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MULTIVARIATE LARGE DEVIATIONS FOR SUMS OF I.I.D. RANDOM VECTORS WITH COMPACTLY SUPPORTED DISTRIBUTION

BY

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Abstract. The sums of i.i.d. random vectors with compactly supported and absolutely continuous distribution are considered. Under some conditions the strong form of the local limit theorem for large deviations is proved. In passing the asymptotic behaviour of the moment generating function as well as possible non-degenerate limit laws for the natural exponential family of distributions are established.

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1. INTRODUCTION

Let $\xi, \xi^{(1)}, \ldots, \xi^{(n)}, \ldots$ be independent identically distributed random vectors taking values in R^d, $d \geqslant 1$. Let us put $S_n = \xi^{(1)} + \ldots + \xi^{(n)}$, $n = 1, 2, \ldots$

If $E|\xi|^2 < \infty$, then from the central limit theorem it follows that for any $A \subset R^d$, being the set of continuity of the Lebesgue measure, the relation

$$(1) \qquad P\left(n^{-1/2}(S_n - na) \in A\right) \to \int_A \varphi_B(u)\,du, \quad n \to \infty,$$

holds. Here $a = E\xi$ while $\varphi_B(u)$ denotes the density of a mean-zero Gaussian vector with the covariance matrix $B = E(\xi - a)(\xi - a)^T$.

If, furthermore, S_{n_0} for some $n_0 \geqslant 1$ has a uniformly bounded density $p_{n_0}(x)$, then the local limit theorem holds, that is,

$$(2) \qquad \sup_{x \in R^d} \left| n^{d/2} p_n(x) - \varphi_B\left(n^{-1/2}(x - na)\right) \right| = o(1), \quad n \to \infty.$$

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A value $x = x_n$ in the range of S_n is called a *large deviation* if

$$n^{-1/2}|x-na| \to \infty, \quad n \to \infty.$$

That is why $P_n(A) = P(S_n - na \in A)$, where $A = A_n$ is such that

$$r_n(A) = n^{-1/2} \inf_{x \in A} |x| \to \infty, \quad n \to \infty,$$

is called a *large deviation probability*. It is convenient to refer to $r_n(A)$ as the order of large deviation. In view of (1), a large deviation probability converges to zero while in view of (2) the term $n^{d/2} p_n(x)$ converges to zero when x enjoys a large deviation. One of the basic problems of the large deviations theory is to establish a precise asymptotic behaviour of $P_n(A)$ or $p_n(x)$ when x enjoys a large deviation. Another class of problems unites those related to so-called *rough* asymptotic behaviour of a large deviation probability, that is, the asymptotic behaviour of $\ln P_n(A)$ (see, e.g., [6]–[8]).

Let P be the distribution of ξ. In what follows we assume that it has a bounded convex support X, which is an essentially d-dimensional open set containing the origin. Moreover, we suppose that X can be written as

$$X = \{x = te \in R^d\colon\ 0 \leqslant t < h(e),\ e \in S^{d-1}\},$$

where $h(e)$ is a positive continuous function on the unit sphere S^{d-1} in R^d.

Let $f(s)$ be the moment generating function corresponding to P, that is

$$f(s) = \int_X e^{\langle s,x\rangle} P(dx),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^d. It is easily seen that $f(s)$ is finite for all $s \in R^d$, that is

$$S = \{s \in R^d\colon\ f(s) < \infty\} = R^d.$$

For any $s \in R^d$ and any Borel set $A \subset R^d$ we define

$$P_s(A) = \big(f(s)\big)^{-1} \int_A e^{\langle s,x\rangle} P(dx).$$

The distribution P_s is called the *Cramér* transformation of the distribution P or the *conjugate* distribution with respect to P (see, e.g., [4], Section 1, § 1). Then $\{P_s,\ s \in R^d\}$ is called the *family of the conjugate distributions* or the *natural exponential family of distributions* generated by P. From now on we assume that P is absolutely continuous and denote its density by $p(x)$.

The classical method of treating large deviations is based on the conjugate distribution techniques. First general results on the topic can be found in [5]. Among them we can find a local limit theorem (see also Theorem 1 in [4], Section 1, §3). Let $\gamma(s)$ and $B(s)$ be the *gradient* and the *hessian* of $\ln f(s)$, respectively. As known, $\gamma(s)$ and $B(s)$ are the mean vector and the covariance matrix of the distribution P_s, respectively. Moreover, the function $\gamma(\cdot)$ establishes a one-to-one correspondence between R^d and $\gamma(R^d) \subset R^d$. Denote by $s(x)$ the inverse function with respect to $\gamma(s)$.

PROPOSITION. *Suppose that for $n_0 \geqslant 1$ there exists a uniformly bounded density $p_{n_0}(x)$ of S_{n_0}. Then*

$$\sup_{x\in\gamma(F)} \left| \frac{p_n(nx)}{\psi_n(x)\left(\varrho(x)\right)^n} - 1 \right| = o(1), \quad n \to \infty,$$

where

$$\psi_n(x) = (2\pi n)^{-d/2} \left(\det B\left(s(x)\right)\right)^{-1/2}, \quad \varrho(x) = \inf_{s\in R^d} f(s) e^{-\langle s,x\rangle} = f\left(s(x)\right) e^{-\langle s(x),x\rangle}$$

and F is any closed bounded set in R^d.

The function

$$H(x) = -\ln \varrho(x) = \sup_{s\in R^d} \left(\langle s, x\rangle - \ln f(s)\right)$$

is called the *Fenchel–Legendre* transformation of $\ln f(s)$ (see, e.g., [6], pp. 26 and 134, or [4], Section 1, §1). This function is also known as the deviation function being the rate function of the large deviation probabilities, so that

$$\left(\varrho(x)\right)^n = \exp\{-nH(x)\}.$$

It is worth noting that in the case considered $r_n(A) = O(n^{1/2})$ and the higher order of large deviations does not arise.

The question arises: *under what additional conditions can the statement of the Proposition be extended to the whole set $\gamma(R^d)$*? In other words, under what conditions does the relation

$$\sup_{x\in\gamma(R^d)} \left| \frac{p_n(nx)}{\psi_n(x)\left(\varrho(x)\right)^n} - 1 \right| = o(1), \quad n \to \infty,$$

hold?

In the case when S is a bounded open set containing the origin, the answer was given in [12] and [13], where P was taken to be the gamma-like distribution in R^d. Here we consider a distribution with a bounded convex support X. In this case it is expected that $\gamma(R^d) = X$. It will be confirmed in the next section.

In general, we follow the scheme proved to be effective in [12] and [13]. We start with establishing the asymptotic behaviour of $f(s)$ when $|s| \to \infty$, that is, with establishing the *Abel* type theorem (see, e.g., [9], Section 13, §5). We make use of the *Laplace* method for its proving. Possible non-degenerate limit laws for the natural exponential family of distributions are discussed in Section 3.

The results obtained are also of interest from the viewpoint of searching for the so-called stable multivariate exponential families (see [2]). Section 4 contains our main result: the strong form of the local limit theorem for large deviations. Some remarks are given in Section 5 while proofs are gathered in the Appendix.

2. A THEOREM OF THE ABEL TYPE

To find an asymptotic expression for $f(s)$ when $|s| \to \infty$, we have the need for assumptions on the distribution P. First, we assume a regular behaviour of its density in a neighbourhood of the boundary ∂X.

(A) *The density $p(x)$ of the distribution P is bounded in any closed subset of X. In a neighbourhood of ∂X it satisfies the condition*

$$\sup_{e \in S^{d-1}} \left| r_{\alpha-1}(\tau^{-1}) p\big((h(e)-\tau)e\big) - \lambda(e)\right| = o(1), \qquad \tau \downarrow 0,$$

or

$$\sup_{e \in S^{d-1}} \left| r_{\alpha-1}\big((h(e)-|x|)^{-1}\big) p(x) - \lambda(e)\right| = o(1), \qquad |x| \uparrow h(e), \qquad e = |x|^{-1} x,$$

where the function $r_{\alpha-1}(u)$, $u > 0$, is $(\alpha-1)$-regularly varying at infinity (cf. [14], Section 5.4.2), $\alpha > 0$, *while $\lambda(e)$ is a positive continuous function on S^{d-1}.*

Assumption (A) is not very restrictive. It means some regularity of the underlying distribution in a neighbourhood of the boundary ∂X but allows even tending the density to infinity when $\alpha \in (0, 1)$. For instance, a uniform distribution on X satisfies (A) with $\alpha = 1$, $r_0(u) \equiv 1$ and $\lambda(e) \equiv \text{const}$.

