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HEAVY-TAILED DEPENDENT QUEUES IN HEAVY TRAFFIC

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Abstract. The paper studies $G/G/1$ queues with heavy-tailed probability distributions of the service times and/or the interarrival times. It relies on the fact that the heavy traffic limiting distribution of the normalized stationary waiting times for such queues is equal to the distribution of the supremum $M = \sup_{0 \leqslant t < \infty} (X(t) - \beta t)$, where X is a Lévy process. This distribution turns out to be exponential if the tail of the distribution of interarrival times is heavier than that of the service times, and it has a more complicated non-exponential shape in the opposite case; if the service times have heavy-tailed distribution in the domain of attraction of a one-sided α-stable distribution, then the limit distribution is Mittag-Leffler's. In the case of a symmetric α-stable process X, the Laplace transform of the distribution of the supremum M is also given. Taking into account the known relationship between the heavy-traffic-regime distribution of queue length and its waiting time, asymptotic results for the former are also provided. Statistical dependence between the sequence of service times and the sequence of interarrival times, as well as between random variables within each of these two sequences, is allowed. Several examples are provided.

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CONTENTS

1. INTRODUCTION

The paper provides a characterization of the limit distributions of an appropriately normalized stationary waiting times for $G/G/1$ queues operating in the heavy traffic regime under the assumption that the service times and/or the interarrival times have heavy-tailed distributions. It relies on the fact that this distribution is equal to the distribution of the supremum $M = \sup_{0 \leqslant t < \infty}(X(t) - \beta t)$, where X is a Lévy process, see Szczotka and Woyczyński (2003). The latter turns out to be exponential if the tail of the distribution of interarrival times is heavier than that of service times, and it has a more complicated non-exponential shape in the opposite case; if service times have a heavy-tailed distribution in the domain of attraction of a one-sided α-stable distribution, then that limit distribution is Mittag-Leffler's. In the case of a symmetric α-stable process X, the Laplace transform of the distribution of the supremum M is also given. Taking into account the known relationship between the heavy-traffic-regime distribution of queue length and its waiting time, asymptotic results for the former are also provided. The paper permits existence of statistical dependence between the sequence of service times and the sequence of interarrival times, as well as between random variables within each of these two sequences. Several examples are provided.

To formulate the problem more precisely let us consider a queueing system of $G/G/1$ type generated by a stationary *input sequence* $\{(v_k, u_k), k = 1, 2, \ldots\}$ of pairs of nonnegative random variables v_k and u_k, where v_k is interpreted as the service time of the k-th customer and u_k as the interarrival time between the k-th and $(k+1)$-st customers. Let $\{(v_k, u_k), k = \ldots, -1, 0, 1, \ldots, \}$ denote a two-sided stationary extension of the input sequence $\{(v_k, u_k), k = 1, 2, \ldots\}$. Although the two sequences are different, there is no danger in using the same notation for both and labelling both of them input sequences.

We shall assume that

$$a = Ev_1 - Eu_1 < 0,$$

and put

$$\xi_k = v_{-k} - u_{-k} \quad \text{and} \quad S_k = \sum_{j=1}^{k} \xi_j, \; k \geqslant 1, \quad S_0 = 0.$$

We shall also require that $S_k \to -\infty$, a.s., as $k \to \infty$. The quantity

$$\omega := \sup_{0 \leqslant k < \infty} S_k$$

is called the *stationary waiting time* for a $G/G/1$ system generated by the input sequence $\{(v_k, u_k), k \geqslant 1\}$ and it is also the limit, in a weak sense, of the sequence w_k, $k \geqslant 1$, of waiting times w_k of the k-th customer.

Our goal is to study the system in the limit

$$a = a_n \uparrow 0,$$

which, in queueing theory jargon, is known as the *heavy traffic* regime. Our notation will thus explicitly reflect the dependence of various quantities on n:

$$(v_k, u_k) = (v_{n,k}, u_{n,k}), \qquad S_k = S_{n,k}, \qquad \xi = \xi(n) = \{\xi_{n,k}, k \geqslant 1\}, \qquad \omega = \omega_n.$$

The basic and well-known fact is that if $a_n \uparrow 0$ and if $\xi(n)$ are ergodic, then $\omega_n \xrightarrow{p} \infty$ as $a_n \uparrow 0$. Formally, our primary goal is to find conditions on the input sequences $\xi(n)$ which guarantee existence of normalizing constants c_n, $c_n \uparrow \infty$, and a non-degenerate random variable M such that

$$\omega_n / c_n \xrightarrow{\mathscr{D}} M \quad \text{as } a_n \uparrow 0,$$

where $\xrightarrow{\mathscr{D}}$ stands for the convergence in distribution. A characterization of possible limit distributions appearing above is a secondary goal.

Our principal tool is the *Heavy Traffic Invariance Principle* (see Szczotka and Woyczyński (2003)), which can be formulated as follows:

HEAVY TRAFFIC INVARIANCE PRINCIPLE. *Let*

$$(1) \qquad X_n(t) = \frac{1}{c_n} \sum_{j=1}^{\lfloor nt \rfloor} (\xi_{n,j} - a_n), \qquad \beta_n(t) = \frac{|a_n| \lfloor nt \rfloor}{c_n}, \qquad \text{and} \qquad \beta_n = \frac{|a_n| n}{c_n},$$

where $t \geqslant 0$, $n \geqslant 1$, and constants $c_n \to \infty$ as $n \to \infty$. If:

(A) *there exists a stochastically continuous process X with stationary increments such that $X_n \xrightarrow{\mathscr{D}} X$ in the Skorokhod topology in $D[0, \infty)$ and $X(t) - ct \to -\infty$ a.s. as $t \to \infty$, for all $c > 0$,*

(B) *there exists β, $0 < \beta < \infty$, such that $\beta_n \to \beta$, and*

(C) *the following sequence is tight:*

$$(2) \qquad \omega_n / c_n = \sup_{0 \leqslant t < \infty} \big(X_n(t) - \beta_n(t)\big), \qquad n \geqslant 1,$$

then, as $n \to \infty$,

$$(3) \qquad \omega_n / c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} \big(X(t) - \beta t\big) \equiv M.$$

Application of the above principle to queues is more fruitful if it is combined with the following observation which is based on the idea of decomposition of processes X_n: Let $\beta_n \to \beta$,

$$X_n = X_n^{(1)} + X_n^{(2)}, \quad X_n^{(1)}(0) = X_n^{(2)}(0) = 0 \text{ a.s.}, \quad X_n \xrightarrow{\mathscr{D}} X, \quad X_n^{(1)} \xrightarrow{\mathscr{D}} X,$$

and assume that the sequences of random variables

$$(1/c_n)\,\omega_n^{(i)} := \sup_{0 \leqslant t < \infty} \big(X_n^{(i)}(t) - p_i \beta(t)\big), \quad n \geqslant 1, \ i = 1, 2, \ p_1 + p_2 = 1,$$

are tight. Then

$$\omega_n / c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} \big(X(t) - \beta t\big).$$

A stronger version of this observation will be formulated later on as the Decomposition Theorem.

If X is the standard Wiener process, then M has an exponential distribution with parameter $\lambda = 2\beta$ (see Karlin and Taylor (1975), p. 361). This asymptotics of the stationary waiting times is encountered in situations when service times and interarrival times form weakly dependent (say, satisfying some mixing conditions) sequences and their distributions have *light tails*, that is

$$\lim_{x \to \infty} x^2 \boldsymbol{P}(\cdot > x) = \gamma < \infty. \tag{4}$$

Indeed, for $GI/GI/1$ queues, Kingman (1961) has shown that if $\boldsymbol{E}u_{n,1} \to \lambda^{-1}$ and $\operatorname{Var}(v_{n,1}) + \operatorname{Var}(u_{n,1}) \to \sigma^2$, $0 < \sigma < \infty$, as $a_n \uparrow 0$, then

$$\lim_n \boldsymbol{P}(\omega_n / c_n \geqslant x) = \exp(-2\lambda x / \sigma^2),$$

where $c_n = (1 - \varrho_n)^{-1} \sigma$, and $\varrho_n = \boldsymbol{E}v_{n,1} / \boldsymbol{E}u_{n,1}$ is the traffic intensity. Kingman's approach was based on an analysis of the limit of the characteristic functions for ω_n / c_n. An analogous result for queueing systems with dependencies between random variables in the input sequences and light tails was obtained by Szczotka (1990), (1999), where functional limit theorems have been utilized.

In this paper we apply the Heavy Traffic Invariance Principle in the situation when X is a Lévy process without Gaussian component, which corresponds to the case of heavy-tailed distributions of service times and/or interarrival times in a $G/G/1$ queueing system. Recall that the distribution of a random variable ζ is said to have a *heavy tail* if there exists $\alpha < 2$ such that

$$\lim_{x \to \infty} x^{\alpha} \boldsymbol{P}(\zeta > x) = \gamma > 0. \tag{5}$$

The special case of heavy-tailed $GI/GI/1$ queues, where the input sequences had independent terms, was considered by Boxma and Cohen (1999), who investigated the limits of the Laplace–Stieltjes transforms of ω_n / c_n. They also assumed that the tails of the distributions of service and interarrival times satisfy some regularity conditions (*ibidem* (2.6–7)). Roughly speaking, these assumptions imply that the input distributions belong to the domains of attraction of stable distributions with different exponents α for service times and interarrival times. They showed that if the distributions of service times have heavier tails than those of interarrival times and if they belong to the domain of attraction of a stable distribution with parameter α, $1 < \alpha < 2$, then the limiting distribution of ω_n/c_n is a Mittag-Leffler distribution (also called sometimes a *Kovalenko* distribution). In this situation the normalizing constants c_n depend only on the distribution of service times. On the other hand, they proved that if the distributions of interarrival times have heavier tails than the service times and if they belong to the domain of attraction of a stable distribution, then the limiting distribution of ω_n/c_n is exponential; here the normalizing constants c_n depend on the distribution of interarrival times. Similar results are given in Whitt (2002).

