PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 24, Fasc. 1 (2004), pp. 47–66

A NOTE ON INVARIANT SETS

BY

RENÉ L. SCHILLING (BRIGHTON)

Abstract. A measurable set A is *invariant* with respect to a not necessarily symmetric sub-Markovian operator T on $L^p(X, m)$ if $T\mathbf{1}_A \leqslant \mathbf{1}_A$, and *strongly invariant* if $T\mathbf{1}_A = \mathbf{1}_A$. We show that these definitions accommodate many of the usual definitions of invariance, e.g., those used in Dirichlet form theory, ergodic theory or for stochastic processes. In finite measure spaces or if T^* is sub-Markovian and recurrent, the notions of invariance and strong invariance coincide. We also show that for certain analytic semigroups of sub-Markovian operators, (strongly) invariant sets are already determined by a single operator, T_1.

Mathematics Subject Classification: 31C15, 47D07, 60J35, 60J45.

Key words and phrases: Invariant set, sub-Markov operator, recurrence, transience, fractional power, analytic semigroup.

The notions of *invariant sets* and *invariant functions* appear in several places, for example in ergodic theory and dynamical systems, in the theory of stochastic processes, in connection with Dirichlet forms and also in the context of positivity-preserving semigroups. Each of these theories uses its own language, approach and definition of *invariance* and, at first glance, they appear to be quite different: In ergodic theory, a measurable set A is *invariant* if for a group of invertible measurable transformations $\{\theta_t\}_{t\in \mathbf{R}}$ we have, modulo null sets,

$$\theta_t^{-1}(A) = A \qquad \forall t \in \mathbf{R}, \tag{1}$$

see, e.g., Jacobs [10] or Da Prato and Zabczyk [1] for a precise definition. For someone working in the field of Markov processes, a set A is *invariant* if almost surely

$$T_t \mathbf{1}_A = \mathbf{1}_A \qquad \forall t > 0, \tag{2}$$

where $T_t u(x) = E^x\big(u(X_t)\big)$ is the Markovian transition semigroup associated with the process $\{X_t\}_{t\geqslant 0}$; see Revuz and Yor [14], pp. 404–405, where one can also find a thorough discussion of how (1) and (2) relate if θ_t, $t > 0$, is the canonical shift on path space. In the theory of symmetric Dirichlet forms, a set A is called *invariant* if almost surely

$$(3)\qquad T_t(\mathbb{1}_A u) = \mathbb{1}_A T_t u \quad \forall u \in L^2,\ \forall t > 0,$$

holds (cf. Fukushima et al. [5], p. 46, or Oshima [12], p. 23), while in connection with a semigroup $\{T_t\}_{t \geqslant 0}$ of (not necessarily symmetric) positivity-preserving operators on an L^p-space, the set A is *invariant* if (modulo null sets)

$$(4)\qquad \{x\colon u(x) \neq 0\} \subset A \Rightarrow \{x\colon T_t u(x) \neq 0\} \subset A \quad \forall u \in L^p,\ \forall t > 0,$$

see Davies [2], p. 174.

The aim of this note is to show that the above definitions (2)–(4) are essentially the same if we see them in a greater context. Since (4) does not require symmetry of the semigroup $\{T_t\}_{t \geqslant 0}$, our approach is useful for non-symmetric Dirichlet spaces where, to our knowledge, invariant sets have not been considered in detail. We will also show that we can restrict ourselves to a time-discrete setting: at least for some analytic semigroups $\{T_t\}_{t \geqslant 0}$, the notion of invariance can be reduced to invariance with respect to one single operator, say, T_1.

PRELIMINARIES, DEFINITIONS AND FIRST PROPERTIES

Throughout this paper X will be a Hausdorff space with some σ-algebra $\mathscr{A}$ and a σ-finite Radon measure m such that $\operatorname{supp} m = X$. By $L^p := L^p(X, m)$, $1 \leqslant p \leqslant \infty$, we denote the spaces of (equivalence classes of) integrable functions, by $C = C(X)$ $[C_b = C_b(X)]$ the space of [bounded] continuous functions. We will understand (in)equalities between L^p-functions always modulo null sets. A "+" as sub- or superscript of a function space denotes its (almost everywhere) positive elements. We write $a \wedge b$ or $a \vee b$ for the minimum or maximum of $a, b \in \boldsymbol{R}$, and $a^+ = a \vee 0$.

A linear operator $T\colon L^p \to L^p$, $1 \leqslant p < \infty$, is called *positivity preserving* if $Tu \geqslant 0$ for all $u \geqslant 0$, and *sub-Markovian* if it is a contraction and if we have $0 \leqslant Tu \leqslant 1$ whenever $0 \leqslant u \leqslant 1$.

Since a sub-Markovian operator $T\colon L^p \to L^p$ preserves positivity, it is bounded in L^p, cf. [9] for a short proof of this fact; moreover, T can be extended onto L^∞ and, by a simple interpolation argument, onto all spaces L^r, $r \in (p, \infty)$. Since we will use the extension argument quite frequently, let us briefly sketch its proof: for $u \in L^\infty_+$ we can find an increasing sequence $u_n \in L^p_+$ with $u = \sup_{n \in N} u_n$. Then

$$(5)\qquad Tu := \sup_{n \in N} Tu_n$$

is well-defined, since for any other sequence $v_m \in L^p_+$ with $u = \sup_{m \in N} v_m$ we find

$$v_m \wedge u_n \uparrow u_n \text{ as } m \to \infty \quad \text{as well as} \quad T(v_m \wedge u_n) \uparrow Tu_n \text{ as } m \to \infty.$$

By monotone convergence, this limit is also valid in L^p. Hence,

$$Tu_n = \sup_{m\in N} T(v_m \wedge u_n) \leqslant \sup_{m\in N} Tv_m$$

and also

$$\sup_{n\in N} Tu_n \leqslant \sup_{m\in N} Tv_m.$$

Interchanging the roles of u_n and v_m shows equality, and the extension to not necessarily positive u follows by linearity. Notice that for all $u\in L^\infty$ the sub-Markov property implies

$$\left| T\frac{u}{\|u\|_\infty} \right| \leqslant 1,$$

i.e., T is an L^∞-contraction. Hence, we have proved

LEMMA 1. *Every sub-Markovian operator $T\colon L^p \to L^p$ extends for every $p \leqslant r \leqslant \infty$ to a sub-Markovian operator $T\colon L^r \to L^r$ which is Daniell-continuous (in the sense that* (5) *holds in L^r-sense) and a contraction on L^∞.*

Note that a positivity-preserving operator T can be defined for every $u\in \bigcup_{p\geqslant 1} L^p_+$ if we allow $Tu\in [0, \infty]$.

The adjoint $T^*\colon L^q \to L^q$, $q = p/(p-1)$, of a sub-Markovian operator T is clearly positivity preserving, but it does, in general, not inherit the sub-Markov property. For completeness we include the proof of the following folklore result.

LEMMA 2. *Let $T\colon L^p \to L^p$ be a linear operator for some $1 < p < \infty$. Then the following assertions are equivalent*:

(a) *The adjoint $T^*\colon L^q \to L^q$, $q = p/(p-1)$, is sub-Markovian.*

(b) *T is positivity-preserving and extends to a contraction $T\colon L^1 \to L^1$.*

In particular, T and T^* are sub-Markovian if and only if both operators extend to positivity-preserving contraction operators on all spaces L^r, $1 \leqslant r \leqslant \infty$.

Proof. We will prove the lemma in four steps. Throughout the proof $\{A_n\}_{n\in N}$ denotes a sequence of sets $A_n \in \mathscr{A}$ such that $A_n \uparrow X$ and $m(A_n) < \infty$.

Step 1. Assume that $T\colon L^p \to L^p$ is positivity preserving. For $v\in L^q_+$ we set $B_n := A_n \cap \{T^* v < 0\}$. Clearly, $\mathbf{1}_{B_n} \in L^p_+$ and we find

$$\int\limits_{A_n\cap\{T^*v<0\}} T^* v\, dm = \int\limits_X T^* v\cdot \mathbf{1}_{B_n}\, dm = \int\limits_X v\cdot T\mathbf{1}_{B_n}\, dm \geqslant 0.$$

This is only possible if $m(A_n \cap \{T^* v < 0\}) = 0$ for all $n\in N$, so T^* is positivity preserving. A similar argument shows that T is positivity preserving if and only if T^* is.

Step 2. Assume that statement (b) holds. We choose $v\in L^q$ such that $0 \leqslant v \leqslant 1$ m-a.e. and set $C_n := A_n \cap \{T^* v > 1\}$. Then

$$\begin{aligned}\int\limits_{A_n\cap\{T^*v>1\}} T^* v\, dm &= \int\limits_X T^* v\cdot \mathbf{1}_{C_n}\, dm = \int\limits_X v\cdot T\mathbf{1}_{C_n}\, dm \\ &\leqslant \|v\|_\infty \|T\mathbf{1}_{C_n}\|_{L^1} \leqslant \|\mathbf{1}_{C_n}\|_{L^1} = \mu(A_n \cap \{T^* v > 1\}).\end{aligned}$$

This is only possible if $m(\{T^* v > 1\}) = 0$, which means, in view of step 1, that T^* is sub-Markovian. This shows (b) $\Rightarrow$ (a).