We also need an assumption on the boundary ∂X. It is formulated in terms of the function $a(e)$, which is the *support function* for X, that is

$$a(e) = \sup_{x \in X} \langle x, e\rangle = \max_{\varepsilon \in S^{d-1}} h(\varepsilon)\langle \varepsilon, e\rangle, \qquad e \in S^{d-1}. \tag{3}$$

The relation (3) can be rewritten as (see, e.g., [12])

$$h(\varepsilon) = \min_{e \in S^{d-1}: \langle \varepsilon, e\rangle > 0} \frac{a(e)}{\langle \varepsilon, e\rangle}, \qquad \varepsilon \in S^{d-1}. \tag{4}$$

For convenience, we partition the assumption onto two parts.

(B) *For any $e \in S^{d-1}$ the set* $\operatorname{argmax}_{\varepsilon \in S^{d-1}} h(\varepsilon)\langle \varepsilon, e\rangle$ *consists of a single point $\varepsilon' = \varepsilon'(e)$.*

Since both functions $a(e)$ and $h(\varepsilon)$ are positive and continuous, from (3) and (4) it follows that $\{\varepsilon'(e),\, e \in S^{d-1}\} = S^{d-1}$. Moreover, if (B) holds, then each point of the boundary ∂X is *regular* (see, e.g., [10], p. 20) and X is a *strictly convex* set.

(C) *For any* $e \in S^{d-1}$ *the function* $h(\varepsilon)\langle\varepsilon, e\rangle$, $\varepsilon \in S^{d-1}$, *in a neighbourhood of* $\varepsilon'(e)$ *admits the representation*

$$h(\varepsilon)\langle\varepsilon, e\rangle = a(e) - \tfrac{1}{2}\big(\varepsilon - \varepsilon'(e)\big)^T \Lambda_e \big(\varepsilon - \varepsilon'(e)\big) + w_e(\varepsilon),$$

where Λ_e *is a non-negative definite symmetric matrix of rank* $d-1$ *which is continuous on* $e \in S^{d-1}$ *and such that* $\Lambda_e \varepsilon'(e) = 0$ *while*

$$\sup_{e \in S^{d-1}} \frac{|w_e(\varepsilon)|}{|\varepsilon - \varepsilon'(e)|^2} = o(1), \qquad \varepsilon \to \varepsilon'(e).$$

The above representation plays a key role in the proof of the Abel theorem by means of the Laplace method.

Moreover, if (C) holds, then X is a *rotund body* (see Section 10.1 in [2]) with a sufficiently smooth boundary.

Let us put

$$\Lambda_e^{(0)} = \operatorname{diag}\big(\lambda_1(e), \ldots, \lambda_{d-1}(e), 0\big), \qquad \bar{\Lambda}_e^{(0)} = \operatorname{diag}\big(\lambda_1(e), \ldots, \lambda_{d-1}(e)\big),$$

where $\lambda_j(e)$, $j = 1, \ldots, d-1$, are non-zero eigenvalues of Λ_e.

THEOREM 1. *If* (A)–(C) *hold, then*

$$\sup_{e \in S^{d-1}} |r_\beta(t) e^{-ta(e)} f(te) - g_\alpha(e)| = o(1), \qquad t \to \infty,$$

where $\beta = \alpha + (d-1)/2$ *and*

$$g_\alpha(e) = (2\pi)^{(d-1)/2} \Gamma(\alpha) \big(h(\varepsilon')\big)^{d-1} \lambda(\varepsilon') \langle\varepsilon', e\rangle^{-\alpha} (\det \bar{\Lambda}_e^{(0)})^{-1/2}.$$

Similarly, one can establish that for $s = te$ as $t \to \infty$ it follows that

$$\sup_{e \in S^{d-1}} \left| r_\beta(t) e^{-ta(e)} \frac{\partial f(s)}{\partial s_i} - g_\alpha(e) h(\varepsilon') \varepsilon_i' \right| = o(1), \qquad i = 1, \ldots, d. \tag{5}$$

Let $p^{(s)}(x)$ be the density of the conjugate distribution P_s, that is

$$p^{(s)}(x) = \frac{e^{\langle s, x\rangle} p(x)}{f(s)}, \qquad x \in X, \; s \in R^d.$$

Obviously, the distribution P_s is supported by the set X. Since $\gamma(s) = \operatorname{grad} \ln f(s)$ is the mean vector of P_s, we infer that $\gamma(s) \in X$.

From Theorem 1 and (5) it follows that

$$\sup_{e \in S^{d-1}} |\gamma(te) - h(\varepsilon')\varepsilon'| = o(1), \qquad t \to \infty.$$

Therefore, if $|s| \to \infty$, then $\gamma(s) \uparrow \partial X$. That is why $\gamma(R^d) = X$ since both sets are bounded, open and convex with the same boundary.

3. LIMIT LAWS FOR CONJUGATE DISTRIBUTIONS

Let ξ_s be a random vector having the distribution P_s. As we know, P_{te} is concentrated in a neighbourhood of $h(\varepsilon')\varepsilon'$ as $t \to \infty$. Therefore, in order to find a non-degenerate limit law for P_s when $|s| \to \infty$, one should transform ξ_s in a proper way.

One transformation follows immediately from the proof of Theorem 1 (see the Appendix). If (A)–(C) hold, then the density $\mu_s(x)$ of the random vector

$$\pi_s(\xi_s) = \left(t^{1/2}(C_e^T e_{\xi_s})_1, \ldots, t^{1/2}(C_e^T e_{\xi_s})_{d-1}, t\left(h(e_{\xi_s}) - |\xi_s|\right)\right),$$

where C_e is the orthogonal matrix reducing Λ_e to the diagonal matrix $\Lambda_e^{(0)}$, satisfies for any $\delta > 0$ and for $s = te$, $t \to \infty$, the relation

$$\sup_{e \in S^{d-1}} \sup_{x \in R^d: x_d \geq \delta} |\mu_s(x_1, \ldots, x_d) - \varphi_{(\bar{\Lambda}_e^{(0)})^{-1}}(x_1, \ldots, x_{d-1}) \cdot \langle e, \varepsilon' \rangle q_\alpha(\langle e, \varepsilon' \rangle x_d)| = o(1).$$

Here $\varphi_{(\bar{\Lambda}_e^{(0)})^{-1}}(x_1, \ldots, x_{d-1})$ stands for the density of a mean-zero $(d-1)$-dimensional normal distribution with the covariance matrix $(\bar{\Lambda}_e^{(0)})^{-1}$, and $q_\alpha(z)$ denotes the density of $\Gamma(\alpha)$ distribution.

However, the transformation π_s is rather useless from the viewpoint of applications since it is non-linear and centering in the last component depends on a random factor. The question arises: *does there exist a linear transformation of ξ_s having a weak non-degenerate limit when $t \to \infty$?*

It turns out that the answer to this question is positive. Let $\eta = (\eta_1, \ldots, \eta_d)$ be a random vector having the density

$$\begin{aligned}\pi_{d,\alpha}(x; a, B) &= \varphi_B(x_1, \ldots, x_{d-1}) \cdot a q_\alpha(a x_d - \bar{x}^T B^{-1} \bar{x}/2) \\ &= \frac{a(a x_d - \bar{x}^T B^{-1} \bar{x}/2)^{\alpha-1} \exp(-a x_d)}{(2\pi)^{(d-1)/2} (\det B)^{1/2} \Gamma(\alpha)},\end{aligned}$$

where $x_d > \bar{x}^T B^{-1} \bar{x}/(2a)$ and $\bar{x} = (x_1, \ldots, x_{d-1})^T$. It is easy to see that $\bar{\eta} = (\eta_1, \ldots, \eta_{d-1})$ has a mean-zero $(d-1)$-normal distribution with the covariance matrix B while $a\eta_d$ has $\Gamma(\beta)$ distribution. The conditional distribution of $a\eta_d$, given $\bar{\eta} = z$, is $\Gamma(\alpha)$ distribution shifted by $z^T B^{-1} z/(2a)$.