The composition of this paper is as follows: Section 2 formulates our main results on limit distributions of stationary waiting times in heavy traffic for $G/G/1$ queues with heavy-tailed distributions of service and/or interarrival times. These results seem to be novel in the queueing theory context but we view them as an illustration of the Heavy Traffic Invariance Principle applied to heavy-tailed $G/G/1$ queues in heavy traffic in presence of some dependence structures. Initially, the dependence structure of the input sequences is that of a martingale difference sequence but then, using the Decomposition Theorem, we are able to relax this restriction to what we call *half-martingale dependence structure*: only one of the two input sequences is required to form a martingale difference sequence. These results also illustrate the general phonomenon of the limit distribution depending only on the input component with heavier distribution tail. A number of corollaries to our two main theorems are also included. They illustrate the possibility of getting the limiting distribution of ω_n/c_n for $G/G/1$ queues with the following dependence structures:

- $GI/GI/1$ queues.

- Queues for which the r.v.'s $v_{n,k}-u_{n,k}$, $-\infty < k < \infty$, are i.i.d., but random variables $v_{n,k}$ and $u_{n,k}$ need not be independent. Moreover, the distributions of $v_{n,k}$, as well as of $u_{n,k}$, may depend on k.

- Queues for which the sequences $\{(v_{n,k}-Ev_{n,1})-(u_{n,k}-Eu_{n,1}),\ k \geqslant 1\}$ form martingale difference sequences for each $n \geqslant 1$.

- Queues for which only one of the two input sequences forms a martingale difference sequence.

In Section 3 we begin to gather tools needed in the proofs of the two main theorems and start with a result on convergence of processes to a Lévy process. This is a well-explored territory but we found that a well-known result from Durrett and Resnik (1978), Theorem 4.1 (see also Jakubowski (1986)), needs some adaptation to be directly applicable for our purposes. We go through a similar process in Section 4, where we adapt known results from Szczotka and Woyczyński on sufficient conditions for tightness of the sequence $\{\omega_n/c_n\}$. Finally, proofs of Theorems 1 and 2, relatively short after all the preparations of Sections 3 and 4, are provided in Section 5.

2. LIMIT DISTRIBUTIONS OF STATIONARY WAITING TIMES IN HEAVY TRAFFIC

The next two subsections present the main results of the paper; the proofs are postponed until the last section. We begin in Subsection 2.1 by considering queueing systems with Lévy input sequences having martingale dependence structure and follow it by the Decomposition Theorem which permits, in Subsection 2.2, an extension of results of Subsection 2.1 to the case where only one of the input sequences has a martingale structure.

In what follows X stands for a Lévy process without Gaussian component and with sample paths in the space $D[0, \infty)$. Its characteristic function can be written in the form

$$\boldsymbol{E}\exp\big(iuX(t)\big) = \exp\big(t\psi_{b,\nu}(u)\big),$$

where

$$\psi_{b,\nu}(u) = iub(r) + \int_{|x|\geqslant r} (e^{iux}-1)\,\nu(dx) + \int_{0<|x|<r} (e^{iux}-1-iux)\,\nu(dx); \tag{6}$$

the *drift* $b(r)$ is a real number, the *spectral measure* ν is a positive measure on $(-\infty, \infty)$ which integrates function $\min(1, x^2)$, and r is a positive number such that points $-r$ and r are continuity points of the spectral measure ν. If spectral measure ν is concentrated on the positive half-line $(0, \infty)$, then we will call process X *spectrally positive* or, loosely, a process with positive jumps. When ν is concentrated on the negative half-line $(-\infty, 0)$, process X will be called *spectrally negative* (process with negative jumps). Let us define

$$b(r, \nu) := -\int_{|x|\geqslant r} x\nu(dx)$$

if it is finite. Then

$$\psi_{b,\nu}(u) = iu\big(b(r)-b(r,\nu)\big) + \int_{0<|x|<\infty} (e^{iux}-1-iux)\,\nu(dx). \tag{7}$$

For an α-stable spectral measure ν defined by the formulas

$$\nu(-\infty, x) = \gamma_1|x|^{-\alpha} \text{ for } x<0 \quad \text{and} \quad \nu(x, \infty) = \gamma_2|x|^{-\alpha} \text{ for } x>0,$$

we have

$$b(r, \nu) = \frac{\alpha}{\alpha-1} r^{1-\alpha}(\gamma_2-\gamma_1).$$

Similarly, for a spectrally negative α-stable ν

$$b(r, \nu) = \gamma_1 \alpha \frac{1}{\alpha - 1} r^{1-\alpha},$$

for a spectrally positive α-stable ν

$$b(r, \nu) = \gamma_2 \alpha \frac{1}{\alpha - 1} r^{1-\alpha},$$

and, for a symmetric α-stable ν, $b(r, \nu) = 0$.

2.1. Input sequences with martingale dependence structure. The following two conditions will play a role in formulation of our main theorems:

CONDITION $C(\tau, \delta)$. Let $\tau \geqslant 2$ be an integer and $\delta > 0$. We say that a sequence $\{c_n\}$ satisfies the *condition* $C(\tau, \delta)$ if, for some n_0,

$$\sum_{k=1}^{\infty} \sup_{n \geqslant n_0} \frac{1}{\tau^{\delta k}} \left(\frac{c_{n\tau^k}}{c_n} \right)^{\delta} < \infty. \tag{8}$$

THE $(\delta, \{c_n\})$-BOUNDEDNESS CONDITION. Let $\delta > 0$ and $\{c_n\}$ be a sequence of positive numbers. An array $\{\eta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ of random variables is said to satisfy the *$(\delta, \{c_n\})$-boundedness condition* if

$$\sup_{n,k} E \left(\frac{1}{c_{n\tau^k}} \sum_{j=1}^{n\tau^k} \eta_{n,j} \right)_+^{\delta} < \infty. \tag{9}$$

Here, $x_+ := \max(0, x)$.

THEOREM 1. *Consider a sequence of $G/G/1$ queues with input sequences $(v_{n,k}, u_{n,k})$, and sequences (a_n) and (c_n) such that, for each $n \geqslant 1$, $\{\xi_{n,k} - a_n,\ k \geqslant 1\}$ is a martingale difference sequence satisfying the following conditions:*

(A) *Processes $X_n \xrightarrow{\mathscr{D}} X$, where X_n are defined in (1) and X is a Lévy process without Gaussian component such that $X(t) - ct \to -\infty$ a.s. as $t \to \infty$, for all $c > 0$, and with characteristic function $\exp[\psi_{b,\nu}(u)]$ in (6), where*

$$b(r) = b_r \equiv \lim_{n \to \infty} \frac{1}{c_n} \sum_{j=1}^{n} E\big((\xi_{n,j} - a_n) \mathbf{1}(|\xi_{n,j} - a_n| < rc_n) | \mathscr{F}_{n,j-1}\big).$$

(B) *As $n \to \infty$, sequence $\beta_n \equiv |a_n| n / c_n \to \beta$, where $0 < \beta < \infty$ and $b_r < \beta$.*

(C) *Sequence $\{c_n\}$ satisfies the condition $C(\tau, \delta)$.*

(D) *Sequence $\{\xi_{n,k} - a_n,\ k \geqslant 1,\ n \geqslant 1\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition.*

Then

$$\omega_n / c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} (X(t) - \beta t) \equiv M.$$

Thus, with the notation

$$\mu = \int_{0+}^{\infty} (e^{-x}-1+x)\,x^{-(\alpha+1)}\,dx,$$

we have the following corollary to the above Theorem 1, and to Theorems 4 and 8 in Szczotka and Woyczyński (2003).

COROLLARY 1. *If the conditions of Theorem* 1 *are satisfied, then the following statements hold true*:

(i) *If X is spectrally negative with spectral measure ν and characteristic function of the form* (6), *where $b(r) = b_r$, and if $\beta + b(r, \nu) - b_r > 0$, then M has an exponential distribution with parameter λ which is the positive root of the equation $\psi(\lambda) = 0$, where*

$$\psi(u) \equiv u(b_r-\beta) + \int_{-\infty}^{-r} (e^{ux}-1)\,\nu(dx) + \int_{-r}^{0-} (e^{ux}-1-ux)\,\nu(dx).$$

In particular, if $\nu(-\infty, x) = \gamma_1 |x|^{-\alpha}$, for $x < 0$, then

$$\lambda = \left(\frac{\beta}{\alpha\gamma_1\mu}\right)^{1/(\alpha-1)}. \tag{10}$$

(ii) *If X is spectrally positive with exponent $\psi_{b,\nu}(u)$, where $b(r) = b_r$, and α-stable, spectrally positive measure $\nu(x, \infty) = \gamma_2 x^{-\alpha}$, for $x > 0$, with $1 < \alpha < 2$, $\gamma_2 > 0$, then M/θ has the Mittag-Leffler distribution with Laplace–Stieltjes transform*

$$E\exp(-sM/\theta) = \frac{1}{1+s^{\alpha-1}}, \quad \textit{where } \theta = \left(\frac{\alpha\gamma_2\mu}{\beta}\right)^{1/(\alpha-1)}. \tag{11}$$

(iii) *If X has a symmetric α-stable measure ν, i.e. $\nu(-\infty, -x) = \nu(x, \infty) = \gamma x^{-\alpha}$ for $x > 0$, $1 < \alpha < 2$, $\gamma > 0$, then M has the Laplace–Stieltjes transform of the form*

$$Ee^{-sM} = e^{-A(s)/\pi}, \quad s \geqslant 0,$$

where

$$A(s) = \int_0^{\infty} \frac{1}{u^{1+1/\alpha}} \int_0^{\infty} (e^{-sz}-1) \int_0^{\infty} \exp(-\lambda t^{\alpha}) \cos\big(t(z+u\beta)\,u^{-1/\alpha}\big)\,dt\,dz\,du.$$

The case of independent increments. It is a useful exercise to reinterpret the above results in the special case of input sequences with independent and identically distributed terms. Note that, for each k, random variables $v_{n,k}$ and $u_{n,k}$ may be dependent.