Step 3. Assume now that (a) holds. By step 1, T is positivity preserving. Therefore we find for $u \in L^p \cap L^1$

$$\int_X |Tu| \cdot \mathbf{1}_{A_n} dm \leqslant \int_X T|u| \cdot \mathbf{1}_{A_n} dm = \int_X |u| \cdot T^* \mathbf{1}_{A_n} dm \leqslant \int_X |u| \, dm.$$

By monotone convergence we can go to the limit $n \to \infty$ and get $\|Tu\|_{L^1} \leqslant \|u\|_{L^1}$ for all $u \in L^p \cap L^1$. Therefore, T extends to an L^1-contraction, i.e., (b) follows from (a).

Step 4. If both T and T^* are sub-Markovian, the above argument together with Lemma 1 show that both operators are L^1- as well as L^∞-contractions. A standard interpolation argument now shows that T and T^* extend to contractions on each of the spaces L^r, $1 \leqslant r \leqslant \infty$. The converse is obvious. ∎

Our definition of invariance is based on (4), while (3) is a stricter condition which we will call *strong invariance.*

Definition 3. Let $T\colon L^p \to L^p$ be a positivity-preserving operator. A set $A \in \mathscr{A}$ is called *(T-)invariant* if

$$\mathbf{1}_A T(\mathbf{1}_A u) = T(\mathbf{1}_A u) \qquad \forall u \in L^p. \tag{6}$$

The set A is *strongly (T-)invariant* if

$$\mathbf{1}_A Tu = T(\mathbf{1}_A u) \qquad \forall u \in L^p. \tag{7}$$

We write $\mathfrak{i}(T)$ and $\mathfrak{si}(T)$ for the families of invariant and strongly invariant sets.

The sets $\mathfrak{i}(T)$ and $\mathfrak{si}(T)$ do not depend on $p \geqslant 1$. This follows easily from the next lemma which is itself a simple consequence of the boundedness of positivity-preserving linear maps (see [9] for a short proof) and an elementary approximation argument.

Lemma 4. *In (6) and (7), the set L^p can be replaced by any norm-dense subset D of L^p_+, e.g., $L^1_+ \cap L^\infty_+$.*

Let us collect some elementary properties of T- and T^*-invariant sets.

Properties 5. *Let $T\colon L^p \to L^p$ be a positivity-preserving operator with adjoint T^*.*

(a) $\mathfrak{si}(T) \subset \mathfrak{i}(T)$.
(b) $N, N^c \in \mathfrak{si}(T)$ *for all m-null sets N.*
(c) $\mathfrak{i}(T)$ *is stable under countable unions and intersections.*
(d) $A \in \mathfrak{i}(T)$ *if and only if* $A^c \in \mathfrak{i}(T^*)$.
(e) $\mathfrak{si}(T) = \mathfrak{i}(T) \cap \mathfrak{i}(T^*) = \mathfrak{i}(T) \cap \mathfrak{i}(T)^c = \mathfrak{si}(T^*)$, *where* $\mathfrak{i}(T)^c := \{A^c\colon A \in \mathfrak{i}(T)\}$.
(f) $\mathfrak{si}(T)$ *is a σ-algebra.*

(g) *The assertions* $\mathfrak{i}(T) = \mathfrak{si}(T)$, $\mathfrak{i}(T) = \mathfrak{i}(T^*)$ *and* $\mathfrak{i}(T^*) = \mathfrak{si}(T^*)$ *are equivalent.*

In particular, for symmetric operators the notions of invariance and strong invariance coincide.

Proof. Throughout the proof u, v denote arbitrary elements of $L^1_+ \cap L^\infty_+$.

(a) This is a direct consequence of the definition of (strong) invariance.

(b) For a null set N we have $\|T(\mathbf{1}_N u)\|_{L^p} \leqslant c\,\|\mathbf{1}_N u\|_{L^p} = 0$, which shows that $T(\mathbf{1}_N u) = 0 = \mathbf{1}_N Tu$ almost everywhere; hence $N \in \mathfrak{si}(T)$. The proof that $N^c \in \mathfrak{si}(T)$ is similar.

(c) Let $A_n, B_n \in \mathfrak{i}(T)$, where $n \in N$ and $A := \bigcup_{k\in N} A_k$, $B := \bigcap_{k\in N} B_k$. By the Daniell continuity of T (Lemma 1) we get

$$T(\mathbf{1}_A u) \leqslant T\big(\sum_{k\in N} \mathbf{1}_{A_k} u\big) = \sum_{k\in N} T(\mathbf{1}_{A_k} u) = \sum_{k\in N} \mathbf{1}_{A_k} T(\mathbf{1}_{A_k} u) \leqslant \big(\sum_{k\in N} \mathbf{1}_{A_k}\big) T(\mathbf{1}_A u).$$

Passing on both sides to the minimum with $T(\mathbf{1}_A u)$ yields

$$T(\mathbf{1}_A u) \leqslant \min\big\{\big(\sum_{k\in N} \mathbf{1}_{A_k}\big) T(\mathbf{1}_A u),\ T(\mathbf{1}_A u)\big\}$$

$$= \min\big\{\sum_{k\in N} \mathbf{1}_{A_k},\ 1\big\}\, T(\mathbf{1}_A u) = \mathbf{1}_A T(\mathbf{1}_A u) \leqslant T(\mathbf{1}_A u)$$

and $A = \bigcup_{k\in N} A_k \in \mathfrak{i}(T)$ follows.

Without loss of generality we can assume that the sets B_k decrease to B. Since $T(\mathbf{1}_B u) \leqslant T(\mathbf{1}_{B_k} u) = \mathbf{1}_{B_k} T(\mathbf{1}_{B_k} u)$, we can use the Daniell property (Lemma 1) to conclude

$$T(\mathbf{1}_B u) \leqslant \inf_{k\in N} \big(\mathbf{1}_{B_k} T(\mathbf{1}_{B_k} u)\big) = \mathbf{1}_B T(\mathbf{1}_B u) \leqslant T(\mathbf{1}_B u).$$

(d) Let $B \in \mathfrak{i}(T)$. Then

$$B \in \mathfrak{i}(T) \Leftrightarrow T(\mathbf{1}_B u) = \mathbf{1}_B T(\mathbf{1}_B u) \Leftrightarrow \mathbf{1}_{B^c} T(\mathbf{1}_B u) = 0. \tag{8}$$

This implies for $A \in \mathfrak{i}(T)$ that $\big(v, \mathbf{1}_{A^c} T(\mathbf{1}_A u)\big)_{L^2} = \big(\mathbf{1}_A T^*(\mathbf{1}_{A^c} v), u\big)_{L^2} = 0$; therefore, $\mathbf{1}_A T^*(\mathbf{1}_{A^c} v) = 0$ and $A^c \in \mathfrak{i}(T^*)$ (use (8) for T^* and with $B = A^c$). The converse direction is similar.

(e) If $A \in \mathfrak{si}(T)$, then

$$T(\mathbf{1}_{A^c} u) = T\{(1 - \mathbf{1}_A) u\} = Tu - T(\mathbf{1}_A u) = Tu - \mathbf{1}_A Tu = \mathbf{1}_{A^c} Tu,$$

i.e., $A^c \in \mathfrak{si}(T)$, and because of (a) and (d) we find for $\mathfrak{i}(T)^c := \{A^c\colon\ A \in \mathfrak{i}(T)\}$ that

$$\mathfrak{si}(T) \subset \mathfrak{i}(T) \cap \mathfrak{i}(T)^c = \mathfrak{i}(T) \cap \mathfrak{i}(T^*).$$

If $A \in \mathfrak{i}(T) \cap \mathfrak{i}(T^*)$, we infer from (8) that $\mathbf{1}_A T(\mathbf{1}_{A^c} u) = \mathbf{1}_{A^c} T(\mathbf{1}_A u) = 0$, and so

$$T(\mathbf{1}_A u) = \mathbf{1}_A T(\mathbf{1}_A u) + \mathbf{1}_{A^c} T(\mathbf{1}_A u) = \mathbf{1}_A T(\mathbf{1}_A u) = \mathbf{1}_A T(\mathbf{1}_A u) + \mathbf{1}_A T(\mathbf{1}_{A^c} u) = \mathbf{1}_A Tu,$$

i.e., $A \in \mathfrak{si}(T)$ and $\mathfrak{i}(T) \cap \mathfrak{i}(T^*) \subset \mathfrak{si}(T)$. The symmetry of the above argument in T and T^* also proves $\mathfrak{si}(T) = \mathfrak{si}(T^*)$.