In what follows, we denote by $e^{(i)}$ the i-th unit vector in R^d, and

$$A_\delta(a, B) = \{x \in R^d\colon\ x_d - \bar{x}^T B^{-1} \bar{x}/(2a) \geq \delta\}.$$

THEOREM 2. *If* (A)–(C) *hold, then the density $\bar{\mu}_s(x)$ of the random vector*

$$\bar{\pi}_s(\xi_s) = D_s C_e^T (\xi_s - h(\varepsilon')\varepsilon')$$

satisfies for any $\delta > 0$ and for $s = te$, $t \to \infty$, the relation

$$\sup_{e \in S^{d-1}} \sup_{x \in A_\delta(\langle \varepsilon', e \rangle, H(e))} \left|\bar{\mu}_s(x) - \pi_{d,\alpha}\left(x; \langle \varepsilon', e \rangle, H(e)\right)\right| = o(1),$$

where $H(e) = \big(h(\varepsilon')\big)^2 (\bar{\Lambda}_e^{(0)})^{-1}$ *while the rows of the matrix* D_s *are* $t^{1/2} e^{(1)}, \ldots, t^{1/2} e^{(d-1)}, -(t/\langle \varepsilon', e\rangle) C_e^T e$, *respectively.*

The linear transformation $\bar{\pi}_s(\xi_s)$ can also be written as

$$\bar{\pi}_s(\xi_s) = \big(t^{1/2}(C_e^T \xi_s)_1, \ldots, t^{1/2}(C_e^T \xi_s)_{d-1}, t\big(h(\varepsilon') - \langle \xi_s, e\rangle/\langle \varepsilon', e\rangle\big)\big).$$

EXAMPLE. Let $d = 2$ and $h(e) \equiv 1$. Thus X is a unit ball centered at the origin. Let P be the uniform distribution on X. Then condition (A) holds with $\alpha = 1$, $r_0(u) \equiv 1$, $\lambda(e) \equiv \pi^{-1}$. One can easily see that $a(e) \equiv 1$, conditions (B) and (C) hold and $\varepsilon'(e) \equiv e$.

Let $s = te$, $e = e^{(2)}$, $t \to \infty$. As we know, the conjugate distribution P_s is concentrated in a neighbourhood of e. The transformations $\pi_s(\xi_s)$ and $\bar{\pi}_s(\xi_s)$ of the random vector $\xi_s = (\xi_{s1}, \xi_{s2})$ with the distribution P_s look like

$$\pi_s(\xi_s) = \left(t^{1/2}\frac{\xi_{s1}}{|\xi_s|}, t(1-|\xi_s|)\right), \qquad \bar{\pi}_s(\xi_s) = \big(t^{1/2}\xi_{s1}, t(1-\xi_{s2})\big),$$

while the densities of the limit distributions are

$$(2\pi)^{-1/2}\exp\big(-(x_1^2/2 + x_2)\big), \qquad x_1 \in R,\ x_2 > 0,$$

and

$$\pi_{2,1}(x; 1, 1) = (2\pi)^{-1/2}\exp(-x_2), \qquad x_1 \in R,\ x_2 > x_1^2/2, \tag{6}$$

respectively.

4. LOCAL LIMIT THEOREMS

Our method of proving the strong form of the local limit theorem for large deviations of sums consists in establishing, firstly, the local limit theorem for conjugate distributions.

One can easily calculate the mean vector γ_e and the covariance matrix Σ_e of the limit distribution from Theorem 2:

$$\gamma_e = (\beta/\langle \varepsilon', e\rangle) e^{(d)}, \qquad \Sigma_e = \begin{pmatrix} H(e) & 0 \\ 0 & \beta/\langle \varepsilon', e\rangle^2 \end{pmatrix},$$

where, as before,

$$\beta = \alpha + \frac{d-1}{2}, \qquad H(e) = \big(h(\varepsilon')\big)^2 (\bar{\Lambda}_e^{(0)})^{-1}, \qquad e^{(d)} = (0, \ldots, 0, 1)^T.$$

From Theorem 2 it follows that the density $\tilde{\mu}_s(x)$ of the random vector

$$\Sigma_e^{-1/2}\bar{\pi}_s(\xi_s) = \Sigma_e^{-1/2} D_s C_e^T \big(\xi_s - h(\varepsilon')\varepsilon'\big)$$

satisfies for any $\delta > 0$ the relation

$$\lim_{|s|\to\infty} \sup_{x\in A_\delta(\beta^{1/2},I)} |\tilde{\mu}_s(x) - \pi_{d,\alpha}(x;\, \beta^{1/2},\, I)| = 0,$$

where I denotes the identity matrix.

Thus, the random vector $\Sigma_e^{-1/2} D_s C_e^T (\xi_s - h(\varepsilon')\varepsilon')$ converges in distribution to a limit vector with the density $\pi_{d,\alpha}(x;\, \beta^{1/2},\, I)$. From the next lemma it follows that the random vector $B^{-1/2}(s)(\xi_s - h(\varepsilon')\varepsilon')$ also converges in distribution to the same limit vector. Here $B(s)$ denotes, as before, the covariance matrix of the distribution P_s.

LEMMA 1. *If* (A)–(C) *hold, then there exists a matrix $B^{1/2}(s)$ with the property $B^{1/2}(s)\big(B^{1/2}(s)\big)^T = B(s)$ such that for any $\delta > 0$*

$$\lim_{|s|\to\infty} \sup_{x\in A_\delta(\beta^{1/2},I)} |\bar{p}^{(s)}(x) - \pi_{d,\alpha}(x;\, \beta^{1/2},\, I)| = 0,$$

where

$$\bar{p}^{(s)}(x) = \big(\det B(s)\big)^{1/2} p^{(s)}\big(B^{1/2}(s)\,x + h(\varepsilon')\,\varepsilon'\big)$$

is the density of $B^{-1/2}(s)\big(\xi_s - h(\varepsilon')\,\varepsilon'\big)$.

Let $\xi_s^{(1)}, \ldots, \xi_s^{(n)}, \ldots$ be independent identically distributed random vectors in R^d with the distribution P_s. By $p_{ns}(x)$ we denote the density of the sum $\xi_s^{(1)} + \ldots + \xi_s^{(n)}$, $n = 1, 2, \ldots$

THEOREM 3. *If* (A)–(C) *hold, then as $n \to \infty$*

$$\sup_{s\in R^d} \sup_{x\in R^d} \left|(2\pi n)^{d/2} \big(\det B(s)\big)^{1/2} p_{ns}\big(n^{1/2} B^{1/2}(s)\,x + n\gamma(s)\big) - \exp(-|x|^2/2)\right| = o(1),$$

where the matrix $B^{1/2}(s)$ is that from Lemma 1.

Now we can establish the main result concerning the large deviations of sums.

THEOREM 4. *If* (A)–(C) *hold, then*

$$\sup_{x\in X} \left|\frac{p_n(nx)}{\psi_n(x)\big(\varrho(x)\big)^n} - 1\right| = o(1), \qquad n \to \infty,$$

where $\psi_n(x)$ and $\varrho(x)$ are as in the Proposition.

Due to the Cramér identity we have

$$p_n(nx) = \big(f(s)\big)^n e^{-n\langle s,x\rangle} p_{ns}(nx), \qquad s \in R^d.$$

For $x = \gamma(s)$ we obtain

$$p_n(nx) = \big(\varrho(x)\big)^n p_{ns}\big(n\gamma(s)\big).$$

The assertion of Theorem 4 follows then from Theorem 3.

5. SOME REMARKS

1. One could note that after the obvious small modifications the main results hold even if the support X of the underlying distribution P is not convex but compact and an essentially d-dimensional set. In this case $\gamma(R^d) \neq X$ but $\gamma(R^d) = \operatorname{conv}(X)$.

