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ON EXACT STRONG LAWS FOR SUMS OF MULTIDIMENSIONALLY INDEXED RANDOM VARIABLES

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Abstract. Let $\{X, X_n, n \in Z_+^d\}$ be independent and identically distributed random variables satisfying $xP(|X| > x) \approx L(x)$ with either $EX = 0$ or $E|X| = \infty$, where $L(x)$ is slowly varying at infinity. This paper proves that there always exist sequences of constants $\{a_n\}$ and $\{B_N\}$ such that an Exact Strong Law holds, that is

$$\sum_{|n| \leq N} a_n X_n / B_N \to 1 \text{ almost surely} \quad \text{as } N \to \infty.$$

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1. INTRODUCTION

Consider independent and identically distributed random variables $\{X, X_n, n \in Z_+^d\}$. Under the condition that $EX = 0$ or $E|X| = \infty$ we study whether there exists an almost sure non-zero limit for some weighted sum of our multidimensionally indexed random variables $\{X_n, n \in Z_+^d\}$. In other words, we examine whether or not

$$\lim_{N \to \infty} \frac{\sum_{|n| \leq N} a_n X_n}{B_N} = 1 \text{ almost surely} \tag{1.1}$$

for some sequences of constants $\{a_n\}$ and $\{B_N\}$. This is known as an *Exact Strong Law*. Several papers, e.g. [1]–[5], have been devoted to exploring the conditions for (1.1) to hold. It was shown in [1] that if the random variables are nonnegative with $EX = \infty$, then

$$\lim_{N \to \infty} \frac{\sum_{|n| \leq N} X_n}{B_N} = 1 \text{ almost surely}$$

fails for any sequence $\{B_N, N \geqslant 1\}$. As a matter of fact, Exact Strong Laws hold only for some special classes of random variables and for only some carefully selected weights $\{a_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$. As mentioned in [5], in order for (1.1) to hold when $d = 1$, we have to require the random variables either to barely have a first moment or to barely just miss having a first moment. To this end it was assumed in previous work that $xP(|X| > x)$ was a slowly varying function. Later, this condition was relaxed in [4] and [5], see (2.1) below. This relaxation allows us to obtain an Exact Strong Law for the St. Petersburg game. Our primary interest in this paper is to show that for any distribution in this class, (2.1), whether or not the Exact Strong Laws (1.1) hold for some sequences $\{a_{\boldsymbol{n}}\}$ and $\{B_N\}$. The answer is affirmative.

In the earlier papers the normalizing constants $\{B_N\}$ were predetermined. As was $\{a_{\boldsymbol{n}}\}$. Although $\{a_{\boldsymbol{n}}\}$ are dependent on the distribution of X, their dependence on $\{c_n\}$ (defined in Section 2) limits the application of the theorems only to a smaller class of distributions contained in (2.1) since an extra condition (2.3) cannot always be established. In this paper we allow more flexibility for the choices of the sequences $\{a_{\boldsymbol{n}}\}$ and $\{B_N\}$.

The paper is organized as follows. In Section 2 we state our two main theorems and then demonstrate an example. In Section 3 we provide all the proofs. Prior to that a few comments about notation are in order. The generic constant C will denote a bound that is not necessarily the same in each appearance. We define $\lg x = \log(\max\{e, x\})$ and $\lg_k x = \lg_{k-1}(\lg x)$ for $k \geqslant 2$. Also, we let $x^{\pm} = \max(\pm x, 0)$.

2. MAIN RESULTS

The assumptions about the distribution of X in this paper are the same as those in [5]. First, we assume

(2.1) $\quad xP(|X| > x) \approx L(x)$, where $L(x)$ is slowly varying at infinity,

where $a(x) \approx b(x)$ implies $c_1 a(x) \leqslant b(x) \leqslant c_2 a(x)$ for some positive constants c_1 and c_2. If $E|X| < \infty$, we assume $EX = 0$ and define

$$\mu(x) = \int_x^\infty P(|X| > t)\,dt.$$

In this case our parameter of symmetry is

$$c = \lim_{x\to\infty} \frac{EX^- I(X^- > x)}{E|X| I(|X| > x)}.$$

If $E|X| = \infty$, we define

$$\mu(x) = \int_0^x P(|X| > t)\,dt$$

and in this situation the parameter of symmetry is

$$c = \lim_{x\to\infty} \frac{EX^- I(X^- \leqslant x)}{E|X| I(|X| \leqslant x)}.$$

Our definition of the parameter of symmetry is different from that in [5]. In [5] the parameter of symmetry is defined as

$$c' = \lim_{x\to\infty} \frac{EX^- I(X^- > x)}{EX^+ I(X^+ > x)}$$

in the finite first moment case and

$$c' = \lim_{x\to\infty} \frac{EX^- I(X^- \leqslant x)}{EX^+ I(X^+ \leqslant x)}$$

in the infinite first moment case. There is no essential difference between the two. There is a simple relationship between c and c': $c = c'/(1+c')$. Obviously, $0 \leqslant c \leqslant 1$ while $0 \leqslant c' \leqslant \infty$. We avoid the case $c = \infty$ in the definition for convenience.