(f) That $X \in \mathfrak{si}(T)$ is clear, the rest follows immediately from (b)–(e).

(g) It is enough to show that $\mathfrak{i}(T) = \mathfrak{si}(T)$ and $\mathfrak{i}(T) = \mathfrak{i}(T^*)$ are equivalent. As we have seen in (e), $\mathfrak{si}(T) = \mathfrak{i}(T) \cap \mathfrak{i}(T^*)$, and "$\Leftarrow$" follows. Conversely, if $\mathfrak{i}(T) = \mathfrak{si}(T)$, part (d) shows $A \in \mathfrak{i}(T) \Leftrightarrow A^c \in \mathfrak{i}(T) \Leftrightarrow A \in \mathfrak{i}(T^*)$ and we are done. ∎

CHARACTERIZATION AND FURTHER PROPERTIES

From now on we will only consider sub-Markovian operators T. This allows us to switch between the spaces L^p and L^∞. If T^* is sub-Markovian, too, we may even work in the full scale L^r, $1 \leqslant r \leqslant \infty$; see Lemmas 1 and 2.

THEOREM 6. *For a sub-Markovian operator $T: L^p \to L^p$ the following assertions are equivalent*:

(a) $A \in \mathfrak{i}(T)$, *i.e.*, $T(\mathbf{1}_A u) = \mathbf{1}_A T(\mathbf{1}_A u)$ $\forall u \in L^1_+ \cap L^\infty_+$.

(b) $\mathbf{1}_A T\mathbf{1}_A = T\mathbf{1}_A$.

(c) $\mathbf{1}_A T\mathbf{1}_{A\cap K} = T\mathbf{1}_{A\cap K}$ $\forall K$ *with* $m(K) < \infty$.

(d) $T\mathbf{1}_A \leqslant \mathbf{1}_A$.

(e) $T\mathbf{1}_A \leqslant \mathbf{1}_A T1$.

Proof. (e)$\Rightarrow$(d). This is clear since $T1 \leqslant 1$.

(d)$\Rightarrow$(c). Note that $T\mathbf{1}_A \leqslant \mathbf{1}_A$ implies $\mathbf{1}_{A^c} T\mathbf{1}_A = 0$, and so $\mathbf{1}_{A^c} T\mathbf{1}_{A\cap K} \leqslant \mathbf{1}_{A^c} T\mathbf{1}_A = 0$. Therefore, $T\mathbf{1}_{A\cap K} = \mathbf{1}_A T\mathbf{1}_{A\cap K}$, and (c) follows.

(c)$\Rightarrow$(b). Since m is σ-finite, we find a sequence of sets $K_n \in \mathscr{A}$ such that $m(K_n) < \infty$, $\sup_{n\in N} \mathbf{1}_{K_n} = \mathbf{1}_A$ and

$$\mathbf{1}_A T\mathbf{1}_A = \sup_{n\in N} \mathbf{1}_A T\mathbf{1}_{A\cap K_n} = \sup_{n\in N} T\mathbf{1}_{A\cap K_n} = T\mathbf{1}_A.$$

(b)$\Rightarrow$(a). Since $u \in L^1_+ \cap L^\infty_+$ is bounded, the sub-Markov property of T shows

$$0 \leqslant T(\mathbf{1}_A u) \leqslant T(\mathbf{1}_A \|u\|_\infty) = \|u\|_\infty T\mathbf{1}_A = \|u\|_\infty \mathbf{1}_A T\mathbf{1}_A,$$

which implies that $\mathbf{1}_{A^c} T(\mathbf{1}_A u) = 0$ and $T(\mathbf{1}_A u) = \mathbf{1}_A T(\mathbf{1}_A u)$.

(a)$\Rightarrow$(e). Choose a sequence $\chi_n \in L^1_+ \cap L^\infty_+$ with $\sup_{n\in N} \chi_n = 1$. Clearly, $T(\mathbf{1}_A \chi_n) = \mathbf{1}_A T(\mathbf{1}_A \chi_n) \leqslant \mathbf{1}_A T1$ and the Daniell continuity of T (cf. Lemma 1) implies that

$$T\mathbf{1}_A = T(\mathbf{1}_A \sup_{n\in N} \chi_n) = \sup_{n\in N} T(\mathbf{1}_A \chi_n) \leqslant \mathbf{1}_A T1. \quad ∎$$

A similar characterization is valid for strongly invariant sets $\mathfrak{si}(T)$.

THEOREM 7. *For a sub-Markovian operator $T: L^p \to L^p$ the following assertions are equivalent*:

(a) $A \in \mathfrak{si}(T)$, *i.e.*, $T(\mathbf{1}_A u) = \mathbf{1}_A Tu$ $\forall u \in L^1_+ \cap L^\infty_+$.

(b) $T\mathbf{1}_A = \mathbf{1}_A T1$.

If T is conservative, that is $T1 = 1$, then (b) *is equivalent to*

(b′) $T\mathbf{1}_A = \mathbf{1}_A$.

Proof. (b) ⇒ (a). Obviously, $T\mathbf{1}_A = \mathbf{1}_A T1$ gives $T\mathbf{1}_A = \mathbf{1}_A T1 \leqslant \mathbf{1}_A$. Assume that $L_+^1 \cap L_+^\infty \ni u \leqslant 1$, and so $L_+^\infty \ni 1-u \leqslant 1$. Then

$$T(\mathbf{1}_A u) \leqslant \min(T\mathbf{1}_A, Tu) \leqslant \min(\mathbf{1}_A, Tu) \leqslant \mathbf{1}_A Tu, \tag{9}$$

$$T\big(\mathbf{1}_A(1-u)\big) \leqslant \min\big(T\mathbf{1}_A, T(1-u)\big) \leqslant \min\big(\mathbf{1}_A, T(1-u)\big) \leqslant \mathbf{1}_A T(1-u), \tag{10}$$

and addition of (9) and (10) shows

$$T\mathbf{1}_A = T(\mathbf{1}_A u) + T\big(\mathbf{1}_A(1-u)\big) \leqslant \mathbf{1}_A Tu + \mathbf{1}_A T(1-u) = \mathbf{1}_A T1. \tag{11}$$

By assumption, $T\mathbf{1}_A = \mathbf{1}_A T1$, which shows that we have equality throughout (11); this is only possible if both (9) and (10) are equalities, and so $T(\mathbf{1}_A u) = \mathbf{1}_A Tu$ for all $u \in L_+^1 \cap L_+^\infty$ with $u \leqslant 1$. The general case follows from a simple scaling argument.

(a) ⇒ (b). Choose a sequence $\chi_n \in L_+^1 \cap L_+^\infty$ with $\sup_{n\in N} \chi_n = 1$. By assumption, $T(\mathbf{1}_A \chi_n) = \mathbf{1}_A T\chi_n$, and the Daniell property of T (see Lemma 1) gives

$$T\mathbf{1}_A = \sup_n T(\mathbf{1}_A \chi_n) = \sup_n \mathbf{1}_A T\chi_n = \mathbf{1}_A T1.$$

The equivalence of (b) and (b′) is obvious. ∎

COROLLARY 8. *Let $T: L^p \to L^p$ be a sub-Markovian operator. We have $Tu \leqslant uT1$ for all $\sigma\big(\mathfrak{i}(T)\big)$-measurable functions u. If u is $\mathfrak{si}(T)$-measurable, we have even $Tu = uT1$.*

Proof. Define $\mathfrak{I} := \{u \in L^p \colon Tu \leqslant uT1\}$. As we have seen in Theorem 6, $\mathbf{1}_A \in \mathfrak{I}$ for all $A \in \mathfrak{i}(T)$. If $(u_n)_{n\in N} \subset \mathfrak{I}^+$ is an increasing sequence such that $\sup_{n\in N} u_n = u$, we see

$$Tu = T(\sup_{n\in N} u_n) = \sup_{n\in N} Tu_n \leqslant \sup_{n\in N} u_n T1 = uT1,$$

i.e., $u \in \mathfrak{I}^+$ and a typical monotone class argument (see, e.g., Ethier and Kurtz [4], Appendix 4, Theorem 4.3, p. 497) shows that *all* $\sigma\big(\mathfrak{i}(T)\big)$-measurable functions are in $\mathfrak{I}$. The same argument applies to $\mathfrak{si}(T)$-measurable functions. ∎

FINITE MEASURE SPACES

If $(X, \mathscr{A}, m)$ is a finite measure space, the notions of invariance and strong invariance coincide. We start with a simple, but useful, observation.

LEMMA 9. *Assume that both T and T^* are sub-Markovian operators. Moreover, let T^* be conservative, i.e., $T^*1 = 1$. Then the following assertions are equivalent for all $u \in L_+^1$:*

(a) $Tu = u$.

(b) $Tu \leqslant u$.