2. Last time there has been increased an interest in the theory of multivariate exponential families on the one hand and exponential families which are invariant under a group of affine transformations on the other. If there exists a linear transformation of the random vector ξ_s with the distribution P_s, $s \in S$, such that it converges in distribution to a non-degenerate limit vector ζ when $s \to \partial S$ (or $|s| \to \infty$), then the exponential family $\{P'_t,\ t \in T\}$ generated by the distribution of ζ is said to be *stable* in the sense that all distributions P'_t are of the same type. The latter means that for any $t \in T$ there exists a linear transformation π_t such that $\pi_t(\zeta)$ has the distribution P'_t (see [2]). In this case it is also said that the distribution of ζ is *stable*. To the contrary with the univariate case (see [1]), in the multivariate case a complete description of the stable distributions still is not available.

One example of the stable distribution was obtained in [12] and [13] as the limit distribution for the exponential family generated by the gamma-like distribution. The density of this stable distribution takes the form

$$\phi_{x_d B_e}(x_1, \ldots, x_{d-1})\, q_\beta(x_d)$$

for some $\beta > 0$ and a diagonal positive-definite matrix B_e.

Another example is given in Theorem 2, where the density of the stable distribution is of the form $\pi_{d,\alpha}(x;\, a,\, B)$. In fact, the same examples were also given in [2].

It is of interest to remind that in the univariate case any stable distribution is either normal or, possibly shifted, gamma (see [1]).

3. It is of interest to compare the transformations $\pi_s(\xi_s)$ and $\bar{\pi}_s(\xi_s)$ aiming to answer the question: *why does there exist a shift in the last component of the limit distribution after the linear transformation* $\bar{\pi}_s$? Note that

$$\big(\bar{\pi}_s(\xi_s)\big)_d - \big(\pi_s(\xi_s)\big)_d = t\big(h(\varepsilon') - h(e_{\xi_s}) + |\xi_s| - |\xi_s| \langle e_{\xi_s}, e\rangle / \langle \varepsilon', e\rangle\big).$$

As we know, if $t \to \infty$, then e_{ξ_s} is in a neighbourhood of ε', and due to (C) and the proof of Theorem 1 we get

$$\langle \varepsilon', e\rangle\, h(\varepsilon') = h(e_{\xi_s}) \langle e_{\xi_s}, e\rangle + \tfrac{1}{2} \bar{e}^T \bar{\Lambda}_e^{(0)} \bar{e} + o(t^{-1}),$$

where $\bar{e}$ is the vector that consists of the first $d-1$ components of $C_e^T e_{\xi_s}$. Then

$$\begin{aligned} &h(\varepsilon') - h(e_{\xi_s}) + |\xi_s| \big(1 - \langle e_{\xi_s}, e\rangle / \langle \varepsilon', e\rangle\big) \\ &\qquad = \big(|\xi_s| - h(e_{\xi_s})\big)\big(1 - \langle e_{\xi_s}, e\rangle / \langle \varepsilon', e\rangle\big) + \frac{1}{2\langle \varepsilon', e\rangle} \bar{e}^T \bar{\Lambda}_e^{(0)} \bar{e} + o(t^{-1}). \end{aligned}$$

From the proof of Theorem 1 it follows that

$$\big||\xi_s|-h(e_{\xi_s})\big| = O(t^{-1}), \quad |1-\langle e_{\xi_s}, e\rangle/\langle \varepsilon', e\rangle| = O(t^{-1/2}).$$

Therefore,

$$t\big(h(\varepsilon')-h(e_{\xi_s})+|\xi_s|-|\xi_s|\langle e_{\xi_s}, e\rangle/\langle \varepsilon', e\rangle\big) = \frac{1}{2\langle \varepsilon', e\rangle}(t^{1/2}\bar{e})^T \bar{\Lambda}_e^{(0)}(t^{1/2}\bar{e})+o(1).$$

4. The next example shows what can happen if (B) or (C) does not hold. Let $d = 2$ and $h(e_1, e_2) = (|e_1|+|e_2|)^{-1}$, $(e_1, e_2) \in S^1$. Thus, X is a square centered at the origin whose vertices are at points $(1, 0), (0, 1), (-1, 0), (0, -1)$. By a direct calculation it can be shown that $a(e_1, e_2) = \max\{|e_1|, |e_2|\}$.

Take $e = (1/\sqrt{2}, 1/\sqrt{2})$. Condition (B) does not hold at this point; any point (e_1, e_2) with $e_1 \geqslant 0$, $e_2 \geqslant 0$ can be taken as $\varepsilon'(e)$. Let P be the uniform distribution on X. Then condition (A) holds with $\alpha = 1$, $r_0(u) \equiv 1$, $\lambda(e) \equiv 1/2$.

By direct calculations one can show that $f(te) = e^{t/\sqrt{2}}/(\sqrt{2}t)\big(1+o(1)\big)$ as $t \to \infty$. The conjugate distribution P_{te} is concentrated in a neighbourhood of the boundary ∂X lying in the first quarter-plane $(x_1 > 0,\ x_2 > 0)$.

A linear transformation of ξ_s having a non-degenerate limit when $s = te$, $t \to \infty$, takes the form

$$\bar{\pi}_s(\xi_s) = \big(\xi_{s1},\ t(a(e)-\langle e, \xi_s\rangle)\big) = \left(\xi_{s1},\ \frac{t}{\sqrt{2}}(1-\xi_{s1}-\xi_{s2})\right).$$

The density of the limit distribution is of the form

$$g(x_1, x_2) = \exp(-x_2), \quad 0 < x_1 < 1,\ x_2 > 0. \tag{7}$$

The limit distribution has two independent components: the first component has a uniform distribution on $(0, 1)$ while the second one has $\Gamma(1)$ (exponential) distribution.

Consider also the second case when $e = (e_1, e_2)$ is such that $e_2 > |e_1|$. Then (B) holds with $\varepsilon'(e) = e^{(2)}$ and $a(e) = e_2$. However, in a neighbourhood of ε' we have

$$h(\varepsilon)\langle \varepsilon, e\rangle = \frac{e_1\varepsilon_1+e_2\varepsilon_2}{|\varepsilon_1|+\varepsilon_2} = a(e)-\frac{e_2|\varepsilon_1|-e_1\varepsilon_1}{|\varepsilon_1|+\varepsilon_2}$$

and (C) does not hold.

By direct calculations one can show that

$$f(te) = \frac{\exp(te_2)}{t^2(e_2^2-e_1^2)}\big(1+o(1)\big), \quad t \to \infty.$$

The conjugate distribution P_{te} is concentrated in a neighbourhood of $e^{(2)}$.

A linear transformation of ξ_s having a non-degenerate limit when $s = te$, $t \to \infty$, takes the form

$$\bar{\pi}_s(\xi_s) = \big(te_2\,\xi_{s1},\, te_2(1-\langle e, \xi_s\rangle/\langle \varepsilon', e\rangle)\big) = \big(te_2\,\xi_{s1},\, t(e_2-e_1\,\xi_{s1}-e_2\,\xi_{s2})\big).$$

The density of the limit distribution is of the form

$$(8)\quad g(x_1, x_2) = \frac{1-e_1^2/e_2^2}{2}\exp(-x_2), \qquad -\frac{x_2}{1+e_1/e_2} < x_1 < \frac{x_2}{1-e_1/e_2},\ x_2 > 0.$$

Again we see that the second component has gamma distribution (it is $\Gamma(2)$ distribution).

It is of interest to note that in all examples considered we have obtained the densities of the same form but with different supports (see (6)–(8)).

APPENDIX

From now on, c denotes any positive constant whose concrete value is of no importance. This means that $c+c = c$, $c^2 = c$, etc. As before, C_e denotes the orthogonal matrix reducing Λ_e to the diagonal matrix $\Lambda_e^{(0)}$, that is, $\Lambda_e = C_e\Lambda_e^{(0)}C_e^T$. By $\omega(t)$ we denote any non-negative function such that $\lim_{t\to\infty}\omega(t) = 0$ while θ varies within $[-1, 1]$.

