complicated or even worse cannot be calculated explicitly, then Theorem 1.1 would not be practical to get α_n and ω_n from the computational point of view. This is the case in examples given in Sections 4 and 5 in [6].

The first purpose of this paper is to present a simpler way to calculate Szegö–Jacobi parameters from a generating function without using explicit information about polynomials P_n's.

It is known that classical orthogonal polynomials associated with continuous measures can be represented in terms of differential operators as follows. Suppose that $w(x)$ is a smooth positive density of a probability measure μ on an interval $I = (a, b) \subset \boldsymbol{R}$ corresponding to classical orthogonal polynomials. Then monic orthogonal polynomials have the following Rodorigues type representation:

$$P_n(x) = \frac{1}{k_n w(x)} D_x^n(X(x)^n w(x)), \qquad D_x^n := \frac{d^n}{dx^n}, \tag{1.5}$$

holds for $n \geqslant 0$ where the conditions

(P1) $X(x)$ is a polynomial with $\deg X \leqslant 2$,
(P2) k_n is the leading coefficient of $w(x)^{-1} D_x^n(X(x)^n w(x))$,
(P3) $D_x^k[X(x)^n w(x)] = 0$ $(0 \leqslant k < n)$ for $x = a$ and $x = b$,

are satisfied (see [9], p. 146). In general, a given function of the form

$$\frac{1}{w(x)} D_x^n (q_n(x) w(x)) \tag{1.6}$$

could not be polynomials and not satisfy the condition (P1).

The second purpose of this paper is to discuss such representations of the polynomials $P_n(x)$ in terms of differential or difference operators for general probability measure μ by using generating functions. This consideration provides not only Rodrigues formulas but also is useful to know under what conditions on our generating functions and the Szegö–Jacobi parameters a function of the type in equation (1.6) becomes orthogonal polynomials.

The present paper is organized as follows: In Section 2, we shall give a new method to directly obtain Szegö–Jacobi parameters from generating functions. In Section 3, we will discuss the relationships between our multiplicative renormalizations of generating functions (pre-generating functions) and the differential operator representations of orthogonal polynomials. In Section 4, the case of the difference operators will be examined. In Section 5, we will apply our method to particular examples including Gaussian, gamma, beta-type, Poisson and negative binomial distributions. We emphasize here that our approach is not based upon known formulas in the literature [9], [23].

2. SZEGÖ–JACOBI PARAMETERS AND GENERATING FUNCTIONS

In the previous section, we have introduced generating functions and discussed how to use them to compute the Szegö–Jacobi parameters α_n and ω_n. As we mentioned, Theorem 1.1 has no problem from the viewpoint of the general theory, but it has practical difficulties for certain particular examples. So we shall introduce another way to calculate them directly from a given generating function.

2.1. General properties. Throughout this paper, μ is supposed to be a probability measure on $\boldsymbol{R}$ with finite moments of all orders such that the linear span of the monomials x^n, $n \geqslant 0$, is dense in $L^2(\mu)$ and the additional assumptions will be stated if necessary.

Suppose that $\psi(t, x)$ is a generating function for monic orthogonal polynomials $\{P_n(x)\}$ given by

$$\psi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n \tag{2.1}$$

converging absolutely around $(t, x) = (0, 0)$.

Remark. It is known [9] that there exists a positive integer N such that $\lambda_m = 0$ for all $m \geqslant N+1$ if and only if the measure μ is supported by the finite set $\{0, 1, \ldots, N\}$. For instance, the binomial distribution is exactly the case. We are not interested in such situations in this paper and they can be treated with some modifications if necessary. So we will consider situations under the condition $a_n > 0$ for any $n \geqslant 0$ in this paper, which is equivalent to the condition $\lambda_n > 0$ for any $n \geqslant 0$. See [2], [4] and the condition (H2) in Section 2.2.

In order to derive the Szegö–Jacobi parameters directly from a generating function (2.1), let us define functions $A(x)$, $B(t)$, $C(t)$ and sequences $\{b_n\}$, $\{c_n\}$ by

$$A(x) = \lim_{t\to\infty} \psi(t, x/t) = \sum_{n=0}^{\infty} a_n x^n, \tag{2.2}$$

$$B(t) = \psi(t, 0) = \sum_{n=0}^{\infty} b_n t^n, \tag{2.3}$$

$$C(t) = \frac{\partial}{\partial x}\psi(t, x)\bigg|_{x=0} = \sum_{n=0}^{\infty} c_n t^n. \tag{2.4}$$

Since $B(t) = \sum_{n=0}^{\infty} a_n P_n(0) t^n$ and $C(t) = \sum_{n=0}^{\infty} a_n P_n'(0) t^n$, we obtain easily

$$P_n(0) = b_n/a_n, \quad P_n'(0) = c_n/a_n.$$

Then it is easy to see the following by the recursion formula for $P_n(x)$ in equation (1.1).