Proof. The direction "(a) $\Rightarrow$ (b)" is trivial. To see "(b) $\Rightarrow$ (a)" we note that for all $N \in \boldsymbol{N}$

$$0 \leqslant N(1, u-Tu)_{L^2} = \Big(\sum_{k=0}^{N} (T^*)^k 1, u-Tu\Big)_{L^2} = (1, u-T^{N+1}u)_{L^2} \leqslant (1, u)_{L^2} = \|u\|_{L^1},$$

which is only possible if $u = Tu$ almost everywhere. ∎

THEOREM 10. *Let $(X, \mathscr{A}, m)$ be a finite measure space, and assume that both T and T^* are sub-Markovian operators and T^* is conservative. Then T is also conservative and $\mathfrak{i}(T) = \mathfrak{i}(T^*) = \mathfrak{si}(T) = \mathfrak{si}(T^*)$, i.e., the notions of invariance and strong invariance coincide.*

In particular, if both T and T^* are sub-Markovian, either both of them or none of them is conservative.

Proof. Since m is finite, we have $1 \in L^1_+$ and Lemma 9 is valid for $u = 1$. Thus, $T1 \leqslant 1$ implies $T1 = 1$, which means that T is conservative. Using the characterization of (strong) invariance obtained in Theorem 6 (d), resp. Theorem 7 (b′), we can apply Lemma 9 again to get $\mathfrak{i}(T) \subset \mathfrak{si}(T)$, which gives $\mathfrak{i}(T) = \mathfrak{si}(T)$. The assertion now follows from Property 5 (g). ∎

POTENTIALS AND RECURRENCE

The proof of Theorem 10 uses two key ingredients: $1 \in L^1$ and $\sum_{n=1}^{\infty} T^* 1 = \infty$. We will now see how we can generalize Theorem 10 to general σ-finite measure spaces $(X, \mathscr{A}, m)$.

DEFINITION 11. Let T be a positivity-preserving operator on some space L^p. For all $u \in L^1_+ \cap L^\infty_+$

$$G_n u := \sum_{k=0}^{n} T^k u \quad \text{and} \quad Gu := \sup_{n \in \boldsymbol{N}} G_n u$$

are well-defined functions with values in $[0, \infty]$. The operator G is called *potential operator* associated with T.

The next theorem is a standard result from (discrete) ergodic theory. For its proof we refer to Revuz [13], Theorem 4.2.3, p. 124.

THEOREM 12 (Hopf decomposition). *Let T be a positivity-preserving operator. For every $u \in L^1_+ \cap L^\infty_+$ the sets*

$$C = \{x\colon\ Gu(x) = 0\} \cup \{x\colon\ Gu(x) = \infty\} \quad \textit{and} \quad D = \{x\colon\ 0 < Gu(x) < \infty\}$$

are unique (modulo null sets) and independent of u if $u > 0$.

In view of Theorem 12 the following definition makes sense.

DEFINITION 13. A positivity-preserving operator T is called *recurrent* if $D = \varnothing$ (modulo null sets) and *transient* if $C = \varnothing$ (modulo null sets).

Remark 14. (A) *T is recurrent if and only if $Gf = \infty$ for some (or all) $f \in L^1_+$, $f > 0$.*

This is an immediate consequence of Hopf's decomposition theorem since $D \cap C = \emptyset$ and $Gf(x) \geqslant f(x) > 0 \Rightarrow Gf(x) = \infty$. Thus, $\{Gf = 0\} = \emptyset$ whenever $f > 0$.

(B) *T is transient if and only if there is some $f \in L^1_+ \cap L^\infty_+$, satisfying $f > 0$ and $Gf < \infty$.*

THEOREM 15. *A positivity-preserving operator T is transient if and only if there exists a sequence $\{A_n\}_{n\in N} \subset \mathscr{A}$ such that $\mathbf{1}_{A_n} \in L^1$, $\sup_{n\in N} \mathbf{1}_{A_n} = 1$ and $G\mathbf{1}_{A_n} < \infty$.*

Proof. If T is transient, we can find some $f \in L^1_+ \cap L^\infty_+$, $f > 0$, such that $Gf < \infty$. Then $A_n := \{x\colon f > 1/n\}$ is an increasing sequence of measurable sets such that for all $n \in N$

$$m(A_n) = m\{f > 1/n\} \leqslant n \int f\,dm < \infty \quad \text{and} \quad G\mathbf{1}_{A_n} \leqslant G(nf) = nGf < \infty.$$

For the converse direction we have to find some $h \in L^1_+ \cap L^\infty_+$ with $h > 0$ and $Gh < \infty$. Set

$$g_n := G\mathbf{1}_{A_n}, \qquad g_{n,j} := g_n \wedge j, \qquad h_{n,j} := g_{n,j} - Tg_{n,j}.$$

Obviously, $\|h_{n,j}\|_\infty \leqslant 2j$ and

$$Gh_{n,j} = \lim_{m\to\infty} (G_m g_{n,j} - G_m Tg_{n,j}) = g_{n,j} - \lim_{m\to\infty} T^m g_{n,j} = g_{n,j}.$$

The last equality follows from $G\mathbf{1}_{A_n} < \infty$ and

$$0 \leqslant \lim_{m\to\infty} T^m g_{n,j} \leqslant \lim_{m\to\infty} T^m G\mathbf{1}_{A_n} = \lim_{m\to\infty} \sum_{k=m}^{\infty} T^k \mathbf{1}_{A_n} = 0.$$

Since A_n exhausts X, we find for each fixed $x \in X$ some A_n with $x \in A_n$. Pick $j > g_n(x)$ and observe that

$$\begin{aligned} h_{n,j}(x) &= g_{n,j}(x) - Tg_{n,j}(x) \geqslant g_{n,j}(x) - Tg_n(x) = g_n(x) - Tg_n(x) \\ &= G\mathbf{1}_{A_n}(x) - TG\mathbf{1}_{A_n}(x) = \mathbf{1}_{A_n}(x) = 1 > 0. \end{aligned}$$

Therefore,

$$h := \sum_{n,j=1}^{\infty} 2^{-n} 2^{-j} h_{n,j}, \qquad h > 0,\ h \in L^1_+ \cap L^\infty_+,$$

$$Gh \leqslant \sum_{n,j=1}^{\infty} 2^{-n} 2^{-j} Gh_{n,j} = \sum_{n,j=1}^{\infty} 2^{-n} 2^{-j} g_{n,j} \leqslant \sum_{n,j=1}^{\infty} 2^{-n} 2^{-j} j < \infty,$$

which even shows that $Gh \in L^\infty$. ∎

For sub-Markovian operators T the following result is, without proof, mentioned in Oshima [12], (1.5.8/9), p. 27.

COROLLARY 16. *A positivity-preserving operator is transient if and only if there is some* $h \in L^1_+ \cap L^\infty_+$ *with* $h > 0$ *and* $\|Gh\|_\infty < \infty$.

In particular, T *is transient if and only if* T^* *is transient.*

Proof. The first part of the corollary follows immediately from the proof of Theorem 15. For the second part, it is enough to show that transience of T implies transience of T^*. Pick h as in the first part. Then we find for the potential operator G^* associated with T^*

$$(h, G^* u)_{L^2} = (Gh, u)_{L^2} \leqslant \|Gh\|_\infty \|u\|_{L^1} < \infty \qquad \forall u \in L^1_+ \cap L^\infty_+.$$

Since $h > 0$, we conclude that $G^* u < \infty$ and that T^* is transient. ∎

Remark 17. (A) In ergodic theory, *recurrence* is usually called *conservativeness*, cf. Revuz [13]. We will not adopt this convention.

(B) Transience and recurrence are only dichotomous if $\mathfrak{i}(T)$ is trivial, i.e., contains only null sets and their complements. This property is often called *irreducibility* (in the theory of Dirichlet forms, see [5] and [12], for semigroups, see [2]).

In this case $\mathfrak{i}(T) = \mathfrak{si}(T) = \mathfrak{i}(T^*) = \mathfrak{si}(T^*)$ and T is recurrent if and only if T^* is recurrent. Note that Theorem 20 will show that T is automatically conservative in the sense $T1 = 1$; this means that our notion of conservativeness follows from conservativeness in the sense of ergodic theory, see (A), if we are irreducible. In finite measure spaces the two notions coincide, see Corollary 23. If both T and T^* are sub-Markovian and T recurrent, m is an invariant measure (see Appendix) and irreducibility is the same as *ergodicity*.

For completeness we mention the following lemma which can be proved with the arguments in Fukushima et al. [5], Lemma 1.6.2, pp. 47–48; see Jacob [8], Lemma 3.5.24, pp. 354–355 for a complete proof.

LEMMA 18. *Let* T *be a sub-Markovian operator. Then for every* $u \in L^1_+$

$$\{Gu < \infty\}, \{Gu = 0\} \in \mathfrak{i}(T^*) \quad \textit{and} \quad \{Gu = \infty\}, \{Gu > 0\} \in \mathfrak{i}(T).$$

To prove the analogue of Theorem 10 we need an auxiliary result.