Proof of Theorem 1. Let $e\in S^{d-1}$ be arbitrary but fixed, $s = te$, $t\to\infty$. First, we prove the theorem in the case when the coordinate axes are associated with the main axes of the matrix Λ_e and the normal direction to the boundary at the point $h(\varepsilon')\varepsilon'$, i.e., we assume that $\varepsilon'(e) = e^{(d)}$ and C_e is the identity matrix.

Let us put

$$X_1 = \{x = r\varepsilon\colon\ h(\varepsilon)-\delta \leqslant r < h(\varepsilon),\ |\varepsilon-\varepsilon'| < Mt^{-1/2}\},$$

$$X_2 = \{x = r\varepsilon\colon\ h(\varepsilon)-\delta \leqslant r < h(\varepsilon),\ Mt^{-1/2} \leqslant |\varepsilon-\varepsilon'| < \delta'\},$$

$$X_3 = \{x = r\varepsilon\colon\ 0 \leqslant r < h(\varepsilon)-\delta,\ |\varepsilon-\varepsilon'| < \delta'\},$$

$$X_4 = \{x = r\varepsilon\colon\ 0 \leqslant r < h(\varepsilon),\ |\varepsilon-\varepsilon'| \geqslant \delta'\},$$

where M is arbitrarily large while $\delta > 0$, $\delta' > 0$ are arbitrarily small. Obviously,

$$(9)\qquad f(s) = f_1(s)+f_2(s)+f_3(s)+f_4(s),$$

where

$$f_i(s) = \int_{X_i} e^{\langle s,x\rangle}p(x)\,dx, \qquad i = 1, 2, 3, 4.$$

We estimate $f_i(s)$ one after another. Let us start with $f_1(s)$ and let X_1 take the form

$$(10)\qquad X_1 = X_{11}\cup X_{12},$$

where

$$X_{11} = \{x \in X_1 \colon h(\varepsilon) - r < Nt^{-1}\}, \qquad X_{12} = \{x \in X_1 \colon h(\varepsilon) - r \geqslant Nt^{-1}\}$$

and N is arbitrarily large. Define

$$f_{1k}(s) = \int_{X_{1k}} e^{\langle s,x\rangle} p(x)\,dx, \qquad k = 1, 2. \tag{11}$$

If $x \in X_1$, then $|\bar{\varepsilon}| = O(t^{-1/2})$ and $1 - \varepsilon_d = O(t^{-1})$, where $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{d-1})^T$. Changing variables

$$x_i = r\varepsilon_i,\ i = 1, \ldots, d-1, \qquad x_d = \operatorname{sign}(x_d)\, r\, |\varepsilon_d| = \operatorname{sign}(x_d)\, r\, (1 - \varepsilon_1^2 - \ldots - \varepsilon_{d-1}^2)^{1/2}$$

with Jacobian $r^{d-1}|\varepsilon_d|^{-1}$ yields

$$f_{11}(s) = \int_{t^{1/2}|\bar{\varepsilon}| < M} |\varepsilon_d|^{-1} \int_{t(h(\varepsilon) - r) < N} r^{d-1} e^{r\langle s,\varepsilon\rangle} p(r\varepsilon)\, dr\, d\bar{\varepsilon}.$$

It is worth noting that $x_d > 0$ in a neighbourhood of $\varepsilon' = e^{(d)}$.

Next the change of variables $u = h(\varepsilon) - r$ leads to the relation

$$\begin{aligned} f_{11}(s) = \int_{t^{1/2}|\bar{\varepsilon}| < M} |\varepsilon_d|^{-1} \int_{tu < N} \big(h(\varepsilon) - u\big)^{d-1} \exp\big(th(\varepsilon)\langle e, \varepsilon\rangle - t\langle e, \varepsilon\rangle u\big) \\ \times p\big(\big(h(\varepsilon) - u\big)\varepsilon\big)\, du\, d\bar{\varepsilon}. \end{aligned}$$

Therefore, as $t \to \infty$ we obtain

$$\begin{aligned} f_{11}(s) = e^{ta(e)} \int_{t^{1/2}|\bar{\varepsilon}| < M} |\varepsilon_d|^{-1} \big(h(\varepsilon)\big)^{d-1} \lambda(\varepsilon) \exp\Big(-\frac{t}{2}(\varepsilon - \varepsilon')^T \Lambda_e (\varepsilon - \varepsilon')\Big) \\ \times \int_{tu < N} \frac{u^{\alpha-1} \exp(-t\langle e, \varepsilon\rangle u)}{l(u^{-1})}\, du\, d\bar{\varepsilon}\big(1 + o(1)\big) \end{aligned}$$

due to (A) and (C), where $l(t)$, $t > 0$, varies slowly at infinity. Since $\langle e, \varepsilon\rangle = e_d + O(t^{-1/2})$ and $(\varepsilon - \varepsilon')^T \Lambda_e^{(0)} (\varepsilon - \varepsilon') = \bar{\varepsilon}^T \bar{\Lambda}_e^{(0)} \bar{\varepsilon}$, we obtain as $t \to \infty$

$$f_{11}(s) = \frac{e^{ta(e)} \big(h(\varepsilon')\big)^{d-1} \lambda(\varepsilon')}{l(t)\, t^{\alpha}} \int_{t^{1/2}|\bar{\varepsilon}| < M} \exp\Big(-\frac{t}{2}\bar{\varepsilon}^T \bar{\Lambda}_e^{(0)} \bar{\varepsilon}\Big)\, d\bar{\varepsilon}\big(\Gamma(\alpha)/e_d^{\alpha} + \omega(N) + o(1)\big)$$

and finally

$$\begin{aligned} f_{11}(s) &= \frac{e^{ta(e)} \big(h(\varepsilon')\big)^{d-1} \lambda(\varepsilon')}{l(t)\, t^{\alpha + (d-1)/2}} \Big(\frac{(2\pi)^{(d-1)/2}}{(\det \bar{\Lambda}_e^{(0)})^{1/2}} + \omega(M)\Big)\big(\Gamma(\alpha)/e_d^{\alpha} + \omega(N) + o(1)\big) \\ &= \frac{e^{ta(e)}}{r_{\beta}(t)}\big(g_{\alpha}(e) + \theta\omega(\min(M, N)) + o(1)\big). \end{aligned} \tag{12}$$

Similarly, for $f_{12}(s)$ we get as $t \to \infty$

$$(13)\quad f_{12}(s) = \int\limits_{t^{1/2}|\bar{\varepsilon}|<M} |\varepsilon_d|^{-1} \int\limits_{Nt^{-1} \leqslant u < \delta} \big(h(\varepsilon)-u\big)^{d-1} \exp\big(th(\varepsilon)\langle e, \varepsilon\rangle - t\langle e, \varepsilon\rangle u\big) \times p\big(\big(h(\varepsilon)-u\big)\varepsilon\big)\, du\, d\bar{\varepsilon}$$

$$\leqslant ce^{ta(e)} \int\limits_{t^{1/2}|\bar{\varepsilon}|<M} \exp\left(-\frac{t}{2}\bar{\varepsilon}^T \bar{\Lambda}_e^{(0)} \bar{\varepsilon}\right) \int\limits_{tu \geqslant N} e^{-t\langle e,\varepsilon\rangle u} p\big(\big(h(\varepsilon)-u\big)\varepsilon\big)\, du\, d\bar{\varepsilon}$$

$$\leqslant \frac{e^{ta(e)}}{r_\beta(t)} \omega(N).$$

From (10)–(13) we obtain

$$(14)\qquad f_1(s) = \frac{e^{ta(e)}}{r_\beta(t)} \big(g_\alpha(e) + \theta\omega\big(\min(M, N)\big) + o(1)\big), \qquad t \to \infty.$$

Before estimating $f_2(s)$ we note that for $x \in X_2$ we have $t^{1/2}|\bar{\varepsilon}| \geqslant 2M$,

$$(15)\qquad |\bar{\varepsilon}| < \delta', \quad 1-\varepsilon_d < c\delta'^2, \quad (\varepsilon-\varepsilon')^T \Lambda_0 (\varepsilon-\varepsilon') \geqslant c\,|\bar{\varepsilon}|^2.$$