PROPOSITION 2.1. *Suppose that $\psi(t, x)$ is a generating function of the form (2.1) for μ. Suppose $b_n c_{n-1} \neq b_{n-1} c_n$. Then the Szegö–Jacobi parameters $\{\alpha_n, \omega_n\}$ are the unique solution of the system of the linear equations*

$$\begin{aligned} \frac{b_n}{a_n}\alpha_n + \frac{b_{n-1}}{a_{n-1}}\omega_n &= -\frac{b_{n+1}}{a_{n+1}}, \\ \frac{c_n}{a_n}\alpha_n + \frac{c_{n-1}}{a_{n-1}}\omega_n &= -\frac{c_{n+1}}{a_{n+1}} + \frac{b_n}{a_n}. \end{aligned} \tag{2.5}$$

2.2. Special cases of pre-generating functions. In Section 5, we shall discuss particular examples associated with well-known probability measures to appeal that our general approach provides a more efficient algorithm to obtain Szegö–Jacobi parameters from generating functions than the method through Theorem 1.1. For this purpose, consider a pre-generating function of the form

$$\varphi(t, x) = h(\varrho(t)x) = \sum_{n=0}^{\infty} h_n \varrho(t)^n x^n \tag{2.6}$$

and the associated multiplicative renormalization given by

$$\psi(t, x) = \frac{h(\varrho(t)x)}{E_\mu[h(\varrho(t)\cdot)]}$$

under the following analytic conditions (H1) and (H2):

(H1) $\varrho(t) = \sum_{n=1}^{\infty} \varrho_n t^n$ is an analytic function near $t = 0$ and $\varrho_1 \neq 0$.

(H2) $h(x) = \sum_{n=0}^{\infty} h_n x^n$ is an analytic function near $x = 0$ such that $h(tx)$ is analytic in x on the support of μ for $|t| < t_1$ with some $t_1 > 0$, $h_0 = 1$ and $h_n \neq 0$ for any $n \geqslant 1$. Furthermore $h(x)$ satisfies

$$\limsup_{n\to\infty} \left(|h_n| \, \|x^n\|_{L^2(\mu)}\right)^{1/n} < \infty. \tag{2.7}$$

We can easily see that the condition (2.7) is equivalent to

$$\int_0^{2\pi} \int_{\mathbf{R}} |h(e^{i\theta} t_0 x)|^2 \, d\mu(x) \, d\theta < \infty$$

for some t_0, $0 < t_0 < t_1$. If $h_n > 0$ for any $n \geqslant 0$, then the condition $E_\mu |h(\pm t_0 x)|^2 < \infty$ for some $t_0 > 0$ implies (2.7).

THEOREM 2.2. *Assume that* $h(x)$ *and* $\varrho(t)$ *satisfy the conditions* (H1) *and* (H2). *Then we have the following assertions*:

(1) $\varphi(t, x) = h(\varrho(t)x)$ *is a pre-generating function for* μ.

(2) *Its renormalization* $\psi(t, x)$ *is a generating function for* μ *if and only if* $E_\mu[\psi(t, \cdot)\psi(s, \cdot)]$ *depends only on ts.*

Proof. We see that

$$\lim_{t\to 0} \frac{1}{t^n} \frac{\partial^{n+1}}{\partial x^{n+1}} \varphi(t, x) = \lim_{t\to 0} \frac{\varrho(t)^{n+1}}{t^n} \varphi(t, x) = 0,$$

$$\lim_{t\to 0} \frac{1}{t^n} \frac{\partial^n}{\partial x^n} \varphi(t, x) = \lim_{t\to 0} \frac{\varrho(t)^n}{t^n} \varphi(t, x) = \varrho_1^n \neq 0.$$

Therefore $g_n(x)$ in equation (1.2) is a polynomial of degree n with the leading coefficient $\varrho_1^n/n!$. Since

$$h(\varrho(t)x) = \sum_{m=0}^{\infty} h_m \varrho(t)^m x^m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Big(\sum_{\substack{j_1+j_2+\ldots+j_k=n,\\ j_1,j_2,\ldots,j_k \geqslant 1}} \varrho_{j_1} \varrho_{j_2} \cdots \varrho_{j_k} h_k x^k \Big) t^n$$

holds, the condition (2.7) implies (1.3). Thus we have the assertion (1). The second assertion is obvious by Theorem 2.5 in [6]. ∎