Like in [5], we partition the space Z^d_+ into disjoint sets $\{A_n, n \geqslant 1\}$ so that the union of $\{A_n, n \geqslant 1\}$ is Z^d_+. One possible partition is

$$A_n = \{\boldsymbol{n}\colon \boldsymbol{n} \in Z^d_+ \text{ and } |\boldsymbol{n}| = n\},$$

that is, the points in A_n are those that are n units from the origin. However, from the proofs below one may find that for any choice of $\{A_n, n \geqslant 1\}$ our theorems are true. Of course, one can also define the distance $|\boldsymbol{n}|$ in an arbitrary way as long as $|\boldsymbol{n}|$ is integer-valued for any $\boldsymbol{n} \in Z^d_+$. Two common choices are

$$|\boldsymbol{n}| = \sum_{i=1}^{d} n_i \quad \text{and} \quad |\boldsymbol{n}| = \max_{1\leqslant i\leqslant d} n_i \quad \text{for } \boldsymbol{n} = (n_1, \ldots, n_d) \in Z^d_+.$$

The purpose of the introduction of $\{A_n, n \geqslant 1\}$ is that we can then set $a_{\boldsymbol{n}} = a_n$ whenever $\boldsymbol{n} \in A_n$, where $\{a_n, n \geqslant 1\}$ are some constants to be defined. We also let $d_n = |A_n|$, the number of points in A_n. Thus $d_n \geqslant 1$ for all $n \geqslant 1$.

Let $\{h_n\}$ be a sequence of positive numbers satisfying

$$\sum_{n=1}^{\infty} h_n = \infty \quad \text{and} \quad \lim_{n\to\infty} h_n = 0. \tag{2.2}$$

Define

$$c_n = \inf\{x\colon x/\mu(x) \geqslant d_n/h_n\}.$$

Since $\mu(x)$ is slowly varying at infinity (see [4]), we have $x/\mu(x) \to \infty$ as $x \to \infty$. Therefore $\{c_n\}$ is well defined and $c_n \sim d_n \mu(c_n)/h_n$ as $n \to \infty$.

Throughout the paper we will let $\{B_n, n \geqslant 1\}$ be non-decreasing and $\{b_n, n \geqslant 1\}$ be strictly increasing (in order to apply Kronecker's lemma) with $\lim_{n\to\infty} b_n = \infty$ and $B_n = \sum_{i=1}^n b_i h_i$. We also set $a_n = b_n/c_n$.

Our first result is the extension of the theorems that can be found in [5].

THEOREM 2.1. *Assume that* (2.1) *holds and*

$$\sum_{n=1}^{\infty} d_n P(|X| > c_n) < \infty. \tag{2.3}$$

If

$$\limsup_{n\to\infty} (b_n/B_n) < \infty, \tag{2.4}$$

then

$$\lim_{N\to\infty} \frac{\sum_{|n|\leqslant N} a_n X_n}{B_N} = 2c-1 \text{ almost surely} \tag{2.5}$$

when $EX = 0$ *and*

$$\lim_{N\to\infty} \frac{\sum_{|n|\leqslant N} a_n X_n}{B_N} = 1-2c \text{ almost surely} \tag{2.6}$$

when $E|X| = \infty$.

Remark 1. If we set $h_n = 1/(n \lg n)$ and $b_n = (\lg n)^b$ for some $b > 0$, then $B_n \sim (\lg n)^b/b = b_n/b$. Thus (2.4) holds. This reduces to the theorems in [5].