LEMMA 19. *Assume that both* T *and* T^* *are sub-Markovian operators. Moreover, let* T^* *be recurrent. Then the following assertions are equivalent for all* $u \in L^\infty$:

(a) $Tu = u$.

(b) $Tu \leqslant u$.

The equivalence still holds for positive measurable, but possibly unbounded, functions.

Proof. The direction "(a) $\Rightarrow$ (b)" is obvious in any case. For "(b) $\Rightarrow$ (a)" we choose a function $f \in L^1_+ \cap L^\infty_+$ such that $\sum_{k=0}^{\infty} (T^*)^k f = \infty$. For all $N \in \boldsymbol{N}$ we find

$$0 \leqslant \Big(\sum_{k=0}^{N} (T^*)^k f,\, u - Tu\Big)_{L^2} = (f, u - T^{N+1} u)_{L^2} \leqslant \|f\|_{L^1} \|u\|_{L^\infty} + \|f\|_{L^1} \|T^{N+1} u\|_{L^\infty}$$
$$\leqslant 2\|f\|_{L^1} \|u\|_{L^\infty},$$

which is only possible if $u = Tu$.

For unbounded $u \geqslant 0$ we find

$$T(u \wedge k) \leqslant Tu \wedge Tk \leqslant u \wedge k \qquad \forall k \in \boldsymbol{N},$$

and the first part of the lemma applied to $u \wedge k \in L^\infty$ shows

$$u \wedge k = T(u \wedge k) \leqslant Tu \qquad \forall k \in \boldsymbol{N}.$$

Letting k tend to infinity, we see $u \leqslant Tu$, and the assertion follows. ∎

THEOREM 20. *Assume that both T and T^* are sub-Markovian operators and that T^* is recurrent. Then $\mathfrak{i}(T) = \mathfrak{si}(T) = \mathfrak{si}(T^*) = \mathfrak{i}(T^*)$. Moreover, T is conservative.*

Proof. The conservativeness of T follows directly from Lemma 19 with $u = 1$. Let $A \in \mathfrak{i}(T)$, which means that $T\mathbf{1}_A \leqslant \mathbf{1}_A$, cf. Theorem 6 (d). Since $\mathbf{1}_A \in L^\infty_+$, another application of Lemma 19 shows $T\mathbf{1}_A = \mathbf{1}_A$, and from Theorem 7 (b′) we conclude that $A \in \mathfrak{si}(T)$. All other assertions follow from the list of Properties 5. ∎

COROLLARY 21. *Assume that both T and T^* are sub-Markovian operators and that T^* is recurrent. For $u \in L^\infty \cup L^p$ such that $u^- \in L^\infty$ (or $u^+ \in L^\infty$) we have $Tu = u$ if and only if u is measurable with respect to $\mathfrak{i}(T)$ $(= \mathfrak{si}(T))$.*

Proof. Theorem 20 shows that T is conservative; therefore, the direction "$\Leftarrow$" follows already from Corollary 8.

Assume now that $u^- \in L^\infty$ (if $u^+ \in L^\infty$, consider $-u$ instead of u) and $Tu = u$. Since $T\alpha = \alpha$ for constants $\alpha \in \boldsymbol{R}$, we have

$$T(u \wedge \alpha) \leqslant Tu \wedge T\alpha = u \wedge \alpha$$

and, by Lemma 19, $T(u \wedge \alpha) = u \wedge \alpha$. Notice that $u + \alpha = u \vee \alpha + u \wedge \alpha$ and $|u| = u \vee 0 - u \wedge 0$; thus $T(u^\pm) = u^\pm$ as well as $T|u| = |u|$. Without loss of generality we can, therefore, always assume $u \geqslant 0$.

If $u \in L^\infty_+ \cup L^p_+$, we can use a variant of Jensen's inequality (see, e.g., Jacob [7], Theorem 4.6.24, pp. 376–377) to get for all $n \in \boldsymbol{N}$

$$(u \wedge 1)^n = \{T(u \wedge 1)\}^n \leqslant T\{(u \wedge 1)^n\}.$$

As $n \to \infty$, the Daniell property of T (Lemma 1) gives $\mathbf{1}_{\{u \geqslant 1\}} \leqslant T\mathbf{1}_{\{u \geqslant 1\}}$, and so

$$T\mathbf{1}_{\{u<1\}} = 1 - T\mathbf{1}_{\{u \geqslant 1\}} \leqslant 1 - \mathbf{1}_{\{u \geqslant 1\}} = \mathbf{1}_{\{u<1\}}.$$

This means that $\{u < 1\} \in \mathfrak{i}(T)$ and, by Theorem 20, also $\{u < 1\} \in \mathfrak{si}(T)$. Considering $\alpha^{-1} u$, $\alpha > 0$, instead of u shows $\{u < \alpha\} = \{\alpha^{-1} u < 1\} \in \mathfrak{i}(T)$. ■

Corollary 22. *Assume that T and T^* are sub-Markovian operators and that T^* is recurrent. Then*

$$\mathfrak{i}(T) = \mathfrak{si}(T) = \Big\{\{x\colon \sum_{k=0}^{\infty} (T^*)^k f(x) = \infty\}\colon f \in L^1_+\Big\} = \mathfrak{i}(T^*) = \mathfrak{si}(T^*).$$

Proof. That $\mathfrak{i}(T) = \mathfrak{si}(T) = \mathfrak{i}(T^*) = \mathfrak{si}(T^*)$ was already shown in Theorem 20.

For $A \in \mathfrak{si}(T^*)$ we have $(T^*)^k (\mathbf{1}_A u) = \mathbf{1}_A (T^*)^k u$ for some (hence, all) strictly positive $u \in L^1_+$. Thus,

$$\sum_{k=0}^{\infty} (T^*)^k (\mathbf{1}_A u) = \mathbf{1}_A \sum_{k=0}^{\infty} (T^*)^k u = \mathbf{1}_A \cdot \infty$$

(where we use the convention $0 \cdot \infty = 0$), and so

$$A = \Big\{\sum_{k=0}^{\infty} (T^*)^k (\mathbf{1}_A u) = \infty\Big\}, \qquad \text{where } \mathbf{1}_A u \in L^1_+.$$

Conversely, Lemma 18 shows that $\{\sum_{k=0}^{\infty} (T^*)^k f = \infty\} \in \mathfrak{i}(T^*)$ for any $f \in L^1_+$. ■

A combination of Theorem 20 and Theorem 10 finally yields

Corollary 23. *Let $(X, \mathscr{A}, m)$ be a finite measure space and assume that both T and T^* are sub-Markovian operators. Then the notions*

T conservative, T^* conservative, T recurrent, T^* recurrent

coincide.

SEMIGROUPS AND RESOLVENTS

From now on we will only consider strongly continuous one-parameter semigroups of sub-Markovian operators $\{T_t\}_{t \geq 0}$ on some space L^p, $1 \leq p < \infty$. Denote by $(\mathfrak{A}, D(\mathfrak{A}))$ the infinitesimal generator. The resolvent $\{R_\lambda\}_{\lambda > 0}$ is given by

$$R_\lambda u = \int_0^{\infty} e^{-\lambda t} T_t u \, dt, \qquad \lambda > 0.$$

Recall that λR_λ, $\lambda > 0$, are again sub-Markovian operators satisfying the *resolvent equation*

$$R_\lambda u - R_\mu u = (\mu - \lambda) R_\lambda R_\mu u = (\mu - \lambda) R_\mu R_\lambda u, \qquad \mu, \lambda > 0.$$

DEFINITION 24. A set $A \in \mathscr{A}$ is (*strongly*) *invariant* with respect to $\{T_t\}_{t \geq 0}$ if A is (strongly) invariant with respect to every T_t, $t > 0$. A set $A \in \mathscr{A}$ is (*strongly*) *invariant* with respect to $\{R_\lambda\}_{\lambda > 0}$ if A is (strongly) invariant with respect to every operator λR_λ, $\lambda > 0$. We write $\mathfrak{i}(T_t, t > 0)$, $\mathfrak{si}(T_t, t > 0)$, resp., $\mathfrak{i}(R_\lambda, \lambda > 0)$, $\mathfrak{si}(R_\lambda, \lambda > 0)$.

LEMMA 25. *We have*

$$\mathfrak{i}(T_t, t > 0) = \mathfrak{i}(R_\lambda, \lambda > 0) \quad \textit{and} \quad \mathfrak{si}(T_t, t > 0) = \mathfrak{si}(R_\lambda, \lambda > 0).$$

Proof. For $A \in \mathfrak{i}(T_t, t > 0)$ we have $T_t \mathbf{1}_A \leq \mathbf{1}_A$ (cf. Theorem 6), and so

$$\lambda R_\lambda \mathbf{1}_A = \lambda \int_0^\infty e^{-\lambda t} T_t \mathbf{1}_A \, dt \leq \lambda \int_0^\infty e^{-\lambda t} \mathbf{1}_A \, dt = \mathbf{1}_A,$$

which means that $A \in \mathfrak{i}(R_\lambda, \lambda > 0)$.