In formulation of the following corollary we will make use of the following definition: A sequence of distribution functions F_n, $n \geqslant 1$, is said to be *attracted by a Lévy distribution* with spectral measure ν on R if the following conditions hold:

$$nF_n(y) \to \nu(-\infty, y) \quad \text{and} \quad n\big(1-F_n(x)\big) \to \nu(x, \infty) \tag{12}$$

for all $y < 0$ and $x > 0$, which are continuity points of ν;

(13) $$\lim_{x\to\infty} \sup_n n\big(F_n(-x)+1-F_n(x)\big) = 0;$$

(14) $$\lim_n n \int_{|x|<r} x dF_n(x) = b_r, \quad |b_r| < \infty;$$

and

(15) $$\lim_{\delta\to 0} \limsup_{n\to\infty} n \int_{|x|<\delta} x^2 dF_n(x) = 0.$$

COROLLARY 2. *Let, for each $n \geqslant 1$, $\{\xi_{n,k}-a_n,\ k \geqslant 1\}$ be a sequence of independent and identically distributed random variables such that the sequence of distribution functions F_n, $n \geqslant 1$, defined as*

$$F_n(x) = \boldsymbol{P}\left(\frac{1}{c_n}(\xi_{n,1}-a_n) \leqslant x\right), \qquad x \in R,$$

is attracted by a Lévy distribution with spectral measure ν. Furthermore, let $\beta_n \to \beta$, $0 < \beta < \infty$, and the distributions of $\xi_{n,1}-a_n$ be majorized by the distribution of ϑ_1 in the convex ordering sense, i.e.

$$\boldsymbol{E}\max(0,\ \xi_{n,1}-a_n-x) \leqslant \boldsymbol{E}\max(0,\ \vartheta_1-x) \quad \textit{for all } x \in R,$$

and let

(16) $$\limsup_{n\to\infty} \boldsymbol{E}\left(\frac{1}{c_n}\sum_{j=1}^{n} \vartheta_j\right)_+^{\delta} < \infty,$$

where $\vartheta_1, \vartheta_2, \ldots$ are i.i.d. random variables with $\boldsymbol{E}\vartheta_1 = 0$. Then

$$\frac{1}{c_n}\omega_n \xrightarrow{\mathscr{D}} \sup_{0\leqslant t<\infty} \big(X(t)-\beta t\big).$$

The normalizing constants c_n, $n \geqslant 1$, are such that the processes $X_n(t) = (1/c_n)\sum_{j=1}^{[nt]}(\xi_{n,j}-a_n)$, $n \geqslant 1$, converge to a Lévy process X. If X is a stable Lévy process with stable spectral measure ν, $\nu(-\infty, -x) = \gamma_1 x^{-\alpha}$, and $\nu(x, \infty) = \gamma_2 x^{-\alpha}$ for $x > 0$, $1 < \alpha < 2$, then $c_n = n^{1/\alpha}h(n)$, where $h(n)$ slowly varies at infinity. Those constants can be evaluated from the conditions (12) and (13), i.e., from the limit conditions

$$n\boldsymbol{P}(v_{n,1}-u_{n,1}-a_n < -xc_n) \to \gamma_1 x^{-\alpha} \quad \text{and} \quad n\boldsymbol{P}(v_{n,1}-u_{n,1}-a_n > xc_n) \to \gamma_2 x^{-\alpha},$$

which hold for all $x > 0$, and from the condition $|a_n| n/c_n \to \beta$, $0 < \beta < \infty$.

It is clear that in the considered case constants c_n depend on the distributions of $u_{n,1}$ (interarrival times) if ν is stable and spectrally negative ($\gamma_1 > 0$, $\gamma_2 = 0$), and on the distributions of $v_{n,1}$ (service times) if ν is stable and spectrally positive ($\gamma_1 = 0$, $\gamma_2 > 0$).

2.2. Input sequences with half-martingale dependence structure. In this subsection we formulate results that extend Theorem 1 to the case when only one of the two input sequences has a martingale structure; we say then that the whole queueing system has the *half-martingale dependence structure.* Again, the proofs are postponed until the last section of the paper.

2.2.1. *Dependence of the distribution of M on the tails of the distributions of service times and interarrival times.* Here we examine situations when the distribution of M depends either on the tail of the distribution of service times or on the tail of the distribution of interarrival times. The emerging picture is explained in the following Decomposition Theorem, a weaker version thereof was formulated as Lemma 4 in Szczotka and Woyczyński (2003). The statement of the theorem is preceded by an adjustment of the notation introduced first in Section 1.

Let

$$V_n(t) = \frac{1}{c_{n,1}} \sum_{j=1}^{\lfloor nt \rfloor} (v_{n,-j} - \bar{v}_n), \qquad U_n(t) = -\frac{1}{c_{n,2}} \sum_{j=1}^{\lfloor nt \rfloor} (u_{n,-j} - \bar{u}_n), \qquad t \geq 0,\ n \geq 1,$$

where $\bar{v}_n = Ev_{n,1}$, $\bar{u}_n = Eu_{n,1}$, $n \geq 1$, and $c_{n,1}$, $c_{n,2}$ are constants tending to infinity. Then $a_n = \bar{v}_n - \bar{u}_n$. For other notation, see (1).

THEOREM 2 (DECOMPOSITION THEOREM). *Suppose that, for each $n \geq 1$, the input sequence $\{(v_{n,k}, u_{n,k}),\ -\infty < k < \infty\}$ is such that $\beta_n \to \beta$, $0 < \beta < \infty$, with $c_n = \max(c_{n,1}, c_{n,2})$, $n \geq 1$, and that the arrays*

$$\{\eta_{n,k}^{(1)} = (v_{n,-k} - \bar{v}_n),\ k \geq 1,\ n \geq 1\} \quad \text{and} \quad \{\eta_{n,k}^{(2)} = -(u_{n,-k} - \bar{u}_n),\ k \geq 1,\ n \geq 1\}$$

are such that the sequences

$$\left\{\frac{1}{c_{n,1}} \omega_n^{(1)} := \sup_{0 \leq t < \infty} \big(V_n(t) - p\beta_n(t)\big)\right\} \quad \text{and} \quad \left\{\frac{1}{c_{n,2}} \omega_n^{(2)} := \sup_{0 \leq t < \infty} \big(U_n(t) - p\beta_n(t)\big)\right\}$$

are tight for some p, $0 < p < 1$, and the sequences $\{\sup_{0 \leq t \leq c} |V_n(t)|,\ n \geq 1\}$ and $\{\sup_{0 \leq t \leq c} |U_n(t)|,\ n \geq 1\}$ are tight for all $c > 0$. Then:

(i) *If $V_n \xrightarrow{\mathscr{D}} V$, where V is a non-degenerate, spectrally positive Lévy process and $c_{n,2}/c_{n,1} \to 0$, then*

$$\omega_n / c_n \xrightarrow{\mathscr{D}} \sup_{0 \leq t < \infty} \big(V(t) - \beta t\big).$$

(ii) *If $U_n \xrightarrow{\mathscr{D}} U$, where U is a non-degenerate, spectrally negative Lévy process and $c_{n,1}/c_{n,2} \to 0$, then*

$$\omega_n / c_n \xrightarrow{\mathscr{D}} \sup_{0 \leq t < \infty} \big(U(t) - \beta t\big).$$

Remark 1. Obviously, the sequence $\{\sup_{0 \leq t \leq c} |V_n(t)|,\ n \geq 1\}$ is tight if $V_n \xrightarrow{P} V$ or if $\{(v_{n,-k} - \bar{v}_n),\ k \geq 1,\ n \geq 1\}$ is a martingale difference array and satisfies the $(\delta, \{c_n\})$-boundedness condition. A similar statement holds true for the sequence $\{\sup_{0 \leq t \leq c} |U_n(t)|,\ n \geq 1\}$.

At this point we are in a position to describe classes of queues to which the Decomposition Theorem is applicable. For convenience, the following terminology will be utilized: A queue generated by a stationary input sequence $\{(v_k, u_k), -\infty < k < \infty\}$ is said to be of *independent-martingale type* (*IM-type*) if the sequence of interarrival times $\{u_k, -\infty < k < \infty\}$ is a sequence of i.i.d. random variables and the sequence of service times $\{v_k, -\infty < k < \infty\}$ is such that $\{v_k - Ev_1, -\infty < k < \infty\}$ forms a martingale difference sequence. Similarly, a queue is said to be of *martingale-independent type* (*MI-type*) if the sequence of interarrival times $\{u_k, -\infty < k < \infty\}$ is such that $\{u_k - Eu_1, -\infty < k < \infty\}$ forms a martingale difference sequence and the sequence of service times $\{v_k, -\infty < k < \infty\}$ is a sequence of i.i.d. random variables. A queue is of *independent-independent type* (*II-type*) if it is both of IM-type and MI-type. Of course, the II-type queueing system need not be $GI/GI/1$ because we do not assume that the sequences of service times and interarrival times are independent. Also, a queue is said to be of *martingale-martingale type* (*MM-type*) if $\{u_k - Eu_1, -\infty < k < \infty\}$ and $\{v_k - Ev_1, -\infty < k < \infty\}$ form martingale difference sequences. Finally, a queue is said to be of *half-martingale type* (*HM-type* or *MH-type*) if only one of the two input sequences forms a martingale difference sequence.

Interarrival times with tails heavier than those of service times. In the case of queues of II-, IM- and MH-type, for which interarrival times have heavier tails than service times, which corresponds to X being spectrally negative, we have the following results:

COROLLARY 3 (The spectrally negative case for queues of II-type and IM-type). *Let, for each* $n \geq 1$, $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ *be an input sequence either* (a) *of a queue of II-type or* (b) *a queue of IM-type, and let* $c_n = n^{1/\alpha} h(n)$, $1 < \alpha < 2$, *where* $h(n)$ *slowly varies at infinity and* $\beta_n \to \beta$, $0 < \beta < \infty$. *Suppose that the following conditions hold*:

(i) $\sup_n E|v_{n,1} - \bar{v}_n|^\varepsilon < \infty$ *for some* ε, $1 < \alpha \leq \varepsilon < 2$, *in case* (a) *and* $\sup_n \operatorname{Var}(v_{n,1}) < \infty$ *in case* (b).

(ii) *The array* $\{\eta_{n,k}^{(2)} := -(u_{n,-k} - \bar{u}_n), k \geq 1, n \geq 1\}$ *satisfies the* $(\delta, \{c_n\})$*-boundedness condition for some* δ, $1 \leq \delta < \alpha$.

(iii) *The sequence of distribution functions* F_n, $n \geq 1$, *defined as*

$$F_n(x) = P\left(-\frac{u_{n,1} - \bar{u}_n}{c_n} \leq x\right), \quad x \in R,$$

is attracted by a stable distribution with spectrally negative measure ν, $\nu(-\infty, x) = \gamma_1 |x|^{-\alpha}$, $x < 0$.

Then $\omega_n / c_n \xrightarrow{\mathscr{D}} M$, *where* M *has an exponential distribution with parameter* $\lambda = (\beta/(\alpha\gamma_1 \mu))^{1/(1-\alpha)}$.