Then, by (A) and (C) we obtain

$$f_2(s) \leqslant ce^{ta(e)} \int\limits_{t^{1/2}|\bar{\varepsilon}| \geqslant 2M} \exp\left(-ct\,|\bar{\varepsilon}|^2\right) \int\limits_0^\delta \frac{u^{\alpha-1} e^{-t\langle e,\varepsilon\rangle u}}{l(u^{-1})}\, du\, d\bar{\varepsilon}.$$

Partitioning the interval $(0, \delta]$ as $\{u \in (0, \delta]\colon tu < N\}$ and $\{u \in (0, \delta]\colon tu \geqslant N\}$, and treating both integrals similarly as we did during the investigation of $f_{11}(s)$ and $f_{12}(s)$, we obtain

$$(16)\qquad f_2(s) \leqslant \frac{ce^{ta(e)}}{l(t)\,t^\alpha} \int\limits_{t^{1/2}|\bar{\varepsilon}| \geqslant 2M} \exp\left(-ct\,|\bar{\varepsilon}|^2\right) d\bar{\varepsilon} = \frac{e^{ta(e)}}{r_\beta(t)} \omega(M), \qquad t \to \infty.$$

Before estimating $f_3(s)$ we note that for $x \in X_3$ again (15) holds and $\langle e, \varepsilon\rangle > e_d - \delta'$. Then, by (C), we have

$$(17)\qquad f_3(s) \leqslant ce^{ta(e)} \int\limits_{|\bar{\varepsilon}|<\delta'} \exp\left(-ct\,|\bar{\varepsilon}|^2 - \delta t\langle e, \varepsilon\rangle\right) \int\limits_0^{h(\varepsilon)-\delta} p(r\varepsilon)\, dr\, d\bar{\varepsilon}$$

$$\leqslant \frac{c \exp\big(ta(e) - t\delta(e_d - \delta')\big)}{t^{(d-1)/2}}.$$

At last, we get

$$(18)\qquad f_4(s) \leqslant c \exp\big(t\big(a(e) - c(\delta')\big)\big) \int\limits_X p(x)\, dx = c \exp\big(ta(e) - tc(\delta')\big),$$

where $c(\delta') > 0$ is such that

$$a(e) - \max_{\varepsilon \in S^{d-1}: |\varepsilon-\varepsilon'| \geqslant \delta'} h(\varepsilon)\langle \varepsilon, e\rangle \geqslant c(\delta') \qquad \text{for all } e \in S^{d-1}.$$

Since M, N, δ and δ' are arbitrary, from (9), (14), (16)–(18) it follows that

$$f(s) = \frac{g_\alpha(e)\, e^{ta(e)}}{r_\beta(t)}(1+o(1)), \quad t \to \infty, \tag{19}$$

uniformly in $e \in S^{d-1}$.

Let us turn to the general case. Clearly, $C_e e^{(d)} = \varepsilon'(e)$. We can write

$$f(s) = \int_{C_e^T X} \exp(\langle s, C_e x\rangle)\, p(C_e x)\, dx, \quad s \in R^d.$$

If $C_e \varepsilon$ is in a neighbourhood of $\varepsilon'(e)$, then ε is in a neighbourhood of $e^{(d)}$ and by (C) we obtain

$$h(C_e \varepsilon)\langle C_e \varepsilon, e\rangle - a(e) = -\tfrac{1}{2}(\varepsilon - e^{(d)})^T A_e^{(0)} (\varepsilon - e^{(d)}) + w_e(C_e \varepsilon).$$

The remainder of the proof repeats that given above. ∎

Proof of Theorem 2. In contrast to the proof of Theorem 1, from the very beginning we consider the general case since the particular one gives no special simplifications.

The transformation $y = C_e D_s^{-1} x + h(\varepsilon')\varepsilon'$ is inverse with respect to $D_s C_e^T (x - h(\varepsilon')\varepsilon')$. Thus

$$\bar{\mu}_s(x) = |\det D_s^{-1}|\, p^{(s)}(C_e D_s^{-1} x + h(\varepsilon')\varepsilon') = \frac{e^{\langle s, y\rangle}\, p(y)}{t^{(d+1)/2} f(s)}, \quad y \in X, \tag{20}$$

where the matrix D_s^{-1} is such that its first $d-1$ rows are

$$t^{-1/2} e^{(1)}, \ldots, t^{-1/2} e^{(d-1)},$$

while the last row is

$$\left(-(t^{-1/2}/\langle \varepsilon', e\rangle)(C_e^T e)_1, \ldots, -(t^{-1/2}/\langle \varepsilon', e\rangle)(C_e^T e)_{d-1}, -t^{-1}\right).$$

Given $\delta > 0$ let us define

$$Z = \sup_{e \in S^{d-1}} \sup_{x \in A_\delta(\langle \varepsilon', e\rangle, H(e))} \left|\bar{\mu}_s(x) - \pi_{d,\alpha}(x; \langle \varepsilon', e\rangle, H(e))\right| = \sup_{e \in S^{d-1}} \max(Z_1, Z_2),$$

where

$$Z_1 = \sup_{x \in Y \cap A_\delta(\langle \varepsilon', e\rangle, H(e))} \left|t^{-(d+1)/2} p^{(s)}(C_e D_s^{-1} x + h(\varepsilon')\varepsilon') - \pi_{d,\alpha}(x; \langle \varepsilon', e\rangle, H(e))\right|,$$

$$Z_2 = \sup_{x \in Y^c \cap A_\delta(\langle \varepsilon', e\rangle, H(e))} \left|t^{-(d+1)/2} p^{(s)}(C_e D_s^{-1} x + h(\varepsilon')\varepsilon') - \pi_{d,\alpha}(x; \langle \varepsilon', e\rangle, H(e))\right|,$$

$$Y = \{x \in R^d \colon |\bar{x}| < M,\ x_d < N\},$$

Y^c is the complement of Y, and M and N are arbitrarily large numbers.

We have

(21) $\langle y, s\rangle - ta(e) = t\langle C_e D_s^{-1} x, e\rangle = t\langle D_s^{-1} x, C_e^T e\rangle = -\langle \varepsilon', e\rangle x_d,$

(22) $C_e^T y = D_s^{-1} x + h(\varepsilon') e^{(d)} = \left(\frac{x_1}{t^{1/2}}, \ldots, \frac{x_{d-1}}{t^{1/2}}, h(\varepsilon') - \frac{\sum_{i=1}^{d-1} x_i (C_e^T e)_i}{t^{1/2}\langle \varepsilon', e\rangle} - \frac{x_d}{t}\right)^T.$

By (22) we obtain as $t \to \infty$

$$|C_e^T y| = h(\varepsilon') - \frac{\sum_{i=1}^{d-1} x_i (C_e^T e)_i}{t^{1/2}\langle \varepsilon', e\rangle} - \frac{x_d}{t} + \frac{\left(\sum_{i=1}^{d-1} x_i (C_e^T e)_i\right)^2}{2h(\varepsilon')\langle \varepsilon', e\rangle^2 t} + \frac{\sum_{i=1}^{d-1} x_i^2}{2h(\varepsilon') t} + o(t^{-1}),$$

(23)

$$C_e^T (e_y - \varepsilon') = e_{C_e^T y} - e^{(d)} = \Bigg(\frac{x_1}{t^{1/2} h(\varepsilon')} + O(t^{-1}), \ldots, \frac{x_{d-1}}{t^{1/2} h(\varepsilon')} + O(t^{-1}),$$
$$-\frac{\left(\sum_{i=1}^{d-1} x_i (C_e^T e)_i\right)^2}{2h^2(\varepsilon')\langle \varepsilon', e\rangle^2 t} - \frac{\sum_{i=1}^{d-1} x_i^2}{2h^2(\varepsilon') t} + O(t^{-3/2})\Bigg)^T.$$

From (21), (23) and (C) it follows that, as $t \to \infty$,

(24) $$0 < h(e_y) - |y| = \frac{h(e_y)\langle e_y, e\rangle - \langle y, e\rangle}{\langle e_y, e\rangle}$$

$$= \frac{h(e_y)\langle e_y, e\rangle - a(e) - \langle D_s^{-1} x, C_e^T e\rangle}{\langle e_y, e\rangle} = \frac{h(e_y)\langle e_y, e\rangle - a(e) + \langle \varepsilon', e\rangle x_d/t}{\langle e_y, e\rangle}$$

$$= \frac{-\frac{1}{2}(e_y - \varepsilon')^T \Lambda_e (e_y - \varepsilon') + \langle \varepsilon', e\rangle x_d/t + o(t^{-1})}{\langle \varepsilon', e\rangle + O(t^{-1/2})}$$

$$= \frac{x_d - \bar{x}^T H(e)^{-1} \bar{x}/(2\langle \varepsilon', e\rangle)}{t}(1 + o(1)).$$

Therefore, if $x \in Y \cap A_\delta(\langle \varepsilon', e\rangle, H(e))$, then $y \in X_{11}$ (X_{11} is from the proof of Theorem 1).