Suppose now that $A \in \mathfrak{i}(R_\lambda, \lambda > 0)$. The Yosida approximation of T_t is given by

$$T_{t,\lambda} u = \sum_{k=0}^\infty \frac{t^k}{k!} (\lambda \mathfrak{A} R_\lambda)^k u = \sum_{k=0}^\infty \frac{\lambda^k t^k}{k!} (\lambda R_\lambda - 1)^k u, \qquad u \in L^1_+ \cap L^\infty_+,$$

see, e.g., Davies [2], p. 49. Since A is invariant with respect to λR_λ,

$$T_{t,\lambda}(\mathbf{1}_A u) = \sum_{k=0}^\infty \frac{\lambda^k t^k}{k!} (\lambda R_\lambda - 1)^k (\mathbf{1}_A u) = \mathbf{1}_A \sum_{k=0}^\infty \frac{\lambda^k t^k}{k!} (\lambda R_\lambda - 1)^k (\mathbf{1}_A u) = \mathbf{1}_A T_{t,\lambda}(\mathbf{1}_A u),$$

and $A \in \mathfrak{i}(T_t)$ follows from the L^p-convergence $\lim_{\lambda \to \infty} T_{t,\lambda}(\mathbf{1}_A u) = T_t(\mathbf{1}_A u)$ of the Yosida approximation.

A similar argument shows that $\mathfrak{si}(T_t, t > 0) = \mathfrak{si}(R_\lambda, \lambda > 0)$. ∎

The following characterization of strong invariance via the infinitesimal generator has its counterpart in the description of strong invariance in terms of a Dirichlet form which can be found in Oshima [12], Theorem 1.5.1, pp. 23–24, or Fukushima et al. [5], Theorem 1.6.1, p. 47.

THEOREM 26. *We have $A \in \mathfrak{si}(T_t, t > 0)$ if and only if for all $u \in D(\mathfrak{A})$ the function $\mathbf{1}_A u \in D(\mathfrak{A})$ and $\mathbf{1}_A \mathfrak{A} u = \mathfrak{A}(\mathbf{1}_A u)$.*

Proof. If $A \in \mathfrak{si}(T_t, t > 0)$, we have by definition $T_t(\mathbf{1}_A u) = \mathbf{1}_A T_t u$ for all $t > 0$ and all $u \in D(\mathfrak{A})$. Therefore,

$$\frac{T_t(\mathbf{1}_A u) - \mathbf{1}_A u}{t} = \frac{\mathbf{1}_A T_t u - \mathbf{1}_A u}{t} = \mathbf{1}_A \frac{T_t u - u}{t} \to \mathbf{1}_A \mathfrak{A} u \quad \text{as } t \to 0$$

strongly. This shows that $\mathbf{1}_A u \in D(\mathfrak{A})$ as well as $\mathfrak{A}(\mathbf{1}_A u) = \mathbf{1}_A \mathfrak{A} u$.

Conversely, assume that $\mathbf{1}_A D(\mathfrak{A}) \subset D(\mathfrak{A})$ and $\mathfrak{A}(\mathbf{1}_A u) = \mathbf{1}_A \mathfrak{A} u$. Then we have for all $\lambda > 0$

$$(\lambda - \mathfrak{A})(\mathbf{1}_A R_\lambda u) = \mathbf{1}_A (\lambda - \mathfrak{A}) R_\lambda u = \mathbf{1}_A u = (\lambda - \mathfrak{A}) R_\lambda (\mathbf{1}_A u).$$

Since $\lambda - \mathfrak{A}$ is injective, we conclude that $\mathbf{1}_A R_\lambda u = R_\lambda(\mathbf{1}_A u)$ for all $\lambda > 0$. This proves that $A \in \mathfrak{si}(R_\lambda, \lambda > 0)$ and, by Lemma 25, $A \in \mathfrak{si}(T_t, t > 0)$. ∎

We will now show that (strong) invariance with respect to *one* operator in the scales $\{T_t\}_{t \geq 0}$ or $\{R_\lambda\}_{\lambda > 0}$ is enough to establish (strong) invariance with respect to the semigroup, resp. resolvent. This is straightforward for $\{R_\lambda\}_{\lambda > 0}$.

THEOREM 27. *We have* $\mathfrak{i}(R_\lambda) = \mathfrak{i}(R_1)$ *and* $\mathfrak{si}(R_\lambda) = \mathfrak{si}(R_1)$ *for all* $\lambda > 0$.

Proof. It is well-known (cf. Hille and Phillips [6], p. 184) that the resolvent $z \mapsto R_z$ is a holomorphic function for all z in any one component of the resolvent set $\varrho(A)$ and that for all $z, w \in \varrho(A)$ with $|z-w| \cdot \|R_z\| < 1$ the formula

$$R_w = R_z\Big(\mathrm{id} + \sum_{k=1}^{\infty} (z-w)^k R_z^k\Big) \tag{12}$$

holds. Assume that $A \in \mathfrak{i}(R_z)$. By induction we find

$$R_z^k(\mathbf{1}_A u) = \mathbf{1}_A R_z^k(\mathbf{1}_A u), \qquad k \in N,$$

so that $A \in \mathfrak{i}(R_w)$ for all $w \in \varrho(A)$ satisfying $|z-w| \cdot \|R_z\| < 1$. For all other w (in the same component of $\varrho(A)$) we can find a finite chain of open balls such that in each of them we have a local expansion of the type (12). Repeating the above argument we get $\mathfrak{i}(R_z) \subset \mathfrak{i}(R_w)$. Since z and w play symmetric roles, we conclude that $\mathfrak{i}(R_z) = \mathfrak{i}(R_w)$ whenever z, w are in the same component of $\varrho(A)$.

Since $(0, \infty) \subset \varrho(A)$ is necessarily in a single component, the assertion follows. The proof for strongly invariant sets is similar. ∎

For the semigroup $\{T_t\}_{t \geq 0}$ the situation is more complicated since we do not have a good substitute for the resolvent equation. The basic idea of the following proof is a relation of the type $T_t = T_1^t$, i.e., where we recover the semigroup $\{T_t\}_{t \geq 0}$ as (fractional) powers of T_1. A major difficulty is the definition of a *unique* fractional power of a rather general operator.

Before we consider the general case, let us outline the argument for symmetric (hence, self-adjoint) semigroups on L^2. Using spectral calculus we *can rigorously* prove $T_t = T_1^t$, $t > 0$ (see [15], no. 141, pp. 390–392): if $E_1(d\lambda)$, resp. $E_t(d\lambda)$, denote the spectral families of T_1, resp. T_t, one can see that

$$T_t u = \int_0^\infty \lambda\, E_t(d\lambda) u = \int_0^\infty \lambda^t E_1(d\lambda) u = T_1^t u.$$

On the other hand, the spectral measure of any open interval (a, b) can be calculated as resolvent integral

$$E_t((a, b)) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \big(R(\tau - \varepsilon i; T_t) - R(\tau + \varepsilon i; T_t)\big)\, d\tau, \tag{13}$$

where $R(z; T_t)$ is the resolvent operator for T_t at z, and the limits exist in the strong operator topology. Since T_t is a strongly continuous semigroup of symmetric operators, $\sigma(T_t) \subset \boldsymbol{R}_+$, which means that $\varrho(T_t)$ is connected. With the argument in the proof of Theorem 27 we conclude that for $n > \|T_t\|$, $n \in \boldsymbol{N}$,

$$\mathfrak{i}(R(z;\, T_t)) = \mathfrak{i}(R(n;\, T_t)) \quad \forall z \in \varrho(T_t).$$

For $n > \|T_t\|$, however, $R(n,\, T_t) = (n - T_t)^{-1}$ has an explicit representation as a Neumann series,

$$R(n,\, T_t) = \sum_{k=1}^{\infty} n^{-k}\, T_t^{k-1},$$

which, in turn, implies that $\mathfrak{i}(R(z,\, T_t)) = \mathfrak{i}(T_t)$ for all $z \in \varrho(A)$.