Another example of the spectrally negative case in which the assertion of the Decomposition Theorem holds true is that of a queue of MH-type with

$\{u_{n,-k}-\bar{u}_n, k \geq 1\}$ assumed to be a martingale difference sequence with a special structure of dependence which we call *chain-dependence*, while no martingale structure is imposed on $\{v_{n,-k}-\bar{v}_n, k \geq 1\}$. The concept of a *chain-dependent* sequence of random variables has been encountered in the queueing context before and is defined as follows:

Let $\{J_k, k \geq 0\}$ be a stationary, irreducible Markov chain (periodic or not) with a finite state space $S = \{1, 2, \ldots, m\}$, the transition matrix $P = \{p_{i,j}, i, j \in S\}$ and stationary distributions $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$. A sequence $\{\zeta_k, k \geq 1\}$ of random variables is called *chain-dependent* with respect to $\{J_k, k \geq 0\}$, with distribution functions $G_1, G_2, \ldots, G_m$ if, for each i and j,

$$(17)\quad \boldsymbol{P}(J_k = j, \zeta_k \leq x \mid J_{k-1} = i, \mathscr{B}_{k-1}) = \boldsymbol{P}(J_k = j, \zeta_k \leq x \mid J_{k-1} = i) = p_{i,j}\cdot G_i(x),$$

where the σ-fields $\mathscr{B}_k = \sigma(J_0, J_1, \ldots, J_k, \zeta_1, \zeta_2, \ldots, \zeta_k)$.

Observe that if $\{\zeta_k, k \geq 1\}$ is a chain-dependent sequence, then $\{\zeta_k, k \geq 1\}$ is stationary, and $\{\zeta_k - \boldsymbol{E}\zeta_k, \mathscr{B}_k, k \geq 1\}$ is a martingale difference sequence, where $\boldsymbol{E}\zeta_k = \sum_{i=1}^m \pi_i d_i$, $d_i = \int_{-\infty}^{\infty} x dG_i(x)$; see Section 4 for details. In the above situation we will simply say that $\{\zeta_k, k \geq 1\}$ is *chain-dependent* with respect to $\{J_k, k \geq 0\}$, with stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ and distribution functions $G_1, G_2, \ldots, G_m$; the irreducibility of $\{J_k, k \geq 0\}$ will be always assumed though.

In what follows we consider a sequence of chain-dependent sequences. Namely, for each $n \geq 1$, $\{\zeta_{n,k}, k \geq 1\}$ is chain-dependent with respect to $\{J_{n,k}, k \geq 0\}$, with stationary distribution $\pi_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,m})$, and distribution functions $G_{n,1}, G_{n,2}, \ldots, G_{n,m}$, and we set

$$\pi_{n,i}(n) = \sum_{j=1}^{n} \mathbf{1}(J_{n,j} = i), \qquad n \geq 1,\ 1 \leq i \leq m.$$

We also need the concept of convex ordering between random variables η_1 and η_2 and their distribution functions F_1 and F_2. Namely, we shall write $\eta_1 \leq_c \eta_2$, and $F_1 \leq_c F_2$ if, for all x,

$$\boldsymbol{E}\max(0, \eta_1 - x) \leq \boldsymbol{E}\max(0, \eta_2 - x).$$

Finally, we shall say that a random variable η has a *Pareto distribution* with parameters (α, γ) if its cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < \gamma^{1/\alpha}, \\ 1-\gamma x^{-\alpha} & \text{for } x \geq \gamma^{1/\alpha}. \end{cases}$$

Of course, $\boldsymbol{E}\eta = \gamma\alpha/(\alpha-1)$.

COROLLARY 4 (The spectrally negative case for queues of MH-type). *Let, for each $n \geq 1$, $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ be an input sequence of a queue of MH-type such that the following conditions hold*:

(i) *For each $n \geqslant 1$ the sequences $\{v_{n,-k}, k \geqslant 1\}$ are either such as in cases* (a) *or* (b) *in Corollary* 3, *or they are ϕ-mixing with the same mixing function $\phi = \{\phi_k, k \geqslant 1\}$, for all n, such that $\sum_{k=1}^{\infty} \sqrt{\phi_k} < \infty$ and $Ev_{n,1}^{2+\varepsilon} < \infty$ for some* $\varepsilon > 0$.

(ii) *For each $n \geqslant 1$, $\{u_{n,-k}, k \geqslant 1\}$ is chain-dependent with respect to $\{J_{n,k}, k \geqslant 0\}$ with stationary distribution $\pi_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,m})$ and distribution functions $\tilde{G}_{n,1}, \tilde{G}_{n,2}, \ldots, \tilde{G}_{n,m}$, where $\tilde{G}_{n,i}(x) = P(u_{n,-k} \leqslant x \mid J_{n,k-1} = i)$.*

Furthermore, assume that the following conditions hold:

(iii) *$\beta_n \to \beta$, $0 < \beta < \infty$, where $c_n = n^{1/\alpha} h(n)$, $1 < \alpha < 2$, $h(n)$ slowly varies at infinity, and $a_n = Ev_{n,1} - Eu_{n,1}$, where $Eu_{n,1} = \sum_{i=1}^{m} \pi_{n,i} \int x d\tilde{G}_{n,i}(x)$.*

(iv) $n^{-1} \pi_{n,i}(n) \xrightarrow{P} \pi_i$.

(v) *The sequences $\{G_{n,i}, n \geqslant 1\}$, $1 \leqslant i \leqslant m$, of distribution functions defined as*

$$G_{n,i}(x) = P\left(-\frac{u_{n,1} - \bar{u}_{n,i}}{c_n} \leqslant x \mid J_{n,0} = i\right), \qquad x \in R,$$

are attracted to stable, spectrally negative probability distributions with spectral measures ν_i, respectively, where $\nu_i(-\infty, x) = \gamma_{1,i} |x|^{-\alpha}$, $x < 0$.

(vi) *$\tilde{G}_{n,i} \leqslant_c \tilde{G}_i$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant m$, where $\tilde{G}_i$ are Pareto distribution functions with parameters $(\alpha, \gamma_{1,i})$, respectively.*

Then $\omega_n/c_n \xrightarrow{\mathscr{D}} M$, where M has an exponential distribution with parameter $\lambda = (\beta/(\alpha\gamma_1 \mu))^{1/(1-\alpha)}$, where $\gamma_1 = \sum_{j=1}^{m} \pi_i \gamma_{1,i}$.

Service times with tails heavier than those of interarrival times. In the case of queues of II-, MI- and HM-type, for which service times have heavier tails than interarrival times, which corresponds to X being spectrally positive, we have the following results:

COROLLARY 5 (The spectrally positive case for queues of II-type and MI-type). *Let, for each $n \geqslant 1$, $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ be an input sequence of a queue of either* (a) *of II-type or* (b) *of MI-type, and let $c_n = n^{1/\alpha} h(n)$, $1 < \alpha < 2$, where $h(n)$ slowly varies at infinity and $\beta_n \to \beta$, $0 < \beta < \infty$. Suppose that the following conditions hold*:

(i) $\sup_n E|u_{n,1} - \bar{u}_n|^{\varepsilon} < \infty$ *for some ε, $1 < \alpha \leqslant \varepsilon < 2$, in case* (a) *and* $\sup_n \operatorname{Var}(u_{n,1}) < \infty$ *in case* (b).

(ii) *The array $\{\eta_{n,k}^{(1)} := (v_{n,-k} - \bar{v}_n), k \geqslant 1, n \geqslant 1\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition for some δ, $1 \leqslant \delta < \alpha$.*

(iii) *The sequence of distribution functions F_n, $n \geqslant 1$, defined as*

$$F_n(x) = P\left(\frac{v_{n,1} - \bar{v}_n}{c_n} \leqslant x\right), \qquad x \in R,$$

is attracted by the stable, spectrally positive probability distribution with spectral measure ν, $\nu(x, \infty) = \gamma_2 x^{-\alpha}$, $x > 0$.

Then $\omega_n/c_n \xrightarrow{\mathscr{D}} M$, where M/θ has a Mittag-Leffler distribution with parameter θ, and $\theta = (\alpha\gamma_2 \mu/\beta)^{1/(1-\alpha)}$.

COROLLARY 6 (The spectrally positive case for queues of HM-type). *Let, for each $n \geqslant 1$, $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$ be an input sequence of a queue of HM-type such that the following conditions hold:*

(i) *For each $n \geqslant 1$, the sequences $\{u_{n,-k}, k \geqslant 1\}$ are either such as in cases* (a) *or* (b) *in Corollary* 5, *or they are ϕ-mixing with the same mixing function $\phi = \{\phi_k, k \geqslant 1\}$, for all n, such that $\sum_{k=1}^{\infty} \sqrt{\phi_k} < \infty$, and $Eu_{n,1}^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$.*

(ii) *For each $n \geqslant 1$, $\{v_{n,-k}, k \geqslant 1\}$ is chain-dependent with respect to $\{J_{n,k}, k \geqslant 0\}$, with stationary distribution $\pi_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,m})$ and distribution functions $\tilde{G}_{n,1}, \tilde{G}_{n,2}, \ldots, \tilde{G}_{n,m}$, where $\tilde{G}_{n,i}(x) = P(v_{n,-k} \leqslant x \mid J_{n,k-1} = i)$.*

Furthermore, assume that the following conditions hold:

(iii) *$\beta_n \to \beta$, $0 < \beta < \infty$, where $c_n = n^{1/\alpha} h(n)$, $1 < \alpha < 2$, and $h(n)$ slowly varies at infinity.*

(iv) *$n^{-1} \pi_{n,i}(n) \xrightarrow{\mathscr{D}} \pi_i$.*

(v) *The sequences $\{G_{n,i}, n \geqslant 1\}$, $1 \leqslant i \leqslant m$, of distribution functions defined as*

$$G_{n,i}(x) = P\left(\frac{v_{n,1} - \bar{v}_{n,i}}{c_n} \leqslant x \mid J_{n,0} = i\right) \equiv \tilde{G}_{n,i}(c_n x + \bar{v}_{n,i}), \qquad x \in R,$$

are attracted to stable, spectrally positive probability distributions with spectral measures v_i, respectively, where $v_i(x, \infty) = \gamma_{2,i} x^{-\alpha}$, $x > 0$.