Since $E_\mu[h(\varrho(t)\cdot)]$ is a function in t with $\varrho(0) = 0$ by (H1) and μ is a probability measure, it is easy to see that

$$\lim_{t\to 0} \frac{1}{E_\mu[h(\varrho(t)\cdot)]} = 1.$$

Hence we get

$$\lim_{t\to 0}\frac{h\big((\varrho(t)/t)\,x\big)}{E_\mu\big[h(\varrho(t)\cdot)\big]} = h(\varrho_1 x).$$

Therefore we have

PROPOSITION 2.3. *Suppose that a pre-generating function $\varphi(t, x)$ is given by* (2.6) *under the conditions* (H1) *and* (H2). *Then*

$$(2.8)\quad A(x) = h(\varrho_1 x),\qquad B(t) = \frac{1}{E_\mu\big[h(\varrho(t)\cdot)\big]}\quad \text{and}\quad C(t) = h_1\varrho(t)B(t),$$

where $\varrho_1 = \varrho'(0)$ and $h_1 = h'(0)$. In addition, $a_n = \varrho_1^n h_n$ and $c_n = h_1\sum_{k=0}^{n}\varrho_k b_{n-k}$.

By Proposition 2.1, we have

THEOREM 2.4. *Suppose that a pre-generating function $\varphi(t, x)$ is given by* (2.6) *under the conditions* (H1) *and* (H2) *and $b_n c_{n-1} \neq b_{n-1} c_n$ for any $n \geqslant 1$. Then the Szegö–Jacobi parameters $\{\alpha_n, \omega_n\}$ are the unique solution of the system of the linear equations*

$$(2.9)\quad \begin{aligned} \frac{\varrho_1 b_n}{h_n}\alpha_n + \frac{\varrho_1^2 b_{n-1}}{h_{n-1}}\omega_n &= -\frac{b_{n+1}}{h_{n+1}},\\ \frac{\varrho_1 c_n}{h_n}\alpha_n + \frac{\varrho_1^2 c_{n-1}}{h_{n-1}}\omega_n &= -\frac{c_{n+1}}{h_{n+1}} + \frac{\varrho_1 b_n}{h_n}, \end{aligned}$$

where $c_n = h_1\sum_{k=0}^{n}\varrho_k b_{n-k}$.

It is well known that the measure μ is symmetric if and only if $\alpha_n = 0$ for all $n \geqslant 0$. In this case, if $\varrho(t)$ is an odd function, then $B(t)$ is even and $C(t)$ is odd. Therefore, the above equations become the following:

$$(2.10)\quad \begin{aligned} \omega_{2m+1} &= -\frac{b_{2m+2}\,h_{2m}}{\varrho_1^2 b_{2m}\,h_{2m+2}},\\ \omega_{2m} &= -\frac{c_{2m+1}\,h_{2m-1}}{\varrho_1^2 c_{2m-1}\,h_{2m+1}} + \frac{b_{2m}\,h_{2m-1}}{\varrho_1 c_{2m-1}\,h_{2m}}. \end{aligned}$$

Examples will be given in Section 5 to use results in this section.

3. ORTHOGONAL POLYNOMIALS AND DIFFERENTIAL OPERATORS

In this section, we discuss representation of orthogonal polynomials associated with continuous measures similar to (1.5) for the classical cases in connection with generating functions. In general cases, we cannot assert the condition (P1). In the next section, we shall consider the corresponding case of discrete measure.

THEOREM 3.1. *Suppose that a probability measure μ on an interval (a, b) has a smooth density function $w(x)$ and $\psi(t, x)$ is a generating function of the form in equation* (2.1). *Furthermore, suppose that a smooth function $q_n(x)$ satisfies the conditions*

$$\frac{1}{w(x)} D_x^n \big(q_n(x) w(x)\big) \in L^2(\mu)$$

and

$$D_x^k \big(q_n(x) w(x)\big) = 0 \qquad \text{at } x = a \text{ and } x = b$$

for any k, $0 \leqslant k < n$. Then $P_n(x)$ can be represented as

$$P_n(x) = \frac{1}{k_n w(x)} D_x^n \big(q_n(x) w(x)\big) \tag{3.1}$$

with some constant k_n if and only if

$$\int_a^b \big(D_x^n \psi(t, x)\big) q_n(x)\, d\mu(x) = d_n t^n \tag{3.2}$$

with some constant d_n. In this case, d_n and k_n are related by

$$d_n = (-1)^n \lambda_n a_n k_n. \tag{3.3}$$

Proof. By the boundary condition, the n iterations of the integration by parts provide