From (20)–(22), (24) and Theorem 1 it follows that, as $t \to \infty$,

(25) $$\bar{\mu}_s(x) = \frac{\exp(\langle y, s\rangle - ta(e))\, t^{\alpha-1} l(t) p(y)}{g_\alpha(e)}(1 + o(1))$$

$$= \frac{\exp(-\langle \varepsilon', e\rangle x_d)\, \lambda(\varepsilon') \left(x_d - \bar{x}^T H(e)^{-1} \bar{x}/(2\langle \varepsilon', e\rangle)\right)^{\alpha-1}}{g_\alpha(e)}(1 + o(1))$$

$$= \frac{\langle \varepsilon', e\rangle^\alpha \left(x_d - \bar{x}^T H(e)^{-1} \bar{x}/(2\langle \varepsilon', e\rangle)\right)^{\alpha-1} \exp(-\langle \varepsilon', e\rangle x_d)(\det \bar{\Lambda}_e^{(0)})^{1/2}}{\Gamma(\alpha)(2\pi)^{(d-1)/2}\left(h(\varepsilon')\right)^{d-1}}$$
$$\times (1 + o(1)),$$

and therefore

$$Z_1 = o(1), \qquad t \to \infty,$$

uniformly in $e \in S^{d-1}$.

As to Z_2, it is easily seen that $Z_2 \leqslant Z_{21} + Z_{22}$, where

$$Z_{21} = \sup_{x \in Y^c \cap A_\delta(\langle \varepsilon', e\rangle, H(e))} \pi_{d,\alpha}\big(x; \langle \varepsilon', e\rangle, H(e)\big),$$

$$Z_{22} = \sup_{x \in Y^c \cap A_\delta(\langle \varepsilon', e\rangle, H(e))} t^{-(d+1)/2} p^{(s)}\big(C_e D_s^{-1} x + h(\varepsilon')\bar{\varepsilon}'\big).$$

Clearly,

$$Z_{21} \leqslant c \sup_{x_d \geqslant N} x_d^{\alpha-1} \exp(-x_d) + c \sup_{|\bar{x}| \geqslant M} \exp\big(-\bar{x}^T H(e)^{-1} \bar{x}/2\big) = \omega(N) + \omega(M).$$

Before estimating Z_{22}, let us define

$$X' = \{x \in R^d\colon\ |\bar{x}| < M,\ x_d \geqslant N\},$$

$$X'' = \{x \in R^d\colon\ |\bar{x}| \geqslant M\} \cap A_\delta\big(\langle \varepsilon', e\rangle, H(e)\big)$$

and note that

$$t^{-(d+1)/2} p^{(s)}\big(C_e D_s^{-1} x + h(\varepsilon')\varepsilon'\big) = \frac{\exp(-x_d)\, t^{\alpha-1} l(t)\, p(y)}{g_\alpha(e)} \big(1 + o(1)\big).$$

Since $y \notin X_{11}$ yields $x \notin Y \cap A_\delta\big(\langle \varepsilon', e\rangle, H(e)\big)$, we have

$$Z_{22} \leqslant c \max(Z', Z''),$$

where

$$Z' = \sup_{x \in X'} \exp(-x_d)\, t^{\alpha-1} l(t)\, p\big(C_e D_s^{-1} x + h(\varepsilon')\varepsilon'\big),$$

$$Z'' = \sup_{x \in X''} \exp(-x_d)\, t^{\alpha-1} l(t)\, p\big(C_e D_s^{-1} x + h(\varepsilon')\varepsilon'\big).$$

By (24) we obtain

$$Z' \leqslant c \sup_{x \in X'} \frac{\exp(-x_d)\, t^{\alpha-1} l(t) \big(h(e_y) - |y|\big)^{\alpha-1}}{l\Big(\big(h(e_y) - |y|\big)^{-1}\Big)} \leqslant c \sup_{x_d \geqslant N} \exp(-x_d)\, x_d^{\alpha-1} = \omega(N),$$

while

$$Z'' \leqslant c \exp(-cM^2) \sup_{x \in X''} \frac{t^{\alpha-1} l(t) \big(h(e_y) - |y|\big)^{\alpha-1}}{l\Big(\big(h(e_y) - |y|\big)^{-1}\Big)} \leqslant c \exp(-cM^2) M^{\alpha-1} = \omega(M).$$

This completes the proof of the theorem. ∎

Proof of Lemma 1. We start with establishing an auxiliary result.

LEMMA 2. *If* (A)–(C) *hold, then for* $s = te$

$$\sup_{e \in S^{d-1}} \|D_s C_e^T B(s) C_e D_s^T - \Sigma_e\| = o(1), \quad t \to \infty,$$

where $\|\cdot\|$ *stands for any matrix norm.*

Proof. In view of Theorem 2 the random vector $\bar{\pi}_s(\xi_s) = D_s C_e^T (\xi_s - h(\varepsilon')\varepsilon')$ having the covariance matrix $D_s C_e^T B(s) C_e D_s^T$ converges in distribution to a limit vector with density $\pi_{d,\alpha}(x; \langle \varepsilon', e\rangle, H(e))$, the covariance matrix of which is Σ_e. Thus, for proving the lemma it is enough to establish that

$$\sup_{s \in R^d} E|\bar{\pi}_s(\xi_s)|^2 < \infty.$$

It is easy to see that

$$E|\bar{\pi}_s(\xi_s)|^2 = (f(s))^{-1} \int_X |D_s C_e^T x + th(\varepsilon') e^{(d)}|^2 e^{\langle s,x\rangle} p(x)\, dx$$

$$= (f(s))^{-1} \int_{C_e^T X} \left(t|\bar{x}|^2 + t^2 \left(h(\varepsilon') - \langle x, \tilde{e}\rangle / \langle \varepsilon', e\rangle\right)^2\right) \exp(\langle s, C_e x\rangle) p(C_e x)\, dx$$

$$= (f(s))^{-1} \bar{f}(s),$$

where $\tilde{e} = C_e^T e$ and $\bar{x} = (x_1, \ldots, x_{d-1})^T$, as before.