(vi) *$\tilde{G}_{n,i} \leqslant_c \tilde{G}_i$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant m$, where $\tilde{G}_i$, $1 \leqslant i \leqslant m$, are Pareto distribution functions with parameters $(\alpha, \gamma_{2,i})$, respectively.*

Then $\omega_n/c_n \xrightarrow{\mathscr{D}} M$, where M/θ has a Mittag-Leffler distribution with parameter θ, $\theta = (\alpha\gamma_2 \mu/\beta)^{1/(1-\alpha)}$, and $\gamma_2 = \sum_{i=1}^{m} \pi_i \gamma_{2,i}$.

3. LIMIT DISTRIBUTIONS OF STATIONARY QUEUE LENGTH

Let l_n denote the stationary queue length in the n-th queue generated by the input sequence $\{(v_{n,k}, u_{n,k}), -\infty < k < \infty\}$. The results for stationary waiting times ω_n formulated in Section 2 immediately give analogous results for the stationary queue length l_n in view of the following result which is due to Szczotka (1990), Theorem 2:

THEOREM 3. *Suppose that there exists a sequence $\{c_n\}$, $c_n \uparrow \infty$, $c_n/n \to 0$, such that, for each $t \geqslant 0$ and $n \to \infty$,*

$$\frac{1}{c_n} \sum_{j=1}^{\lfloor c_n t \rfloor} v_{n,-j} \xrightarrow{P} \bar{v}t \quad \text{and} \quad \frac{1}{c_n} \sum_{j=1}^{\lfloor c_n t \rfloor} u_{n,-j} \xrightarrow{P} \bar{v}t, \tag{18}$$

and

$$\omega_n/c_n \xrightarrow{\mathscr{D}} M \quad \text{as } n \to \infty. \tag{19}$$

Then

(20) $$(1/c_n)(l_n - \bar{v}\omega_n) \xrightarrow{P} 0$$

and, consequently,

$$(1/c_n)\, l_n \xrightarrow{\mathscr{D}} \bar{v}M.$$

Notice that the convergence in conditions (18) takes place if processes

$$\tilde{V}_n(t) = \frac{1}{c_n}\sum_{j=1}^{\lfloor nt\rfloor}(v_{n,-j} - \bar{v}_n), \qquad \tilde{U}_n(t) = \frac{1}{c_n}\sum_{j=1}^{\lfloor nt\rfloor}(u_{n,-j} - \bar{u}_n), \qquad t \geqslant 0,\ n \geqslant 1,$$

converge to processes $\tilde{V}$ and $\tilde{U}$, respectively, with $\tilde{V}(0) = \tilde{U}(0) = 0$ a.s. (which is usually assumed), and $\bar{v}_n \to \bar{v}$, $\bar{u}_n \to \bar{v}$.

Indeed, in view of the equalities

$$\frac{1}{c_n}\sum_{j=1}^{\lfloor c_n t\rfloor} v_{n,-j} = \frac{1}{c_n}\sum_{j=1}^{\lfloor c_n t\rfloor}(v_{n,-j} - \bar{v}_n) + \frac{\lfloor c_n t\rfloor}{c_n}\bar{v}_n = \frac{1}{c_n}\sum_{j=1}^{\lfloor nc_n t/n\rfloor}(v_{n,-j} - \bar{v}_n) + \frac{\lfloor c_n t\rfloor}{c_n}\bar{v}_n,$$

the assumptions $\tilde{V}_n \xrightarrow{\mathscr{D}} \tilde{V}$, $c_n/n \to 0$, and $\bar{v}_n \to \bar{v}$, and the Continuous Mapping Theorem for the topology of weak convergence (see Theorem 5.1 in Billingsley (1968)) imply the first convergence in (18). The second convergence in (18) can be verified in a similar fashion.

4. PROOF PRELIMINARIES

This section gathers facts needed in the proofs of results stated in Section 2. Although these facts are essentially known, we need to adapt them for our use in Section 5. The first subsection deals with the issue of convergence of a sequence of processes to a Lévy process while the second subsection collects results about tightness.

4.1. Convergence to a Lévy process. A Lévy process can be viewed as the limiting process, $n \to \infty$, of the interpolated sums processes $Y_n(t) = \sum_{k=1}^{\lfloor nt\rfloor}\zeta_{n,k}$, $t \geqslant 0$, $n \geqslant 1$, where $\{\zeta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ is an array of random variables. We begin in Proposition 1 by rewriting a result due to Durrett and Resnik (1978), Theorem 4.1, in the case when $\{\zeta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ is a martingale difference array. It gives sufficient conditions for the convergence $Y_n \xrightarrow{\mathscr{D}} X$, when X is a Lévy process with spectral measure ν. As a corollary we formulate the classical Prokhorov's result for the convergence $Y_n \xrightarrow{\mathscr{D}} X$, when, for each $n \geqslant 1$, $\{\zeta_{n,k},\ k \geqslant 1\}$ is a sequence of i.i.d. random variables. Finally, we rewrite Durrett and Resnik's result in the case where, for each $n \geqslant 1$, random variables $\{\zeta_{n,k},\ k \geqslant 1\}$ form a chain-dependent sequence.

The martingale case. An array $\{\zeta_{n,k},\ k \geqslant 1, n \geqslant 1\}$ is called a *martingale difference array* if, for each $n \geqslant 1$, the random variables $\zeta_{n,k}$, $k \geqslant 1$, are defined on a common probability space $(\Omega_n, \mathscr{F}_n, P_n)$ on which there is an increasing sequence $\{\mathscr{F}_{n,k}, k \geqslant 1\}$ of σ-fields contained in the σ-field $\mathscr{F}_n$, and $\zeta_{n,k}$ is $\mathscr{F}_{n,k}$-measurable, $k \geqslant 1$, while the conditional expectation $E(\zeta_{n,k} | \mathscr{F}_{n,k-1}) = 0$. In the described situation we also say that $\{(\zeta_{n,k}, \mathscr{F}_{n,k}), k \geqslant 1, n \geqslant 1\}$ is a martingale difference array.

Denote by $\mathbf{1}(A)$ the indicator function of event A and let

$$b_{n,r}(t) = \sum_{j=1}^{\lfloor nt \rfloor} E\big(\zeta_{n,j}\mathbf{1}(|\zeta_{n,j}| < r) \mid \mathscr{F}_{n,j-1}\big), \qquad t \geqslant 0,\ n \geqslant 1, \tag{21}$$

$$Z_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \zeta_{n,j} - b_{n,r}(t), \qquad t \geqslant 0,\ n \geqslant 1,$$

$$\bar{\bar{\zeta}}_{n,j} = \zeta_{n,j}\mathbf{1}(|\zeta_{n,j}| < \delta) - E\big(\zeta_{n,j}\mathbf{1}(|\zeta_{n,j}| < \delta) \mid \mathscr{F}_{n,j-1}\big), \qquad \delta > 0. \tag{22}$$

PROPOSITION 1. *Let $\{\zeta_{n,k}, k \geqslant 1, n \geqslant 1\}$ be a martingale difference array satisfying the following conditions*:

(a) *For all $t \geqslant 0$ and all $x > 0$, $y < 0$, which are the continuity points of the spectral measure ν, as $n \to \infty$,*

$$\sum_{j=1}^{[nt]} P(\zeta_{n,j} > x \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} t\nu(x, \infty), \qquad \sum_{j=1}^{[nt]} P(\zeta_{n,j} < y \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} t\nu(-\infty, y).$$

(b) *For all $\varepsilon > 0$*

$$\max_{1 \leqslant j \leqslant n} P(|\zeta_{n,j}| > \varepsilon \mid \mathscr{F}_{n,j-1}) \xrightarrow{P} 0.$$

(c) *For all $\varepsilon > 0$*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\Big(\sum_{j=1}^{n} E\big((\bar{\bar{\zeta}}_{n,j})^2 \mid \mathscr{F}_{n,j-1}\big) > \varepsilon\Big) = 0.$$

Then $Z_n \xrightarrow{\mathscr{D}} X$ in $D[0, \infty)$ with J_1 Skorokhod topology, where X is a Lévy process with characteristic function $E\exp\big(iuX(t)\big) = \exp\big(it\psi_{0,\nu}(u)\big)$ with exponent $\psi_{b,\nu}(u)$ of the form (6) *with $b(r) = 0$, and spectral measure ν given in condition* (a).

Furthermore:

(d) *If b_r is a number such that, for $n \to \infty$,*

$$\sup_{0 \leqslant t \leqslant c} |b_{n,r}(t) - tb_r| \xrightarrow{P} 0 \ \text{ as } n \to \infty \quad \text{for all } c > 0, \tag{23}$$

then $Y_n = Z_n + b_{n,r} \xrightarrow{\mathscr{D}} X$ in $D[0, \infty)$ with J_1 Skorokhod topology, where X is a Lévy process with characteristic exponent $\psi_{b,\nu}(u)$ and $b(r) = b_r$.

Now let us consider condition (d) of Proposition 1 and the form of the limit b_r in the case when the spectral measure is continuous with $\int_r^\infty \nu(x, \infty)\,dx < \infty$ and $\int_{-\infty}^{-r} \nu(-\infty, x)\,dx < \infty$. For this purpose let us put

$$F_{n,j}(x \mid \mathscr{F}_{n,j-1}) = P(\zeta_{n,j} \leqslant x \mid \mathscr{F}_{n,j-1}), \qquad j \geqslant 1,\ n \geqslant 1,$$

and assume that $F_{n,j}(x \mid \mathscr{F}_{n,j-1})$ are regular conditional distribution functions.