$$\int_a^b \psi(t, x) \frac{1}{w(x)} D_x^n \big(q_n(x) w(x)\big) d\mu(x) = \int_a^b \psi(t, x) D_x^n \big(q_n(x) w(x)\big) dx$$

$$= (-1)^n \int_a^b \big(D_x^n \psi(t, x)\big) q_n(x)\, d\mu(x).$$

Hence, by the relation

$$\int_a^b \psi(t, x) P_n(x)\, d\mu(x) = \lambda_n a_n t^n,$$

we see that (3.2) is equivalent to

$$\int_a^b \psi(t, x) \frac{1}{w(x)} D_x^n \big(q_n(x) w(x)\big) d\mu(x) = (-1)^n \frac{d_n}{\lambda_n a_n} \int_a^b \psi(t, x) P_n(x)\, d\mu(x),$$

which means

$$\frac{1}{w(x)} D_x^n \big(q_n(x) w(x)\big) = (-1)^n \frac{d_n}{\lambda_n a_n} P_n(x)$$

since $\{P_n\}$ is a complete system of orthogonal polynomials. Thus we have (3.1). ∎

Remark. We make an important remark on Theorem 3.1. The fact, a function

$$\frac{1}{w(x)} D_x^n (q_n(x) w(x))$$

is a polynomial of degree n, is not an assumption but a conclusion. If we know this fact *a priori*, then the assumption, it belongs to $L^2(\mu)$, is automatically satisfied. However, when the above Rodrigues type expression is given first, it cannot tell in general if the condition (P1) is fulfilled or not.

Now we consider the special case when $\varphi(t, x) = h(\varrho(t)x)$ as given by equation (2.6). In this case, the generating function is given by $\psi(t, x) = B(t) h(\varrho(t)x)$ and equation (3.2) becomes the following equation:

$$\int h^{(n)}(\varrho(t)x) q_n(x) d\mu(x) = \frac{d_n}{B(t)} \left(\frac{t}{\varrho(t)}\right)^n.$$

Next, we apply Theorem 3.1 to this special case with $q_n(x)$ of the form $q_n(x) = w_n(x)/w(x)$. We derive the following theorem.

THEOREM 3.2. *Suppose that μ is a probability measure on an interval (a, b) with density function $w(x)$ and $\psi(t, x)$ is a generating function for μ arising from $\varrho(t)$ and $h(x)$ satisfying* (H1) *and* (H2), *respectively. Assume that $w_n(x)$ is a smooth function with support in $[a, b]$ such that*

$$\frac{1}{w(x)} D_x^n w_n(x) \in L^2(\mu)$$

and

$$D_x^k (w_n(x)) = 0 \quad \text{at } x = a \text{ and } x = b$$

for any $0 \leqslant k < n$. Then the orthogonal polynomials $\{P_n(x)\}$ for the measure μ can be represented by

$$P_n(x) = \frac{1}{k_n w(x)} D_x^n (w_n(x)) \tag{3.4}$$

with some constant k_n if and only if the equality

$$\int_a^b h^{(n)}(\varrho(t)x) w_n(x) dx = \frac{d_n}{B(t)} \left(\frac{t}{\varrho(t)}\right)^n \tag{3.5}$$

holds for some constant d_n. In this case, d_n and k_n are given by

$$d_n = n! \varrho_1^n h_n \quad \text{and} \quad k_n = (-1)^n \frac{n!}{\lambda_n}. \tag{3.6}$$

Proof. The equivalence of the conditions in equations (3.4) and (3.5) follows from Theorem 3.1 and the remark preceding the statement of the theorem. To find the value of d_n, note that the left-hand side of equation (3.5) has the limit

$$\lim_{t\to 0}\int_a^b h^{(n)}(\varrho(t)x)\,w_n(x)\,dx = \int_a^b h^{(n)}(0)\,w_n(x)\,dx = h^{(n)}(0) = n!\,h_n.$$

On the other hand, note that $B(0) = 1$ and $\lim_{t\to 0}\varrho(t)/t = \varrho_1$. Hence the right-hand side of equation (3.5) has the limit

$$\lim_{t\to 0}\frac{d_n}{B(t)}\left(\frac{t}{\varrho(t)}\right)^n = \frac{d_n}{\varrho_1^n}.$$

Therefore, we get $n!\,h_n = d_n/\varrho_1^n$, and so $d_n = n!\,\varrho_1^n h_n$. With this value of d_n we can use equation (3.3) to find that

$$k_n = (-1)^n\frac{n!}{\lambda_n}. \blacksquare$$

This theorem will be useful to study particular examples in Section 5.