Remark 2. The condition (2.4) provides a general rule for the selection of b_n, and thus B_n. For any sequence $\{h_n\}$ with property (2.2) we can carefully select $\{b_n\}$ so that (2.4) is satisfied. For example, we can set $b_n = S_n + S_{n-1}$, where $S_n = \sum_{i=1}^{n} h_i$ for $n \geqslant 1$ and $S_0 = 0$. Then $B_n = S_n^2$. So (2.4) is trivial in this case. This follows from the calculations

$$B_n = \sum_{i=1}^{n} b_i h_i = \sum_{i=1}^{n} (S_i + S_{i-1})(S_i - S_{i-1}) = \sum_{i=1}^{n} (S_i^2 - S_{i-1}^2) = S_n^2.$$

Remark 3. In general, if $g(x)$ is a non-decreasing function on $(0, \infty)$ satisfying

$$\lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{g'(x)}{g(x)} = \beta \in [0, \infty) \tag{2.7}$$

and $b_n = g(S_n)$, we have

$$B_n = \sum_{i=1}^{n} b_i h_i = \sum_{i=1}^{n} g(S_i)(S_i - S_{i-1}) \geqslant \int_0^{S_n} g(x)\,dx,$$

and thus, by L'Hospital's rule,

$$\limsup_{n\to\infty} \frac{b_n}{B_n} \leqslant \lim_{n\to\infty} \frac{g(S_n)}{\int_0^{S_n} g(x)\,dx} = \lim_{x\to\infty} \frac{g'(x)}{g(x)} = \beta,$$

yielding (2.4). A large class of functions satisfies (2.4). For example, $g(x) = x^{\alpha}$ for $\alpha > 0$ and $g(x) = \exp(\beta x)$ for $\beta > 0$ are two such functions.

From our remarks the existence of b_n for (2.4) to hold is no longer a problem as long as one can find a sequence $\{h_n\}$ that ensures both (2.2) and (2.3). It is our task to show the existence of such a sequence $\{h_n\}$ in the following theorem.

THEOREM 2.2. *Under* (2.1) *there exists a sequence* $\{h_n\}$ *so that both* (2.2) *and* (2.3) *hold, and hence* (2.5) *or* (2.6) *holds for some sequences* $\{a_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ *and* $\{B_n, n \geq 1\}$.

Several examples were given in the earlier papers. As we mentioned in Remark 1, only a special case was considered in [5], that is, $h_n = 1/(n \lg n)$. In the situation when the theorems in [5] hold our Theorem 2.1 can give more choices about the weights as pointed out in Remarks 2 and 3. In the situation where the theorems in [5] do not hold, Theorem 2.2 guarantees the existence of some appropriate weights for our Exact Strong Law. Following the proof of Theorem 2.2 one gets the idea on how to find such weights. We will just present one example (Example 3) in [5] where the lack of choice in selecting $\{h_n\}$ complicated matters.

EXAMPLE. Let $\{X, X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ be i.i.d. random variables with

$$xP(X > x) \sim \exp(\lg x / \lg_2 x)/\lg_2 x.$$

For simplicity assume that $X > 0$. In [5] it has been shown that an Exact Strong Law does not apply in the one-dimensional case, i.e., $d = 1$, when $\{h_n\}$ was equal to $1/(n \lg n)$. However, when $d > 1$, an additional condition $d_n = a^n$ was needed in order to establish an Exact Strong Law.

Without any additional conditions we will show how to define $\{h_n\}$ and $\{b_n\}$ so that an Exact Strong Law can be established for any $d \geq 1$ and any choices of d_n. From [5] we have

$$L(x) = \exp(\lg x/\lg_2 x)/\lg_2 x, \quad \mu(x) \sim \exp(\lg x/\lg_2 x), \quad L(x)/\mu(x) \sim 1/\lg_2 x.$$

Define $h_n = 1/(n(\lg n)(\lg_2 n))$. Then (2.2) is trivial. Since $c_n \sim d_n \mu(c_n)/h_n > n$, it follows that

$$d_n P(X > c_n) \sim \frac{d_n L(c_n)}{c_n} \sim \frac{L(c_n) h_n}{\mu(c_n)}$$
$$\sim \frac{1}{n \lg n \lg_2 n \lg_2 c_n} < \frac{1}{n \lg n (\lg_2 n)^2},$$

which yields (2.3). Observe that $\sum_{i=1}^n h_i \sim \lg_3 n$. From Remark 3 we can take $b_n = (\lg_3 n)^\alpha$ and $B_N \sim (\lg_3 N)^{\alpha+1}/(\alpha+1)$, where $\alpha > 0$. Or we can set $b_n = (\lg_2 n)^\beta$ and $B_N \sim (\lg_2 N)^\beta/\beta$ for some $\beta > 0$. With either of these choices we have

$$\lim_{N \to \infty} \frac{\sum_{|\boldsymbol{n}| \leq N} a_{\boldsymbol{n}} X_{\boldsymbol{n}}}{B_N} = 1 \text{ almost surely.}$$

3. PROOFS

The proof of our first theorem is mainly based on the techniques utilized in [5]. However, the proof of the second theorem concerning the existence of the constants in (1.1) is new and more difficult.