PROPOSITION 2. *Let* $\{\zeta_{n,k},\ k \geq 1,\ n \geq 1\}$ *be a martingale difference array satisfying conditions* (a) *and* (b) *of Proposition* 1 *and let a spectral measure* ν *be continuous at all points with* $\int_r^\infty \nu(x, \infty)\,dx < \infty$ *and* $\int_{-\infty}^{-r} \nu(-\infty, x)\,dx < \infty$. *Then condition* (d) *of Proposition* 1 *holds and*

$$b_r = -\int_r^\infty x\nu(dx) - \int_{-\infty}^{-r} x\nu(dx).$$

Proof. Since functions $\sum_{j=1}^{\lfloor nt\rfloor}(1-F_{n,j}(x\,|\,\mathscr{F}_{n,j-1}))$ of variable $x > 0$ are monotonic and the limiting function $t\nu(x, \infty)$ of variable $x > 0$ is continuous, we have

$$\sup_{r \leq x \leq c} \Big|\sum_{j=1}^{\lfloor nt\rfloor} (1-F_{n,j}(x\,|\,\mathscr{F}_{n,j-1})) - t\nu(x, \infty)\Big| \xrightarrow{P} 0 \quad \text{for any } c > 0.$$

Since, for any $\varepsilon > 0$, there exists $x_0 > r$ such that, for $x > x_0$, $\nu(x, \infty) < \varepsilon$, the above uniform convergence holds on the interval $[r, \infty)$, i.e.

$$\sup_{x \geq r} \Big|\sum_{j=1}^{\lfloor nt\rfloor} (1-F_{n,j}(x\,|\,\mathscr{F}_{n,j-1})) - t\nu(x, \infty)\Big| \xrightarrow{P} 0. \tag{24}$$

In a similar way we show that

$$\sup_{y \leq -r} \Big|\sum_{j=1}^{\lfloor nt\rfloor} F_{n,j}(y\,|\,\mathscr{F}_{n,j-1}) - t\nu(-\infty, y)\Big| \xrightarrow{P} 0. \tag{25}$$

Now, since $\boldsymbol{E}(\zeta_{n,j}\,|\,\mathscr{F}_{n,j-1}) = 0$, we get

$$\begin{aligned}
\boldsymbol{E}\big(\zeta_{n,j}\mathbf{1}(|\zeta_{n,j}| < r)\,|\,\mathscr{F}_{n,j-1}\big) &= -\boldsymbol{E}\big(\zeta_{n,j}\mathbf{1}(|\zeta_{n,j}| \geq r)\,|\,\mathscr{F}_{n,j-1}\big)\\
&= -\boldsymbol{E}\big(\zeta_{n,j}\mathbf{1}(\zeta_{n,j} \geq r)\,|\,\mathscr{F}_{n,j-1}\big) - \boldsymbol{E}\big(\zeta_{n,j}\mathbf{1}(\zeta_{n,j} \leq -r)\,|\,\mathscr{F}_{n,j-1}\big)\\
&= \int_r^\infty x\,d(1-F_{n,j}(x\,|\,\mathscr{F}_{n,j-1})) - \int_{-\infty}^{-r} x\,dF_{n,j}(x\,|\,\mathscr{F}_{n,j-1})\\
&= -r(1-F_{n,j}(r\,|\,\mathscr{F}_{n,j-1})) - \int_r^\infty (1-F_{n,j}(x\,|\,\mathscr{F}_{n,j-1}))\,dx\\
&\quad + rF_{n,j}(-r\,|\,\mathscr{F}_{n,j-1}) + \int_{-\infty}^{-r} F_{n,j}(x\,|\,\mathscr{F}_{n,j-1})\,dx.
\end{aligned}$$

Hence, using the definition of $b_{n,r}(t)$ from (21), we get

$$\begin{aligned}
b_{n,r}(t) &= -r\sum_{j=1}^{\lfloor nt\rfloor} (1-F_{n,j}(r\,|\,\mathscr{F}_{n,j-1})) - \int_r^\infty \Big(\sum_{j=1}^{\lfloor nt\rfloor} (1-F_{n,j}(x\,|\,\mathscr{F}_{n,j-1}))\Big)\,dx\\
&\quad + r\sum_{j=1}^{[nt]} F_{n,j}(-r\,|\,\mathscr{F}_{n,j-1}) + \int_{-\infty}^{-r} \Big(\sum_{j=1}^{\lfloor nt\rfloor} F_{n,j}(x\,|\,\mathscr{F}_{n,j-1})\Big)\,dx.
\end{aligned}$$

Therefore, using the convergences (24) and (25), we get, for $n \to \infty$, the following convergence:

$$b_{n,r}(t) \xrightarrow{P} -trv(r, \infty) - t\int_r^\infty v(x, \infty)\,dx + trv(-\infty, -r) + t\int_{-\infty}^{-r} v(-\infty, y)\,dy.$$

Integrating by parts the right-hand side of the above equality we get the form of b_r asserted in the proposition. Since the point convergence of monotonic functions to a continuous function implies the uniform convergence on compact sets, we get the statement of the proposition for all four factors in the formula for $b_{n,r}(t)$. This concludes the proof of the proposition. ∎

The i.i.d. case. In the case when random variables in $\{\zeta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ are row-wise i.i.d. we have the following corollary to Proposition 1, a result due to Prokhorov (1956):

COROLLARY 7. *Let, for each $n \geqslant 1$, $\{\zeta_{n,k},\ k \geqslant 1\}$ be a sequence of i.i.d. random variables with $\boldsymbol{E}\zeta_{n,k} = 0$ and distribution functions F_n, $n \geqslant 1$, such that the sequence $\{F_n\}$ is attracted to a Lévy distribution with spectral measure v. Then $Z_n \xrightarrow{\mathscr{D}} X$, in $D[0, \infty)$ with J_1 Skorokhod topology, where X is a Lévy process with $b(r) = 0$ in the exponent $\psi_{b,v}(u)$ in (6).*

Furthermore, if $\bar{b}_{n,r} \stackrel{df}{=} n\boldsymbol{E}\zeta_{n,1}\mathbf{1}(|\zeta_{n,1}| < r) \to b_r$, and $b_{n,r}(t) = (\lfloor nt \rfloor/n)\bar{b}_{n,r}$, then $Y_n = Z_n + b_{n,r} \xrightarrow{\mathscr{D}} X$, in $D[0, \infty)$ with J_1 Skorokhod topology, where X is a Lévy process with exponent $\psi_{b,v}(u)$ and $b(r) = b_r$.

The chain-dependent case. Another special case of martingale difference arrays $\{\zeta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$, for which we obtain sufficient conditions for processes Y_n to converge to a Lévy process, are chain-dependent sequences introduced in Section 2. Our model here is Example 4.1 from Durrett and Resnik (1978) which we adapt to the case of a series of chain-dependent sequences.

Note that the definition of chain-dependent sequence implies that $\boldsymbol{P}(\zeta_k \leqslant x \mid J_{k-1} = i) = G_i(x)$ and

$$\boldsymbol{P}\Big(\bigcap_{i=1}^{k} \{\zeta_i \leqslant x_i \mid J_0 = j_0,\ J_1 = j_1, \ldots, J_{k-1} = j_{k-1}\}\Big) = \prod_{i=1}^{k} G_{j_{i-1}}(x_i).$$

Hence

$$\boldsymbol{E}\zeta_k = \sum_{i=1}^{m} \boldsymbol{E}(\zeta_k \mid J_{k-1} = i)\boldsymbol{P}(J_{k-1} = i) = \sum_{i=1}^{m} \pi_i \int_{-\infty}^{\infty} x\,dG_i(x) \equiv a,$$

and

$$\boldsymbol{P}\Big(\bigcap_{l=1}^{s} \{\zeta_{k+l} \leqslant x_l\}\Big)$$

$$= \sum_{j_0, j_1, \ldots, j_{s-1}} \boldsymbol{P}\Big(\bigcap_{l=1}^{s} \{\zeta_{k+l} \leqslant x_l \mid J_k = j_0, \ldots, J_{k+s-1} = j_{s-1}\}\Big) \times$$

$$\times P(J_k = j_0, \ldots, J_{k+s-1} = j_{s-1})$$

$$= \sum_{j_0, j_1, \ldots, j_{s-1}} P\Big(\bigcap_{l=1}^{s} \{\zeta_l \leqslant x_l \mid J_0 = j_0, \ldots, J_{s-1} = j_{s-1}\}\Big)$$

$$\times P(J_0 = j_0, \ldots, J_{s-1} = j_{s-1})$$

$$= P\Big(\bigcap_{l=1}^{s} \{\zeta_l \leqslant x_l\}\Big).$$

The above reasoning yields the statements of Section 2 to the effect that the sequence $\{\zeta_k - a,\ k \geqslant 1\}$ is stationary and $\{(\zeta_k - a, \mathscr{B}_k),\ k \geqslant 1\}$ is a martingale difference sequence.

Let, for each $n \geqslant 1$, $\{J_{n,k},\ k \geqslant 0\}$ be a stationary, irreducible Markov chain (periodic or not) with a finite state space $S = \{1, 2, \ldots, m\}$, the transition probability matrix $P^{(n)} = \{p_{i,j}^{(n)},\ i, j \in S\}$, stationary distribution $\pi_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,m})$ and let $\{\zeta_{n,k},\ k \geqslant 1\}$ be a chain-dependent sequence with respect to $\{J_{n,k},\ k \geqslant 0\}$, and with distributions functions $G_{n,1}, G_{n,2}, \ldots, G_{n,m}$. Furthermore, let $\mathscr{B}_{n,k} = \sigma(J_{n,0}, J_{n,1}, J_{n,2}, \ldots, J_{n,k}, \zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,k})$ denote σ-fields, and $a_n = E\zeta_{n,k}$. Then, for each $n \geqslant 1$, the sequence $\{\zeta_{n,k} - a_n,\ k \geqslant 1\}$ is stationary and $\{(\zeta_{n,k} - a_n, \mathscr{B}_{n,k}),\ k \geqslant 1,\ n \geqslant 1\}$ is a martingale array.