4. ORTHOGONAL POLYNOMIALS AND DIFFERENCE OPERATORS

Analogously to the case of continuous measure in Section 3, one can represent orthogonal polynomials associated with a discrete probability measure with positive point masses $w(x)$ at $x \in N_0 := \{0\}\cup N$ in terms of difference operators. For this purpose, we need to introduce the forward difference Δ_+ and the backward difference Δ_- by

$$\Delta_+ u(x) = u(x+1)-u(x) \quad \text{and} \quad \Delta_- u(x) = u(x)-u(x-1),$$

where we use the convention $u(x) = 0$ for $x \leqslant -1$ for any function u. It is not hard to see the formulas

$$\Delta_+^n u(x-n) = \Delta_-^n u(x) \tag{4.1}$$

and

$$\Delta_+^n [u(x)v(x)] = \sum_{k=0}^{n}\binom{n}{k}\Delta_+^k u(x+n-k)\,\Delta_-^{n-k} v(x+n-k). \tag{4.2}$$

Then the following theorem is derived.

THEOREM 4.1. *Let $\psi(t, x)$ be a generating function for a probability measure μ on N_0 and $w(x) = \mu(\{x\})$ for $x \in N_0$. Suppose that a function $q_n(x)$ on N_0 satisfies the conditions*

$$\frac{1}{w(x)}(\Delta_-^n q_n w)(x) \in L^2(\mu)$$

and

$$\Delta_+^k (q_n w)(x) = 0 \qquad \text{at } x = 0 \text{ and } x = \infty$$

for any $0 \leqslant k < n$. *Then* P_n *can be represented as*

$$P_n(x) = \frac{1}{k_n w(x)} (\Delta_-^n q_n w)(x)$$

with some constants k_n *if and only if*

$$\sum_{x=0}^{\infty} (\Delta_+^n \psi)(t, x) q_n(x) w(x) = d_n t^n$$

with some constant d_n. *In this case,* d_n *and* k_n *are related by* $d_n = (-1)^n \lambda_n a_n k_n$.

Proof. By the same reason as in the continuous case, for a function $Q_n \in L^2(\mu)$ the condition

$$\sum_{x=0}^{\infty} \psi(t, x) Q_n(x) w(x) = a_n \lambda_n t^n$$

is equivalent to $Q_n = P_n$. (It means that a function Q_n is a monic orthogonal polynomial with respect to the measure μ.) Therefore,

$$P_n(x) = \frac{1}{k_n w(x)} (\Delta_-^n q_n w)(x)$$

holds if and only if

$$\lambda_n a_n t^n = \sum_{x=0}^{\infty} \psi(t, x+n) \Delta_-^n (q_n w)(x+n) = (-1)^n \sum_{x=0}^{\infty} (\Delta_+^n \psi)(t, x) q_n(x) w(x)$$

holds with the help of the formulas in equations (4.1) and (4.2). This completes the proof. ■

For the discrete measure case, we shall restrict our consideration to the case of $h(x) = e^x$ from now on, which can cover the cases of Poisson and negative binomial distributions discussed in Section 5. Similarly to the continuous cases, we have the following theorem.

THEOREM 4.2. *Let* μ $(w(x) = \mu(\{x\}))$ *be a probability measure on* N_0 *such that*

$$\psi(t, x) = B(t) e^{\varrho(t)x}, \qquad \text{where } B(t) = \frac{1}{E_\mu[e^{\varrho(t)x}]},$$

gives a generating function for μ *with* $\varrho(t)$ *satisfying* (H1). *Suppose that there exists a family of probability measures* μ_n $(w_n(x) = \mu_n(\{x\}))$ *on* N_0 *such that*

$$\frac{1}{w(x)} \Delta_-^n w_n(x) \in L^2(\mu)$$

and

$$\Delta_+^k (w_n(x)) = 0 \quad \text{at } x = 0 \text{ and } x = \infty$$

for any $0 \leqslant k < n$. *Then the orthogonal polynomials* $\{P_n(x)\}$ *for the measure* μ *can be represented as*

$$P_n(x) = \frac{1}{k_n w(x)} \Delta_-^n (w_n(x))$$

with some constant k_n *if and only if the equality*

$$\sum_{x=0}^{\infty} e^{\varrho(t)x} q_n(x) w(x) = \frac{d_n t^n}{B(t)(e^{\varrho(t)}-1)^n} \tag{4.3}$$

holds for some constant d_n. *In this case,* d_n *and* k_n *are given by*

$$d_n = \varrho_1^n \quad \text{and} \quad k_n = (-1)^n \frac{n!}{\lambda_n}. \tag{4.4}$$