Proof of Theorem 2.1. Since the proof is almost the same as those in [5], we will only consider the case of $EX = 0$. As usual we use the partition

$$\frac{1}{B_N}\sum_{|\mathbf{n}|\leqslant N} a_{\mathbf{n}} X_{\mathbf{n}} = \left(\frac{b_N}{B_N}\right)\frac{1}{b_N}\sum_{|\mathbf{n}|\leqslant N} a_{\mathbf{n}}[X_{\mathbf{n}} I(|X_{\mathbf{n}}| \leqslant c_{\mathbf{n}}) - EX_{\mathbf{n}} I(|X_{\mathbf{n}}| \leqslant c_{\mathbf{n}})]$$
$$+\left(\frac{b_N}{B_N}\right)\frac{1}{b_N}\sum_{|\mathbf{n}|\leqslant N} a_{\mathbf{n}} X_{\mathbf{n}} I(|X_{\mathbf{n}}| > c_{\mathbf{n}}) + \frac{1}{B_N}\sum_{|\mathbf{n}|\leqslant N} a_{\mathbf{n}} EX_{\mathbf{n}} I(|X_{\mathbf{n}}| \leqslant c_{\mathbf{n}}),$$

where $c_{\mathbf{n}} = c_n$ whenever $\mathbf{n} \in A_n$. Note that $a_n = b_n/c_n$ and b_N/B_N is bounded by (2.4). In order to show that the first two terms converge almost surely to zero we need to verify that

$$\sum_{n=1}^{\infty} d_n c_n^{-2} EX^2 I(|X| \leqslant c_n) < \infty \tag{3.1}$$

as in [5], by applying the Khintchine–Kolmogorov convergence theorem and Kronecker's lemma (see, e.g., [7]). The proof of (3.1) is the same as that in [5]. For completeness of the proof we display it here:

$$\sum_{n=1}^{\infty} d_n c_n^{-2} EX^2 I(|X| \leqslant c_n) \leqslant C \sum_{n=1}^{\infty} d_n c_n^{-2} \int_0^{c_n} tP(|X| > t)\,dt$$
$$\leqslant C \sum_{n=1}^{\infty} d_n c_n^{-2} \int_0^{c_n} L(t)\,dt \leqslant C \sum_{n=1}^{\infty} d_n c_n^{-1} L(c_n) \leqslant C \sum_{n=1}^{\infty} d_n P(|X| > c_n) < \infty.$$

To complete the proof we still need to show that the third term converges to $2c-1$. Since $EX = 0$, we have

$$EXI(|X| \leqslant c_n) = -EXI(|X| > c_n) \sim (2c-1)E|X|I(|X| > c_n) = (2c-1)\mu(c_n).$$

Therefore

$$\frac{1}{B_N}\sum_{|\mathbf{n}|\leqslant N} a_{\mathbf{n}} EX_{\mathbf{n}} I(|X_{\mathbf{n}}| \leqslant c_{\mathbf{n}}) \sim \frac{2c-1}{B_N}\sum_{n=1}^{N}\frac{d_n b_n \mu(c_n)}{c_n}$$
$$\sim \frac{2c-1}{B_N}\sum_{n=1}^{N}\frac{d_n b_n \mu(c_n)}{d_n \mu(c_n)/h_n} \sim \frac{2c-1}{B_N}\sum_{n=1}^{N} b_n h_n \sim 2c-1.$$

This completes the proof of Theorem 2.1. ∎

Proof of Theorem 2.2. From [4] it follows that $\mu(x)$ is a slowly varying function and $l_1(x) = L(x)/\mu(x) \to 0$. By Karamata's representation theorem ([6], Theorem 1.3.1), since $l_1(x)$ is also a slowly varying function, it can be expressed as

$$l_1(x) = q(x)\exp\left\{\int_1^x \frac{\varepsilon(t)}{t}\,dt\right\},$$

where $q(x) \to q > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$. Set $l_3(x) = \sup_{t>x} l_2(t)$, where

$$l_2(x) = \exp\left\{\int_1^x \frac{\varepsilon(t)}{t}\,dt\right\}.$$

Since the continuous function $l_2(x)$ vanishes as $x \to \infty$, $l_3(x)$ is well defined and $l_3(x)\downarrow 0$ as $x \to \infty$.