Define

$$\pi_{n,i}(n) := \sum_{j=0}^{n} \mathbf{1}(J_{n,j} = i), \qquad 1 \leqslant i \leqslant m,\ n \geqslant 1.$$

Notice that

$$\sum_{j=1}^{\lfloor nt \rfloor} E\big(\zeta_{n,j} \mathbf{1}(|\zeta_{n,j}| < r) \,\big|\, \mathscr{B}_{n,j-1}\big)$$

$$= \sum_{j=1}^{\lfloor nt \rfloor} \sum_{i=1}^{m} E\big(\zeta_{n,j} \mathbf{1}(|\zeta_{n,j}| < r) \,\big|\, J_{n,j-1} = i\big) \mathbf{1}(J_{n,j-1} = i)$$

$$= \sum_{j=1}^{\lfloor nt \rfloor} \sum_{i=1}^{m} \int_{|x|<r} x dG_{n,i}(x) \mathbf{1}(J_{n,j-1} = i) = \sum_{i=1}^{m} \Big(n \int_{|x|<r} x dG_{n,i}(x)\Big) \Big(\frac{1}{n} \pi_{n,i}([nt])\Big).$$

Hence using the form of $b_{n,r}(t)$, defined in (21), we get

$$b_{n,r}(t) = \sum_{i=1}^{m} \Big(n \int_{|x|<r} x dG_{n,i}(x)\Big) \Big(\frac{1}{n} \pi_{n,i}([nt])\Big).$$

COROLLARY 8. *Let Markov chains $\{J_{n,k},\ k \geqslant 1\}$, $n \geqslant 1$, be such that*

$$n^{-1} \pi_{n,i}(n) \xrightarrow{P} \pi_i \ \text{ as } n \to \infty, \qquad 1 \leqslant i \leqslant m, \tag{26}$$

and assume that each of the sequences $\{G_{n,i},\ n \geqslant 1\}$, $1 \leqslant i \leqslant m$, of the distribution functions is attracted to a Lévy distribution with spectral measure ν_i, respectively, *i.e. conditions* (12)–(15) *hold with* $\{F_n\} = \{G_{n,i}\}$ *for* $1 \leqslant i \leqslant m$ *and with* $b_r = b_r^{(i)}$ *in* (14), *i.e.*

$$n \int_{|x|<r} x dG_{n,i}(x) \to b_r^{(i)} \text{ as } n \to \infty, \quad i = 1, 2, \ldots, m. \tag{27}$$

Then the processes $Z_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \zeta_{n,j} - b_{n,r}(t)$, $t \geqslant 0$, $n \geqslant 1$, *converge in* $D[0, \infty)$ *with* J_1 *Skorokhod topology to a Lévy process* X *with exponent* $\psi_{b,v}(u)$, *where* $b(r) = 0$, *and* $v = \sum_{i=1}^{m} \pi_i v_i$, *while* $Y_n = Z_n + b_{n,r}$ *converge to a Lévy process* X *with exponent* $\psi_{b,v}$, *where* $b(r) = b_r = \sum_{i=1}^{m} \pi_i b_r^{(i)}$.

Proof. To prove the corollary we will verify the conditions of Proposition 1. Notice that

$$P(\zeta_{n,j} > x \mid \mathscr{B}_{n,j-1}) = \sum_{i=1}^{m} P(\zeta_{n,j} > x \mid J_{n,j-1} = i) \mathbf{1}(J_{n,j-1} = i).$$

Therefore, for $x > 0$, we have

$$\begin{aligned}\sum_{j=1}^{\lfloor nt \rfloor} P(\zeta_{n,j} > x \mid \mathscr{B}_{n,j-1}) &= \sum_{i=1}^{m} \sum_{j=1}^{\lfloor nt \rfloor} P(\zeta_{n,j} > x \mid J_{n,j-1} = i) \mathbf{1}(J_{n,j-1} = i) \\ &= \sum_{i=1}^{m} \big(1 - G_{n,i}(x)\big) \sum_{j=1}^{\lfloor nt \rfloor} \mathbf{1}(J_{n,j-1} = i) \\ &= \sum_{i=1}^{m} n\big(1 - G_{n,i}(x)\big) \frac{1}{n} \pi_{n,i}([nt]) \xrightarrow{P} \sum_{i=1}^{m} \pi_i t v_i(x, \infty)\end{aligned}$$

for all $t \geqslant 0$ and all continuity points $x > 0$ of all spectral measures v_i.

In a similar way we show that, for all $t \geqslant 0$ and all continuity points $y < 0$ of all v_i,

$$\sum_{j=1}^{\lfloor nt \rfloor} P(\zeta_{n,j} < y \mid \mathscr{B}_{n,j-1}) = \sum_{i=1}^{m} n G_{n,i}(y) \frac{1}{n} \pi_{n,i}(\lfloor nt \rfloor) \xrightarrow{P} \sum_{i=1}^{m} \pi_i t v_i(-\infty, y), \qquad y < 0.$$

Therefore condition (a) of Proposition 1 is satisfied.

Now notice that

$$\begin{aligned}\max_{1 \leqslant j \leqslant n} P(|\zeta_{n,j}| > \varepsilon \mid \mathscr{B}_{n,j-1}) &= \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} P(|\zeta_{n,j}| > \varepsilon \mid J_{n,j-1} = i) \mathbf{1}(J_{n,j-1} = i) \\ &\leqslant \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} \big(1 - G_{n,i}(\varepsilon) + G_{n,i}(-\varepsilon)\big) \mathbf{1}(J_{n,j-1} = i) \\ &\leqslant \sum_{i=1}^{m} \big(1 - G_{n,i}(\varepsilon) + G_{n,i}(-\varepsilon)\big) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This implies that condition (b) of Proposition 1 is satisfied.

To check condition (c) recall that $\bar{\bar{\zeta}}_{n,j}$ are defined in (22). Reasoning as in Durrett and Resnik (1978) we have the following relations:

$$\sum_{j=1}^{n} E\big((\bar{\bar{\zeta}}_{n,j})^2 \,|\, \mathscr{B}_{n,j-1}\big) \leqslant \sum_{j=1}^{n} E\big(\zeta_{n,j}^2 \mathbf{1}(|\zeta_{n,j}| < \delta) \,\big|\, \mathscr{B}_{n,j-1}\big).$$

Therefore

$$\sum_{j=1}^{n} E\big((\bar{\bar{\zeta}}_{n,j})^2 \,|\, \mathscr{B}_{n,j-1}\big) \leqslant \sum_{i=1}^{m} \Big(n \int_{|x| \leqslant \delta} x^2\, dG_{n,i}(x)\Big)\Big(\frac{1}{n}\pi_{n,i}(n)\Big),$$

which shows that condition (c) of Proposition 1 is fulfilled. Therefore, applying the first assertion of Proposition 1 we get the first assertion of the corollary.

To check condition (d) of Proposition 1 notice that $n^{-1}\pi_{n,i}([nt]) \xrightarrow{p} \pi_i t$ and the limiting function is a continuous function of variable t. This and the monotonicity of the sample functions of $\pi_{n,i}(t)$ imply that

$$\sup_{0 \leqslant t \leqslant c} |n^{-1}\pi_{n,i}([nt]) - \pi_i t| \xrightarrow{p} 0 \text{ as } n \to \infty \qquad \text{for all } c > 0.$$

Now, using the form of $b_{n,r}(t)$ and the relation $b_r = \sum_{i=1}^{m} \pi_i b_r^{(i)}$, we get the inequality

$$\sup_{0 \leqslant t \leqslant c} |b_{n,r}(t) - b_r t| \leqslant \sum_{i=1}^{m} n \int_{|x| < r} x dG_{n,i}(x) \sup_{0 \leqslant t \leqslant c} |n^{-1}\pi_{n,i}([nt]) - \pi_i t|$$
$$+ c \sum_{i=1}^{m} \pi_i \Big| n \int_{|x| < r} x dG_{n,i}(x) - b_{n,r}^{(i)} \Big|,$$

which, in view of the above convergence and assumption (27), gives condition (d) of Proposition 1. This, in turn, gives the second assertion of the corollary. ∎

If the spectral measures ν_i in Corollary 8 are stable, with the same exponent α, i.e.

$$\nu_i(-\infty, -x) = \gamma_{i,1} x^{-\alpha} \qquad \text{and} \qquad \nu_i(x, \infty) = \gamma_{i,2} x^{-\alpha} \qquad \text{for all } x > 0,$$

then the spectral measure $\nu = \sum_{i=1}^{m} \nu_i$ is stable with exponent α, while $\gamma_1 = \sum_{i=1}^{m} \gamma_{i,1}$ and $\gamma_2 = \sum_{i=1}^{m} \gamma_{i,2}$.

4.2. Tightness conditions. In Proposition 3 we rewrite Theorem 3 from Szczotka and Woyczyński (2003), which gives sufficient conditions for the tightness of the sequence

$$\Big\{\frac{1}{c_n}\omega_n\Big\} := \Big\{\sup_{0 \leqslant t < \infty} \Big(X_n(t) - \frac{|a_n| \lfloor nt \rfloor}{c_n}\Big)\Big\}$$

with $X_n(t) = (1/c_n)\sum_{j=1}^{\lfloor nt \rfloor} \eta_{n,j}$, $t \geqslant 0$, where the sequence $\{\eta_{n,k}\}$ is row-wise stationary and $E\eta_{n,k} = 0$. Then we rewrite this proposition in two special cases: when $\{\eta_{n,k},\, k \geqslant 1,\, n \geqslant 1\}$ is a row-wise i.i.d. random array, and when it is a martingale difference array.

PROPOSITION 3. *Let the array* $\{\eta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ *be row-wise stationary with* $E\eta_{n,k} = 0$, *and* $|a_n| n/c_n \to \beta$, $0 < \beta < \infty$. *If*

$$\sum_{k=1}^{\infty} \sup_{n \geqslant n_0} P\left(\sup_{0 \leqslant t \leqslant \tau^k} \frac{1}{c_n} \sum_{j=1}^{\lfloor nt \rfloor} \eta_{n,j} > \kappa\tau^k\right) < \infty, \tag{28}$$

where $\tau \geqslant 2$ *and* n_0 *are integers such that* $|a_n|\, n/c_n \geqslant \beta/2$ *for* $n \geqslant n_0$ *and* $\kappa \equiv (1/2\tau)\beta$, *then the sequence* $\{(1/c_n)\,\omega_n\}$ *is tight.*

PROPOSITION 4. *Condition* (28) *is satisfied if, for some integers* $\tau \geqslant 2$ *and* n_0 *as in Proposition* 3, *one of the following three conditions is satisfied*:

(i) *For each* $n \geqslant 1$, $\{\eta_{n,k},\ k \geqslant 1\}$ *is a sequence of i.i.d. random variables such that* $d := \inf_{n \geqslant n_0} \inf_{k \geqslant 1} P\left(\sum_{j=1}^{k} \eta_{n,j} > 0\right) > 0$, *and*

$$\sum_{k=1}^{\infty} \sup_{n \geqslant n_0} P\left(\frac{1}{c_n} \sum_{j=1}^{n\tau^k} \eta_{n,j} \geqslant \kappa\tau^k\right) < \infty. \tag{29}$$

(ii) *The array* $\{\eta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ *is a martingale difference array such that, for some* δ, $1 \leqslant \delta < 2$, *the following condition holds*:

$$\sum_{k=1}^{\infty} \sup_{n \geqslant n_0} \frac{1}{\tau^{k\delta}} E\left(\frac{1}{c_n} \sum_{j=1}^{n\tau^k} \eta_{n,j}\right)_+^{\delta} < \infty, \quad \text{where } (x)_+ = \max(0, x). \tag{30}$$

(iii) $\{\eta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ *is a martingale difference array satisfying the* $(\delta, \{c_n\})$*-boundedness condition with* $\{c_n\}$ *satisfying the* $C(\tau, \delta)$*-condition.*

Proof. Applying Lemma 1.1.6, p. 9, in Iosifescu and Theodorescu (1969), with $x = 0$, to the sequence $\{\eta_{n,k},\ k \geqslant 1\}$ of i.i.d. random variables we get

$$P\left(\max_{1 \leqslant i \leqslant n\tau^k} \frac{1}{c_n} \sum_{j=1}^{i} \eta_{n,j} > \kappa\tau^k\right) \leqslant \frac{1}{d} P\left(\frac{1}{c_n} \sum_{j=1}^{n\tau^k} \eta_{n,j} > \kappa\tau^k\right),$$

which, by (29), implies (28).