5. EXAMPLES

5.1. Continuous measure case.

EXAMPLE 5.1 (Gaussian distribution). Let us consider first the case of Gaussian measure with mean 0 and variance σ^2 with density

$$w(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

It has been proved in [5] and [6] that a generating function $\psi(t, x)$ for μ is given by a pre-generating function $\varphi(t, x) = e^{tx}$ with $\varrho(t) = t$ and $h(x) = e^x$. Then $A(x)$ in equation (2.2), $B(t)$ in equation (2.3) and $C(t)$ in equation (2.4) are

$$A(x) = e^x, \qquad a_n = h_n = \frac{1}{n!},$$

$$B(t) = \exp\left(-\tfrac{1}{2}\sigma^2 t^2\right), \qquad b_{2m} = \frac{(-\sigma^2)^m}{2^m m!}, \; b_{2m+1} = 0,$$

$$C(t) = t\exp\left(-\tfrac{1}{2}\sigma^2 t^2\right), \qquad c_{2m} = 0, \; c_{2m+1} = \frac{(-\sigma^2)^m}{2^m m!}$$

by equation (2.8), and then a generating function for μ is given by

$$\psi(t, x) = \exp\left(tx - \tfrac{1}{2}\sigma^2 t^2\right).$$

Since μ is symmetric and $\varrho_1 = 1$, we have

$$\alpha_n = 0, \qquad \omega_n = \sigma^2 n \quad \text{and} \quad \lambda_n = \sigma^{2n} n!$$

for $n \geqslant 1$ ($\alpha_0 = 0$, $\omega_0 = 1$, $\lambda_0 = 1$) by equation (2.10).

Now let us apply Theorem 3.2. Equation (3.5) is satisfied with $w_n = w$ and $1/k_n = (-1)^n \lambda_n/n! = (-1)^n \sigma^{2n}$ by the second equality in (3.6). Therefore we have

$$P_n(x) = (-\sigma^2)^n \exp\left(\frac{x^2}{2\sigma^2}\right) D_x^n \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

by equation (3.4).

Example 5.2 (gamma distribution). Let μ_α be the gamma distribution with parameter $\alpha > 0$ and let

$$w_\alpha(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$$

be its density on $(0, \infty)$. We have shown in [5] and [6] that a generating function $\psi_\alpha(t, x)$ is given by

$$\varrho(t) = \frac{t}{1+t}, \qquad h(x) = e^x \quad \text{and} \quad B_\alpha(t) = (1+t)^{-\alpha}.$$

So we have

$$\psi_\alpha(t, x) = (1+t)^{-\alpha} \exp\left(\frac{t}{1+t} x\right).$$

Then we can see that

$$A(x) = e^x, \qquad a_n = h_n = \frac{1}{n!},$$

$$B_\alpha(t) = (1+t)^{-\alpha}, \qquad b_n = (-1)^n \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!},$$

$$C_\alpha(t) = t(1+t)^{-\alpha-1}, \quad c_n = (-1)^{n-1} \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)(n-1)!}.$$

Therefore equation (2.9) becomes

$$(n+\alpha-1)\alpha_n - \omega_n = (n+\alpha)(n+\alpha-1),$$

$$n(n+\alpha-1)\alpha_n - (n-1)\omega_n = n(n+\alpha+1)(n+\alpha-1).$$

Then we have

$$\alpha_n = 2n+\alpha, \quad \omega_n = n(n+\alpha-1) \quad \text{and} \quad \lambda_n = \frac{n!\,\Gamma(n+\alpha)}{\Gamma(\alpha)}.$$

Since we have

$$\left(\frac{\varrho(t)}{t}\right)^n B_\alpha(t) = (1+t)^{-n} B_\alpha(t) = (1+t)^{-n-\alpha} = B_{n+\alpha}(t),$$

the equality in equation (3.5) is satisfied with $w_{n+\alpha}$. Hence by Theorem 3.2 we have the representation

$$P_n(\alpha, x) = (-1)^n x^{-\alpha+1} e^x D_x^n (x^{n+\alpha-1} e^{-x}).$$

EXAMPLE 5.3 (beta-type distribution). Let μ_β be the beta-type distribution with parameter $\beta > -1/2$ and the density

$$w_\beta(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})} (1-x^2)^{\beta-1/2}$$

on $(-1, 1)$. In [5] and [6] we have proved that a generating function $\psi_\beta(t, x)$ is given by

$$\varrho(t) = \frac{2t}{1+t^2}, \quad h_\beta(x) = (1-x)^{-\beta} \quad \text{and} \quad B_\beta(t) = (1+t^2)^{-\beta}.$$