Since the limits in (13) are in the strong operator topology, we see that for $A \in \mathfrak{i}(T_t)$ and $u \in L^2$

$$E_t((a,\, b))(\mathbf{1}_A u) = \mathbf{1}_A E_t((a,\, b))(\mathbf{1}_A u) \tag{14}$$

holds. The converse assertion, that (14) implies $T_t(\mathbf{1}_A u) = \mathbf{1}_A T_t(\mathbf{1}_A u)$, that is, $A \in \mathfrak{i}(T_t)$, is clear from the spectral theorem. We have thus proved

LEMMA 28. *Let $\{T_t\}_{t \geqslant 0}$ be a strongly continuous semigroup of symmetric sub-Markovian operators on L^2. Then $A \in \mathfrak{i}(T_t)$ if and only if* (14) *holds for the spectral family $\{E_t(d\lambda)\}_{t > 0}$ and all open intervals $(a,\, b) \subset \boldsymbol{R}$.*

Since we can express $E_1(d\lambda)$ and $E_t(d\lambda)$ as image measures of each other (see [3], Chapter XII.2, Theorem 9, pp. 1200–1202) – for $f_s(x) := x^s$, $x > 0$, we have

$$E_t(d\lambda) = E_1(f_t^{-1}(d\lambda)) \quad \text{and} \quad E_1(d\lambda) = E_t(f_{1/t}^{-1}(d\lambda))$$

– we see that (14) holds for the family $\{E_t(d\lambda)\}_{t > 0}$ if and only if it holds for $E_1(d\lambda)$. This and Lemma 28 finally prove

THEOREM 29. *Let $\{T_t\}_{t \geqslant 0}$ be a strongly continuous semigroup of symmetric sub-Markovian operators on L^2. Then $\mathfrak{i}(T_t) = \mathfrak{i}(T_1)$.*

As already mentioned, it is in general not clear how to define a (unique) fractional power of T_1. Since T_1 comes from a semigroup, the mere existence of fractional powers is clear; it is the re-embedding into the scale $\{T_t\}_{t \geqslant 0}$ which is the problem: why should T_1^t coincide with T_t? For exponentially bounded, *analytic semigroups* in the half-space $\operatorname{Re} z > 0$ we can overcome this difficulty. The key ingredient is the following result of Hille (see [6], Theorem 17.6.1, p. 489), which is a consequence of a deep result on Newton interpolation series due to Nörlund [11], VIII.6.122, pp. 236–237.

THEOREM 30 (Hille). *Let $\{T_z\colon \operatorname{Re} z > 0\}$ be an analytic strongly continuous sub-Markovian semigroup on L^p such that $\|T_z\|_{L^p \to L^p} \leqslant e^{\kappa|z|}$ for some $\kappa \geqslant 0$ and all* $\operatorname{Re} z > 0$. *Then*

$$T_z = \sum_{j=0}^{\infty} \binom{z}{j} (T_1 - 1)^j. \tag{15}$$

The assumptions imposed on $\{T_z\colon \operatorname{Re} z > 0\}$ in Theorem 30 are non-trivial. Typically, the opening angle of the region of analyticity is smaller than π and the exponential boundedness condition means that the spectrum of the infinitesimal generator is in a horizontal strip of finite width.

Due to a result of Stein [17], III.§1, Theorem 1, p. 67, we know that symmetric strongly continuous sub-Markovian semigroups on L^2 are analytic for $\operatorname{Re} z > 0$ and satisfy $\|T_z\|_{L^2\to L^2} \leqslant 1$; hence the conditions of Theorem 30 are met for such semigroups.

COROLLARY 31. *Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous semigroup of sub-Markovian operators on L^p which is, in some L^q, $q\in[p, \infty)$, analytic on the half-space* $\operatorname{Re} z > 0$ *and satisfies there $\|T_z\|_{L^q\to L^q} \leqslant e^{\kappa|z|}$, $\kappa \geqslant 0$. Then $\mathfrak{i}(T_t, t > 0) = \mathfrak{i}(T_1)$ and $\mathfrak{si}(T_t, t > 0) = \mathfrak{si}(T_1)$.*

This holds always for symmetric strongly continuous L^2-sub-Markovian semigroups.

Proof. If $A\in\mathfrak{i}(T_1)$ and $u\in L^1_+\cap L^\infty_+$, we find

$$(T_1-1)^{j+1}(\mathbf{1}_A u) = (T_1-1)^j\{\mathbf{1}_A(T_1-1)(\mathbf{1}_A u)\},$$

and, by induction, $(T_1-1)^j(\mathbf{1}_A u) = \mathbf{1}_A(T_1-1)^j(\mathbf{1}_A u)$ for all $j\in N_0$. We conclude from (15) that

$$T_z(\mathbf{1}_A u) = \sum_{j=0}^{\infty}\binom{z}{j}(T_1-1)^j(\mathbf{1}_A u) = \sum_{j=0}^{\infty}\binom{z}{j}\mathbf{1}_A(T_1-1)^j(\mathbf{1}_A u) = \mathbf{1}_A T_z(\mathbf{1}_A u),$$

i.e., $\mathfrak{i}(T_1)\subset\bigcap_{t>0}\mathfrak{i}(T_t)$; hence $\mathfrak{i}(T_1) = \mathfrak{i}(T_t, t > 0)$. The proof for strongly invariant sets is similar. ∎

Remark 32. (A) The representation formula (15) allows us to get uniform (in the parameter z) exceptional sets for expressions of the type "$T_z u = \dots$ almost everywhere".

(B) Assume that $\{T_t\}_{t\geqslant 0}$ is a symmetric and recurrent L^2-sub-Markovian semigroup. If u is a positive measurable function which is *supermedian* with respect to T_1, i.e., $T_1 u \leqslant u$, then u is already *excessive* for $\{T_t\}_{t\geqslant 0}$ — that is $\sup_{t>0} T_t u = u$ — and even $\{T_t\}_{t\geqslant 0}$-invariant.

This follows easily from a combination of Lemma 19 showing first $T_1 u = u$, (15) and Corollary 31 which gives $T_t u = u$, $t > 0$, and (A) which allows us to choose a uniform exceptional set for all $t > 0$.

We remark that, if T_1 (hence, by Corollary 31, all T_t) is irreducible, this entails that all supermedian functions $u \geqslant 0$ are almost surely constant.

EXAMPLES

One can often associate a Markov chain or Markov process with a sub-Markovian operator, resp. semigroup. In the discrete time setting every sub-Markovian operator T yields a Markov chain $\{X_t\}_{t\in N}$ with transition semigroup $\{T^t\}_{t\in N}$. In continuous time, we get a Markov process $\{X_t\}_{t\geqslant 0}$ if, for example, $\{T_t\}_{t\geqslant 0}$ is Feller (i.e., maps continuous functions into continuous functions), cf. [4] and [7], or if $\{T_t\}_{t\geqslant 0}$ is an L^2-sub-Markovian semigroup, cf. [5] and [12].

The advantage of this is the following intuitive understanding of (strict) invariance. To keep things simple, let us assume that the state space is $X = \boldsymbol{R}^n$. By construction we know that $T_t \mathbf{1}_A(x)$ is the probability for the process to start at $X_0 = x$ and to be in A by epoch t, i.e., $T_t \mathbf{1}_A(x) = \boldsymbol{P}^x(X_t \in A)$ for every (resp. quasi-every, in the L^2-setting) $x \in \boldsymbol{R}^n$. Thus the set A is invariant if and only if for almost all x and all $t > 0$

$$T_t \mathbf{1}_A(x) = \boldsymbol{P}^x(X_t \in A) \leqslant \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This means that the process X_t cannot enter an invariant set if it is started outside of A, $X_0 = x \notin A$, but that X_t may, if started in A, leave A at any time with positive probability. On the other hand, if A is strictly invariant, so is A^c and the above interpretation shows that a process X_t can neither leave nor enter a strictly invariant set.

This allows us to give many examples where invariant sets and strictly invariant sets do not coincide. Here is the prototype for such constructions.

EXAMPLE 33. Let $\{B_t\}_{t\geqslant 0}$ be a Brownian motion on $\boldsymbol{R}$ and let $\{N_t\}_{t\geqslant 0}$ be a Poisson process on $[0, \infty)$ with jumps of size 1 and starting point $N_0 = 0$. We denote the corresponding Markovian semigroups by $\{P_t\}_{t\geqslant 0}$, resp. $\{S_t\}_{t\geqslant 0}$. Without loss of generality we can assume that $B_{\bullet}$ and $N_{\bullet}$ are stochastically independent. Denote by

$$\tau_0 := \inf\{s \geqslant 0\colon\ B_s \geqslant 0\}$$

the first entrance time of the Brownian motion into $[0, \infty)$ and consider the process

$$X_t := B_{t\wedge\tau_0} + N_{(t-\tau_0)\vee 0}, \qquad t \geqslant 0,$$

which is the concatenation of a Brownian motion on the left half-axis $(0, \infty)$ and a Poisson process on the right half-axis $[0, \infty)$. A simple direct calculation (see also the book by Sharpe [16], §II.14, pp. 77–84, for a more general approach) shows that $\{X_t\}_{t\geqslant 0}$ is a Markov process with transition semigroup

$$\begin{aligned} T_t u(x) &= \boldsymbol{E}^x u(X_t) \\ &= \mathbf{1}_{[0,\infty)}(x)\, \boldsymbol{E}^x\big(u(N_{t-\tau_0})\, \mathbf{1}_{\{t\geqslant\tau_0\}}\big) + \mathbf{1}_{(-\infty,0)}(x)\, \boldsymbol{E}^x\big(u(B_t)\, \mathbf{1}_{\{t<\tau_0\}} + u(N_{t-\tau_0})\, \mathbf{1}_{\{t\geqslant\tau_0\}}\big) \\ &= \mathbf{1}_{[0,\infty)}(x)\, S_t(\mathbf{1}_{[0,\infty)} u)(x) + \mathbf{1}_{(-\infty,0)}(x)\big[P_t(\mathbf{1}_{(-\infty,0)} u) + \boldsymbol{E}^x\big(\mathbf{1}_{\{t\geqslant\tau_0\}} S_{t-\tau_0}(\mathbf{1}_{[0,\infty)} u)\big)\big]. \end{aligned}$$

In particular,

$$T_t \mathbf{1}_{(-\infty,0)}(x) = \boldsymbol{P}^x(X_t < 0) = \begin{cases} 0 & \text{if } x \geqslant 0, \\ \boldsymbol{P}^x(t < \tau_0) = \boldsymbol{P}^x(\sup_{s\leqslant t} B_s < 0) & \text{if } x < 0, \end{cases}$$

so that $T_t \mathbf{1}_{(-\infty,0)} = \mathbf{1}_{(-\infty,0)} T_t \mathbf{1}_{(-\infty,0)}$ but $T_t \mathbf{1}_{(-\infty,0)} \neq \mathbf{1}_{(-\infty,0)}$. This shows that $(-\infty, 0)$ is invariant but not strongly invariant.