Define $e_n = l_3(n^{1/2})$ for $n \geqslant 1$ and $e_0 = e_1$. Then $e_n > 0$ for $n \geqslant 1$ and $e_n \downarrow 0$ as $n \to \infty$. Define $h_n = e_n^{-1/2} - e_{n-1}^{-1/2} + n^{-2}$ for $n \geqslant 1$. Thus $h_n > 0$ and it follows that

$$\sum_{n=1}^{N} h_n > \sum_{n=1}^{N} [e_n^{-1/2} - e_{n-1}^{-1/2}] = e_N^{-1/2} - e_1^{-1/2} \to \infty$$

and

$$\sum_{n=1}^{N} e_n h_n \leqslant e_1 \sum_{n=1}^{N} n^{-2} + \sum_{n=1}^{N} e_n^{1/2} e_{n-1}^{1/2} [e_n^{-1/2} - e_{n-1}^{-1/2}]$$
$$= e_1 \sum_{n=1}^{N} n^{-2} + \sum_{n=1}^{N} [e_{n-1}^{1/2} - e_n^{1/2}] \leqslant e_1 \sum_{n=1}^{\infty} n^{-2} + e_1^{1/2} < \infty.$$

Hence we can conclude that

$$\sum_{n=1}^{\infty} h_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} e_n h_n < \infty. \tag{3.2}$$

Since $l_2(x)$ is slowly varying, we have ([8], p. 277)

$$l_3(x) \geqslant l_2(x) \geqslant Cx^{-1/3},$$

which implies $e_n \geqslant Cn^{-1/6}$ for all large n. Note that $e_{n-1}/e_n \geqslant 1$. We want to prove that

$$e_{n-1}/e_n = 1 + o(1/n) \tag{3.3}$$

as $n \to \infty$. By definition, $e_{n-1} = \sup_{t \geqslant (n-1)^{1/2}} l_2(t)$. If the supremum is achieved in $[n^{1/2}, \infty)$, then $e_{n-1}/e_n = 1$. Otherwise, $e_{n-1} = \sup_{t\in[(n-1)^{1/2}, n^{1/2})} l_2(t)$ and $e_n \geqslant l_2(n^{1/2})$, and thus as $n \to \infty$

$$1 \leqslant \frac{e_{n-1}}{e_n} \leqslant \sup_{t \in [(n-1)^{1/2}, n^{1/2})} \frac{l_2(t)}{l_2(n^{1/2})}$$
$$\leqslant \exp\left\{\int_{\sqrt{n-1}}^{\sqrt{n}} \frac{|e(t)|}{t} dt\right\} = \exp\left\{o(1) \int_{\sqrt{n-1}}^{\sqrt{n}} \frac{1}{t} dt\right\}$$
$$= \exp\left\{\log \frac{n}{n-1}\right\} = 1 + o\left(\frac{1}{n}\right),$$

proving (3.3). Therefore

$$0 \leqslant e_n^{-1/2} - e_{n-1}^{-1/2} = \frac{1}{\sqrt{e_{n-1}}}\left(\sqrt{\frac{e_{n-1}}{e_n}} - 1\right) = o\left(\frac{1}{n\sqrt{e_n}}\right) = o(n^{-11/12}).$$

This allows us to conclude that $n^{-2} \leqslant h_n = o(n^{-11/12})$, which together with (3.2) yields (2.2).

Our remaining task is to show (2.3). Notice that

$$\frac{d_n}{h_n n^{2/3}} \geqslant \frac{1}{h_n n^{2/3}} \to \infty \quad \text{as } n \to \infty.$$

From [4] it follows that $\mu(x)$ is a slowly varying function. Thus from [8], p. 277, we have $\mu(x) > x^{-1/3}$ for all large x. Since $c_n \sim d_n \mu(c_n)/h_n$, for large n we have

$$\frac{c_n^{4/3}}{n^{2/3}} > \frac{c_n}{n^{2/3}\mu(c_n)} \sim \frac{d_n}{h_n n^{2/3}} \to \infty,$$

which implies that

$$c_n/n^{1/2} \to \infty.$$

Hence, for large n, $l_3(c_n) \leqslant l_3(n^{1/2}) = e_n$. Moreover, we have

$$d_n P(|X| > c_n) \leqslant \frac{C d_n L(c_n)}{c_n} \sim \frac{C d_n L(c_n)}{d_n \mu(c_n)/h_n}$$
$$= \frac{C L(c_n) h_n}{\mu(c_n)} = C h_n l_1(c_n) \leqslant C h_n l_2(c_n) \leqslant C h_n l_3(c_n) \leqslant C h_n e_n,$$

which combined with (3.2) gives (2.3). This completes the proof of Theorem 2.2. ∎

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