So we have

$$\psi_\beta(t, x) = (1-2tx+t^2)^{-\beta},$$

$$A_\beta(x) = (1-2x)^{-\beta}, \qquad a_n = 2^n h_n = \frac{2^n \Gamma(n+\beta)}{n!\, \Gamma(\beta)},$$

$$B_\beta(t) = (1+t^2)^{-\beta}, \qquad b_{2m} = (-1)^m \frac{\Gamma(m+\beta)}{\Gamma(\beta)\, m!}, \; b_{2m+1} = 0,$$

$$C_\beta(t) = t(1+t^2)^{-\beta-1}, \quad c_{2m} = 0, \; c_{2m+1} = (-1)^m \frac{\Gamma(m+\beta+1)}{\Gamma(\beta+1)\, m!}.$$

Since μ_β is symmetric, $\alpha_n = 0$ for all $n \geqslant 0$. By equation (2.10), we have

$$\omega_n = \frac{n(n+2\beta-1)}{4(n+\beta-1)(n+\beta)} \quad \text{and} \quad \lambda_n = \frac{\Gamma(\beta)\Gamma(\beta+1)\Gamma(n+2\beta)\, n!}{2^{2n}\Gamma(2\beta)\Gamma(n+\beta)\Gamma(n+\beta+1)}.$$

Since $\varrho_1 = 2$, we have

$$\left(\frac{\varrho(t)}{\varrho_1 t}\right)^n B_\beta(t) = (1+t^2)^{-n} B_\beta(t) = (1+t^2)^{-n-\beta} = B_{n+\beta}(t).$$

So the equality in equation (3.5) is satisfied with $w_{n+\beta}$. Therefore by Theorem 3.2 we get

$$P_n(\beta, x) = (-1)^n \frac{\Gamma(n+2\beta)}{\Gamma(2n+2\beta)} (1-x^2)^{-\beta+1/2} D_x^n (1-x^2)^{n+\beta-1/2}.$$

5.2. Discrete measure case.

EXAMPLE 5.4 (Poisson distribution). Let μ be the Poisson measure with parameter $\lambda > 0$;

$$\mu(\{x\}) = w(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

on N_0. In [5] and [6] it is proved that a generating function $\psi(t, x)$ is given by

$$\varrho(t)=\log(1+t), \quad h(x)=e^x \quad \text{and} \quad B(t)=e^{-\lambda t}.$$

Hence we have $\psi(t, x)=e^{-\lambda t}(1+t)^x$ and

$$A(x)=e^x, \qquad a_n=h_n=\frac{1}{n!},$$

$$B(t)=e^{-\lambda t}, \qquad b_n=(-1)^n\frac{\lambda^n}{n!},$$

$$C(t)=e^{-\lambda t}\log(1+t), \quad c_n=(-1)^n\sum_{k=0}^{n-1}\frac{\lambda^k}{k!(n-k)}.$$

Applying the first equality in (2.9), we have the relation

$$\alpha_n=\lambda\omega_n+\lambda^2.$$

From the second equation in (2.9) we have

$$\alpha_n=\lambda+n, \quad \omega_n=\lambda n \quad \text{and} \quad \lambda_n=\lambda^n n!$$

after the computations for simplification.

Since $e^{\varrho(t)}-1=t$ and $\varrho_1=1$, we have the relation

$$\left(\frac{e^{\varrho(t)}-1}{t}\right)^n B(t)=B(t).$$

Hence the equality in equation (4.3) is satisfied with $w_n=w$. Thus by Theorem 4.2 we have

$$P_n(x)=(-1)^n\lambda^{-x}\Gamma(x+1)\Delta_-^n\frac{\lambda^x}{\Gamma(x+1)}=(-1)^n\lambda^{-x}\Gamma(x+1)\Delta_+^n\frac{\lambda^{x-n}}{\Gamma(x-n+1)}.$$