Similar considerations prove that, modulo null-sets, $\mathfrak{si}(T_t, t > 0) = \{\emptyset, \boldsymbol{R}\}$ while $\mathfrak{i}(T_t, t > 0) = \{\emptyset, (-\infty, 0), \boldsymbol{R}\}$.

If the semigroup is not analytic, it may happen that the invariant sets depend on the parameter $t > 0$. This implies that Corollary 31 need not hold for non-analytic semigroups.

EXAMPLE 34. Consider on the half-line $[0, \infty)$ the semigroup of right-translations

$$T_t u(x) := u(t+x), \qquad u \in L^p \text{ or } u \in C_b.$$

It is obvious that $z \mapsto T_z u$ is not analytic. The process corresponding to $\{T_t\}_{t \geqslant 0}$ is a deterministic uniform motion to the right, $X_t = X_0 + t$. Consider now the interval $[a, b) \subset [0, \infty)$ and let $t \geqslant b$. Then

$$T_t \mathbf{1}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(x+t) = \mathbf{1}_{[a-t,b-t)}(x) \equiv 0 \qquad \forall x \geqslant 0;$$

hence, $T_t \mathbf{1}_{[a,b)} \leqslant \mathbf{1}_{[a,b)}$ and we see $[a, b) \in \mathfrak{i}(T_t, t \geqslant b)$. On the other hand, if $0 < s < b$,

$$T_s \mathbf{1}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(x+s) = \mathbf{1}_{[(a-s) \vee 0, b-s)}(x)$$

but $\mathbf{1}_{[(a-s) \vee 0, b-s)} \leqslant \mathbf{1}_{[a,b)}$ is only possible if $a = 0$. Therefore $[a, b) \notin \mathfrak{i}(T_t, 0 < t < b)$ for all $0 < a < b$.

Corollary 31 also fails in genuinely discrete situations.

EXAMPLE 35. The proof of Corollary 31 can be modified to give $\mathfrak{si}(T_2) = \mathfrak{si}(T_1^2)$. This is, in general, wrong in discrete situations. To see this, let P be the transition operator of a simple random walk on $\boldsymbol{Z}$,

$$Pf(j) = \tfrac{1}{2}\big(f(j+1) + f(j-1)\big), \qquad \{f(j)\}_{j \in \boldsymbol{Z}} \in l^2(\boldsymbol{Z}),$$

and observe that P^2 is the transition operator of a two-step random walk. Clearly, P is self-adjoint and $\mathfrak{si}(P) = \{\emptyset, \boldsymbol{Z}\}$ while $\mathfrak{si}(P^2) = \{\emptyset, 2\boldsymbol{Z}, (2\boldsymbol{Z})^c, \boldsymbol{Z}\}$!

This does not contradict (the modification of) Corollary 31 since P has not only positive spectral values, while a sub-Markovian semigroup is necessarily spectrally positive. Indeed, $T_t = T_{t/2}^2$ while

$$(Pf, f)_{l^2} = \frac{1}{2} \sum_{j=-\infty}^{\infty} \big(f(j-1) f(j) - f(j+1) f(j)\big) = \sum_{j=-\infty}^{\infty} f(j-1) f(j)$$

is strictly negative for, say, $f(j) = (-1)^{|j|} a_j$ and $\{a_j\} \in l^2$, $a_j > 0$. This means that P can never be embedded in a semigroup — but this was crucial for Corollary 31.

The four-step random walk still satisfies $\mathfrak{si}(P^4) = \{\emptyset, 2\boldsymbol{Z}, (2\boldsymbol{Z})^c, \boldsymbol{Z}\} = \mathfrak{si}(P^2)$ but in this case we do have $P^2 = \sqrt{P^4}$ in the sense of Corollary 31.

APPENDIX

We claimed in Remark 17 (B) that m is an invariant measure, i.e., that $\int u\,dm = \int Tu\,dm$. This follows from yet another variant of Lemma 19.

PROPOSITION. *Let $(X, \mathscr{A}, m)$ be a σ-finite measure space and assume that T and T^* are sub-Markovian and T recurrent. Then m is an invariant measure for T.*

Proof. Since T^* is sub-Markovian, we have $\int Tu\,dm = \int uT^*1\,dm \leqslant \int u\,dm$ for all $u \in L_+^1 \cap L_+^\infty$, which means that m is sub-invariant. Using the left-action notation, $mT(A) := \int T1_A\,dm$, we can rewrite this in the form $mT \leqslant m$.

Since T is recurrent, we find some $f \in L_+^\infty \cap L_+^1$, $f > 0$, such that $\sum_{k=0}^\infty T^k f = \infty$ both m-a.e. and mT-a.e. (because of $mT \leqslant m$). Therefore we have for the positive measure $m - mT$ and all $N \in N$

$$0 \leqslant \langle m-mT, f\rangle \leqslant \langle m-mT, \sum_{k=0}^N T^k f\rangle = \langle (m-mT)\sum_{k=0}^N T^k, f\rangle$$
$$= \langle m-mT^{N+1}, f\rangle \leqslant \langle m, f\rangle = \|f\|_{L^1(m)} < \infty.$$

This implies $m(A) = mT(A)$ for all $m(A) > 0$, and hence $m = mT$. ∎

REFERENCES

[1] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge Univ. Press, LMS Lecture Notes Ser. Vol. 229, Cambridge 1996.

[2] E. B. Davies, *One-Parameter Semigroups*, Academic Press, L.M.S. Monographs Vol. 15, London 1980.

[3] N. Dunford and J. T. Schwartz, *Linear Operators I–III*, Interscience, Pure Appl. Math. Vol. 7, New York 1957.

[4] St. E. Ethier and Th. G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley Ser. Probab. Math. Statist. New York 1986.

[5] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Stud. Math. Vol. 19, Berlin 1994.

[6] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Providence (RI) 1957 (2nd ed.).

[7] N. Jacob, *Pseudo Differential Operators and Markov Processes.* Vol. 1: *Fourier Analysis and Semigroups*, Imperial College Press, London 2001.

[8] N. Jacob, *Pseudo Differential Operators and Markov Processes.* Vol. 2: *Generators and Their Potential Theory*, Imperial College Press, London 2003.

[9] N. Jacob and R. L. Schilling, *Towards an L^p-potential theory for sub-Markovian semigroups: kernels and capacities*, submitted.

[10] K. Jacobs, *Neuere Methoden und Ergebnisse der Ergodentheorie*, Springer, Ergeb. Math. Grenzgeb. (Neue Folge) Heft 29, Berlin 1960.

[11] N. E. Nörlund, *Differenzenrechnung*, Springer, Grundlehren Math. Wiss. Bd. 13, Berlin 1924.

[12] Y. Oshima, *Dirichlet Spaces*, Lecture Notes, Universität Erlangen-Nürnberg, Summer Term 1988.

[13] D. Revuz, *Markov Chains* (revised ed.), Elsevier, North-Holland Math. Library Vol. 11, Amsterdam 1984.
[14] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, Grundlehren Math. Wiss. Bd. 293, Berlin 1994 (2nd ed.).
[15] F. Riesz and B. Sz-Nagy, *Leçons d'analyse fonctionnelle*, Akadémiai Kiadó, Budapeszt 1952.
[16] M. Sharpe, *General Theory of Markov Processes*, Academic Press, Pure Appl. Math. Vol. 133, Boston (MA) 1988.
[17] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Princeton Univ. Press, Ann. of Math. Stud. Vol. 63, Princeton (NJ) 1970.

Department of Mathematics
University of Sussex
Brighton BN1 9RF, U.K.
E-mail: r.schilling@sussex.ac.uk

Received on 5.9.2002;
revised version on 24.2.2004