Let X_i, $i = 1, 2, 3, 4$, and X_{1k}, $k = 1, 2$, be the same sets as in the proof of Theorem 1. Denote by $\bar{f}_i(s)$, $\bar{f}_{1k}(s)$ the parts of the integral $\bar{f}(s)$ corresponding to those sets. One can show that the ratios $\bar{f}_i(s)/f(s)$, $i = 2, 3, 4$, and $\bar{f}_{12}(s)/f(s)$ can be made arbitrarily small while $\bar{f}_{11}(s)/f(s) < \infty$ when $s = te$, $t \to \infty$. Note that the integrals $\bar{f}_i(s)$, $\bar{f}_{1k}(s)$ are estimated in the same way as $f_i(s)$, $f_{1k}(s)$ in the proof of Theorem 1. Let us estimate, for example, $\bar{f}_2(s)$. For $x = r\varepsilon$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)^T$, we have

$$t|\bar{x}|^2 + t^2 \left(h(\varepsilon') - \langle x, \tilde{e}\rangle / \langle \varepsilon', e\rangle\right)^2 = tr^2 |\bar{\varepsilon}|^2 + t^2 \left(h(\varepsilon') - r\langle \varepsilon, \tilde{e}\rangle / \langle \varepsilon', e\rangle\right)^2.$$

Thus (cf. (16))

$$\bar{f}_2(s) \leqslant \frac{ce^{ta(e)}}{l(t)t^\alpha} \int_{t^{1/2}|\bar{\varepsilon}| \geqslant 2M} (t^{1/2}|\bar{\varepsilon}|)^2 \exp(-ct|\bar{\varepsilon}|^2)\, d\bar{\varepsilon}$$

$$+ ce^{ta(e)} \int_{t^{1/2}|\bar{\varepsilon}| \geqslant 2M} \exp(-ct|\bar{\varepsilon}|^2) \int_0^\delta t^2 u^2 \exp(-tu\langle e, C_e \varepsilon\rangle)$$

$$\times p\left(\left(h(C_e \varepsilon) - u\right) C_e \varepsilon\right) du\, d\bar{\varepsilon}$$

$$= \frac{e^{ta(e)}}{r_\beta(t)} \omega(M).$$

From Theorem 1 it follows that

$$\bar{f}_2(s)/f(s) \leqslant \omega(M).$$

Similarly one can estimate all other ratios. ■

Now we come back to the proof of Lemma 1. Define $R(s) = \Sigma_e^{-1/2} D_s C_e^T$. From Lemma 2 it follows that for $|s| \to \infty$

$$R(s) B(s) R^T(s) = \begin{bmatrix} 1+o(1) & o(1) & \dots & o(1) \\ o(1) & 1+o(1) & \dots & o(1) \\ \dots & \dots & \dots & \dots \\ o(1) & o(1) & \dots & 1+o(1) \end{bmatrix}$$

and

$$\lim_{|s| \to \infty} (\det B(s))^{1/2} \det R(s) = 1. \tag{26}$$

Let $B^{1/2}(s)$ be any matrix with the property

$$B^{1/2}(s) (B^{1/2}(s))^T = B(s) \tag{27}$$

and let $\mathscr{C}$ be the set of all orthogonal $d \times d$-matrices while $\mathscr{C}_0 \subset \mathscr{C}$ consists of those matrices whose last column is $e^{(d)}$. Note that the density $\pi_{d,\alpha}(x;\, \beta^{1/2},\, I)$ is invariant with respect to the transformations from $\mathscr{C}_0$. Clearly,

$$\lim_{|s| \to \infty} \min_{C \in \mathscr{C}} \|R(s) B^{1/2}(s) - C\| = 0.$$

We have a degree of freedom in choosing $B^{1/2}(s)$ since for any $C \in \mathscr{C}$ the matrix $B^{1/2}(s) C$ also satisfies (27). Therefore, there exists $B^{1/2}(s)$ such that

$$\lim_{|s| \to \infty} \min_{C \in \mathscr{C}_0} \|R(s) B^{1/2}(s) - C\| = 0. \tag{28}$$

Finally, by (26), (28) and Theorem 2 we obtain

$$\begin{aligned} &\lim_{|s| \to \infty} \sup_{x \in A_\delta(\beta^{1/2}, I)} |\bar{p}^{(s)}(x) - \pi_{d,\alpha}(x;\, \beta^{1/2},\, I)| \\ &\leqslant \lim_{|s| \to \infty} \sup_{x \in A_\delta(\beta^{1/2}, I)} \Big|(\det B(s))^{1/2} \det R(s) (\det R(s))^{-1} p^{(s)}\big(R^{-1}(s) R(s) B^{1/2}(s) x \\ &\qquad\qquad + h(\varepsilon')\varepsilon'\big) \\ &\quad - (\det B(s))^{1/2} \det R(s)\, \pi_{d,\alpha}\big(R(s) B^{1/2}(s) x;\, \beta^{1/2},\, I\big)\Big| \\ &\quad + \lim_{|s| \to \infty} \sup_{x \in A_\delta(\beta^{1/2}, I)} \Big|(\det B(s))^{1/2} \det R(s)\, \pi_{d,\alpha}\big(R(s) B^{1/2}(s) x;\, \beta^{1/2},\, I\big) \\ &\qquad\qquad - \pi_{d,\alpha}(x;\, \beta^{1/2},\, I)\Big| = 0. \end{aligned}$$ ■

Proof of Theorem 3. Consider the characteristic function

$$\psi_s(u) = \int_{R^d} e^{i\langle u,x\rangle} \bar{p}^{(s)}(x)\,dx, \qquad u \in R^d,$$

corresponding to the density $\bar{p}^{(s)}(x)$ from Lemma 1. From this lemma it follows that:

(i) for any $C > 0$

(29) $$\sup_{s\in R^d} \sup_{|u|<C} |\psi_s^n(u/\sqrt{n}) - \exp(-|u|^2/2)| = o(1), \qquad n \to \infty;$$

(ii) for any sufficiently small $\delta > 0$ there exists $c_\delta > 0$ such that

(30) $$\sup_{s\in R^d} |\psi_s(u)| \leqslant 1 - c_\delta |u|^2, \qquad 0 < |u| \leqslant \delta;$$

(iii) for any $\delta > 0$

(31) $$\sup_{s\in R^d} \sup_{|u|>\delta} |\psi_s(u)| = \varrho_\delta < 1.$$

It remains to show that there exists $n_0 > 1$ such that $|\psi_s(u)|^{n_0}$ is integrable uniformly in $s \in R^d$. Let $s = te$. Since

$$\bar{p}^{(s)}(x) = \big(\det B(s)\big)^{1/2} \frac{\exp\big(ta(e) + t\langle e, B^{1/2}(s)x\rangle\big) p\big(B^{1/2}(s)x + h(\varepsilon')\varepsilon'\big)}{f(te)},$$

from (A) it follows that for any $C > 0$

$$\sup_{e\in S^{d-1}} \sup_{t<C} \sup_{x\in R^d} \bar{p}^{(s)}(x) < \infty.$$

Hence, for any $C > 0$

$$\sup_{e\in S^{d-1}} \sup_{t<C} \int_{R^d} |\psi_s(u)|^2\,du < \infty.$$

Let us set

$$I_q(s) = \int_X \big(e^{\langle s,x\rangle} p(x)\big)^q\,dx, \qquad q > 1.$$

Arguing as in the proof of Theorem 1, one can show that

(32) $$\sup_{e\in S^{d-1}} \left|t^{q(\alpha-1)+(d+1)/2} \big(l(t)\big)^q e^{-ta(e)q} I_q(te) - g_{\alpha,q}(e)\right| = o(1), \qquad t \to \infty,$$

where

$$g_{\alpha,q}(e) = \frac{(2\pi)^{(d-1)/2} \big(h(\varepsilon')\big)^{d-1} \big(\lambda(\varepsilon')\big)^q \Gamma\big(q(\alpha-1)+1\big)}{q^{q(\alpha-1)+(d+1)/2} \langle \varepsilon, e\rangle^{q(\alpha-1)+1} (\det \bar{\Lambda}_e^{(0)})^{1/2}}.$$

Furthermore, for $q \in (1, 2)$ we have

$$\int_{R^d} (\bar{p}^{(s)}(x))^q \, dx = (\det B(s))^{(q-1)/2} (f(s))^{-q} I_q(s).$$

From Theorem 1, (26) and (32) it follows that

$$\lim_{|s| \to \infty} \int_{R^d} (\bar{p}^{(s)}(x))^q \, dx < \infty.$$

Therefore (see, e.g., [3], §19),

$$\lim_{|s| \to \infty} \int_{R^d} |\psi_s(u)|^{q/(q-1)} \, du < \infty.$$

Thus for $n_0 \geqslant q/(q-1)$ we obtain

$$\sup_{s \in R^d} \int_{R^d} |\psi_s(u)|^{n_0} \, du < \infty. \tag{33}$$

The assertion of the theorem follows from (29)–(31) and (33) (see, e.g., [10], Section 4, § 3). ∎

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