EXAMPLE 5.5 (negative binomial distribution). Let μ_r be the negative binomial distribution with parameter $r>0$ and $0<p<1$;

$$\mu_r(\{x\})=w_r(x)=p^r\binom{-r}{x}(-1)^x(1-p)^x=p^r\frac{\Gamma(x+r)}{\Gamma(r)\Gamma(x+1)}q^x$$

on N_0, where $p+q=1$. In [5] and [6] we have proved that a generating function $\psi_r(t, x)$ is given by

$$\varrho(t)=\log(1+t)(1+qt)^{-1}, \quad h(x)=e^x \quad \text{and} \quad B_r(t)=(1+qt)^{-r}.$$

Hence we have $\psi_r(t,x) = (1+t)^x(1+qt)^{-x-r}$ and

$$A(x) = e^x, \qquad a_n = p^n h_n = \frac{p^n}{n!},$$

$$B_r(t) = (1+qt)^{-r}, \qquad b_n = (-q)^n \frac{\Gamma(n+r)}{\Gamma(r)\,n!},$$

$$C_r(t) = (1+qt)^{-r}\log\frac{1+t}{1+qt}, \qquad c_n = (-1)^n \sum_{k=0}^{n-1} \frac{\lambda^k(1-q^{n-k})}{k!\,(n-k)}.$$

Applying the first equality in (2.9), we have the relation

$$\alpha_n = -\frac{p}{q(n-1+r)}\omega_n + \frac{q}{p}(n+r).$$

From the second equation in (2.9) we have

$$\alpha_n = \frac{(1+q)n+rq}{p}, \qquad \omega_n = \frac{qn(n-1+r)}{p^2} \quad \text{and} \quad \lambda_n = \frac{q^n}{p^{2n}}\frac{\Gamma(n+r)\,n!}{\Gamma(r)}$$

after the computations for simplification.

Since $e^{\varrho(t)}-1 = pt/(1+qt)$ and $\varrho_1 = p$, we have the relation

$$\left(\frac{e^{\varrho(t)}-1}{\varrho_1 t}\right)^n B(t) = (1+qt)^{-n} B_r(t) = B_{n+r}(t).$$

Thus the equality in equation (4.3) is satisfied with w_{n+r}. Consequently, by Theorem 4.2, we have the representation

$$P_n(x) = (-1)^n \frac{1}{p^n}\frac{\Gamma(x+1)}{\Gamma(x+r)} q^{-x} \Delta_+^n \left(\frac{\Gamma(x+r)}{\Gamma(x-n+1)} q^x\right).$$

5.3. Remarks. It is possible to give another method to derive the Szegö–Jacobi parameters. In fact, by equations (3.6) or (4.4), and the first equality in (2.5), we obtain

$$\omega_n = -\frac{k_{n-1}}{k_n} n,$$

(5.1)

$$\alpha_n = \frac{b_{n-1}a_n}{b_n a_{n-1}}\omega_n - \frac{b_{n+1}a_n}{b_n a_{n+1}}.$$

Actually, for all examples in Section 5, it is not hard to compute directly the leading coefficient k_n.

EXAMPLE 5.6 (leading coefficient and λ_n). For the negative binomial case, it is not difficult to calculate the Szegö–Jacobi parameters in principle, but a bit of a long calculation is required. However, since the leading coefficient of

$$\frac{\Gamma(x+1)}{\Gamma(x+r)} q^{-x} \Delta_+^n \left(\frac{\Gamma(x+r)}{\Gamma(x-n+1)} q^x\right)$$

is equal to $(q-1)^n = (-1)^n p^n$, it is easy to see that the leading coefficient k_n of

$$\frac{1}{w_r(x)}(\Delta_+^n w_{n+r})(x-n) = p^n q^{-n} \frac{\Gamma(n+r)\Gamma(x+1)}{\Gamma(r)\Gamma(x+r)} q^{-x} \Delta_+^n \left(\frac{\Gamma(x+r)}{\Gamma(x-n+1)} q^x\right)$$

is equal to

$$k_n = (-1)^n \frac{p^{2n}}{q^n} \frac{\Gamma(r)}{\Gamma(n+r)}.$$

By equation (4.4), we have

$$\lambda_n = \frac{q^n}{p^{2n}} \frac{\Gamma(n+r)\,n!}{\Gamma(r)}.$$

When we obtain orthogonal polynomials by applying the theory of [5] and [6], we have $E_\mu[\psi^2(t, x)]$ on the calculation. Hence λ_n is obtained rather easily from it [5]. Then α_n can be calculated from equation (5.1).

Example 5.7 (deriving α_n from λ_n). For the Poisson case, it is not hard to get $\lambda_n = \lambda^n n!$ by equation (1.4) in Theorem 1.1 and $\omega_n = \lambda n$. By equation (5.1), we have $\alpha_n = n + \lambda$.

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