The assertion in case (ii) follows immediately from Doob's inequality.

The assertion in case (iii) follows from the inequality

$$\frac{1}{\tau^{\delta k}} E\left(\frac{1}{c_n} \sum_{j=1}^{n\tau^k} \eta_{n,j}\right)_+^{\delta} \leqslant \frac{1}{\tau^{\delta k}} \left(\frac{c_{n\tau^k}}{c_n}\right)^{\delta} E\left(\frac{1}{c_{n\tau^k}} \sum_{j=1}^{n\tau^k} \eta_{n,j}\right)_+^{\delta}.$$

This concludes the proof. ∎

The following remark gives examples of sequences $\{c_n\}$ satisfying the condition $C(\tau, \delta)$.

Remark 2. The sequence $\{c_n = n^{1/\alpha} h(n)\}$, with $1 < \alpha \leqslant 2$ and a function $\{h(n)\}$ slowly varying at infinity, satisfies the condition $C(\tau, \delta)$. In particular, the sequences $\{c_n = n^{1/\alpha}\}$ and $\{c_n = n^{1/\alpha} \log n\}$ satisfy the condition $C(\tau, \delta)$.

Indeed, choose a positive ε such that $1+\varepsilon < \tau^{(\alpha-1)/\alpha}$ and an integer n_1 such that $h(n\tau)/h(n) \leqslant 1+\varepsilon$ for $n \geqslant n_1$. Then

$$\frac{h(n\tau^k)}{h(n)} = \frac{h(n\tau^{k-1}\tau)}{h(n\tau^{k-1})} \cdot \frac{h(n\tau^{k-2}\tau)}{h(n\tau^{k-2})} \cdots \frac{h(n\tau)}{h(n)} \leqslant (1+\varepsilon)^k \quad \text{for } n \geqslant n_1.$$

Hence

$$\frac{1}{\tau^{\delta k}}\left(\frac{c_{n\tau^k}}{c_n}\right)^{\delta} = \tau^{-k\delta(\alpha-1)/\alpha}\left(\frac{h(n\tau^k)}{h(n)}\right)^{\delta} \leqslant \left(\frac{1+\varepsilon}{\tau^{(\alpha-1)/\alpha}}\right)^{k\delta},$$

which, in view of the inequality $1+\varepsilon < \tau^{(\alpha-1)/\alpha}$, implies that the sequence $\{c_n = n^{1/\alpha} h(n)\}$ satisfies the condition $C(\tau, \delta)$.

PROPOSITION 5 (Sufficient conditions for the $(\delta, \{c_n\})$-boundedness condition).

(i) *If $\{c_n\}$ satisfies the condition $C(\tau, \delta)$ and, for each n, k, $E\left(\sum_{j=1}^{k} \eta_{n,j}\right)_+^{\delta} \leqslant E\left(\sum_{j=1}^{k} \vartheta_j\right)_+^{\delta}$, then the condition*

$$\limsup_{n\to\infty} E\left(\frac{1}{c_n}\sum_{j=1}^{n} \vartheta_j\right)_+^{\delta} < \infty \tag{31}$$

implies (30).

(ii) *If $\vartheta_1, \vartheta_2, \ldots$ are i.i.d. random variables with distribution belonging to the domain of attraction of the α-stable distribution, then* (31) *holds with $c_n = n^{1/\alpha} h(n)$, where $h(n)$ slowly varies at infinity, and with any δ such that $1 \leqslant \delta < \alpha < 2$.*

(iii) *If for each $n \geqslant 1$, $\{\eta_{n,k}, k \geqslant 1\}$ is a sequence of i.i.d. random variables with $\sup_n E|\eta_{n,1}|^{\delta} < \infty$ for some δ, $1 < \alpha \leqslant \delta < 2$, then $\{\eta_{n,k}, k \geqslant 1, n \geqslant 1\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition with $c_n = n^{1/\alpha} h(n)$, where $h(n)$ slowly varies at infinity.*

(iv) *If $\{\eta_{n,k}, k \geqslant 1, n \geqslant 1\}$ is a martingale difference array with $\mathrm{Var}(\eta_{n,k}) = \sigma_n^2$, $\sup_n \sigma_n < \infty$ and $c_n = n^{1/2} h(n)$, where $h(n)$ slowly varies at infinity, then $\{\eta_{n,k}, k \geqslant 1, n \geqslant 1\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition with $\delta = 2$.*

Proof. The proof of case (i) is obvious. The proof of case (ii) follows from Kwapień and Woyczyński (1992), p. 36. To prove case (iii) notice that from point 30 in Petrov (1975), p. 98, we get the following inequalities:

$$E\left|\frac{1}{c_{n\tau^k}}\sum_{j=1}^{n\tau^k} \eta_{n,j}\right|^{\delta} \leqslant \frac{2n\tau^k}{n^{\delta/\alpha}\tau^{k\delta/\alpha}\left(h(n\tau^k)\right)^{\delta}} E|\eta_{n,1}|^{\delta} \leqslant 2n^{1-\delta/\alpha}\tau^{k(1-\delta/\alpha)} \frac{1}{\left(h(n\tau^k)\right)^{\delta}} E|\eta_{n,1}|^{\delta}.$$

This and $\alpha \leqslant \delta$ imply the assertion of point (iii).

The proof of case (iv) is an immediate consequence of the identity

$$\sup_{n,k} E\left(\frac{1}{c_{n\tau^k}}\sum_{j=1}^{n\tau^k} \eta_{n,j}\right)^2 = \sup_{n,k} \frac{n\tau^k}{n\tau^k h^2(n\tau^k)} \sigma_n^2 < \infty.$$

This completes the proof of the proposition. ∎

To verify condition (i) of Proposition 5 one can take advantage of the concept of majorization of the distributions of $\eta_{n,1}$, $n \geqslant 1$, by the distribution of ϑ. We shall demonstrate how this approach works in the context of convex ordering $\leqslant_c$, introduced in Section 2. Recall that if η and ϑ are random variables with finite expectations and with distribution functions G and F, respectively, then we write $\eta \leqslant_c \vartheta$ and $G \leqslant_c F$, if $E(\eta - x)_+ \leqslant E(\vartheta - x)_+$ for all x, where $(x)_+ = \max(0, x)$. A convenient sufficient condition for $\eta \leqslant_c \vartheta$ is the so-called *cut criterion*; random variables η and ϑ with finite expectations are said to satisfy the *cut criterion of Karlin and Novikoff* (see Stoyan (1983), p. 12) if $E\eta \leqslant E\vartheta$ and if there exists an $x_0 < \infty$ such that the following inequalities hold: $G(x) \leqslant F(x)$ for all $x \leqslant x_0$, and $G(x) \geqslant F(x)$ for all $x \geqslant x_0$. If η and ϑ satisfy the cut criterion, then $\eta \leqslant_c \vartheta$, i.e. $G \leqslant_c F$ (see Proposition 1.5.1 in Stoyan (1983), p. 13).

Summarizing the above discussion we get the following results:

PROPOSITION 6. *Let $\{\vartheta_k, k \geqslant 1\}$ and $\{\eta_{n,k}, k \geqslant 1\}$, $n \geqslant 1$, be sequences of i.i.d. random variables with expectations zero and $\eta_{n,1} \leqslant_c \vartheta_1$ for all $n \geqslant 1$. If the sequence $\{\vartheta_k, k \geqslant 1\}$ satisfies condition* (31), *then the array $\{\eta_{n,k}, k \geqslant 1, n \geqslant 1\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition.*

Proof. The proposition follows immediately from the fact that the convex ordering $\leqslant_c$ is closed with respect to the operation of convolution of distributions, and from Proposition 5 (i). ∎

PROPOSITION 7. *Let, for each $n \geqslant 1$, $\{\eta_{n,k}, k \geqslant 1\}$ be chain-dependent with respect to irreducible stationary Markov chain $\{J_{n,k}, k \geqslant 0\}$ with stationary distribution $\pi_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,m})$, and distribution functions $G_{n,1}, G_{n,2}, \ldots, G_{n,m}$ with expectation zero. Furthermore, let G_i, $1 \leqslant i \leqslant m$, be the distribution functions of the centered (zero-mean) Pareto distributions with parameters (γ_i, α), $1 \leqslant i \leqslant m$, $1 < \alpha < 2$, respectively, such that*

$$G_{n,i} \leqslant_c G_i \quad \text{for all } n \geqslant 1,\ 1 \leqslant i \leqslant m.$$

Then the array $\{\eta_{n,k}, k \geqslant 1, n \geqslant 1\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition with $c_n = n^{1/\alpha}$ and any δ, $1 < \delta < \alpha$.

Proof. Let $\{\tilde{\vartheta}_{i,j}, j \geqslant 1, 1 \leqslant i \leqslant m\}$ be an array of mutually independent random variables such that the distribution function $\tilde{G}_i$ of $\tilde{\vartheta}_{i,j}$ is a Pareto distribution with parameters (γ_i, α). Since $\tilde{G}_{i_1}(x) \geqslant \tilde{G}_{i_2}(x)$ whenever $\gamma_{i_1} \leqslant \gamma_{i_2}$ (stochastic ordering $\tilde{\vartheta}_{i_1,j} \leqslant_s \tilde{\vartheta}_{i_2,j}$) and since stochastic ordering of distribution functions is closed under their convolution, we have

$$\tilde{G}_{i_1} * \tilde{G}_{i_2} * \ldots * \tilde{G}_{i_n}(x) \geqslant \tilde{G}^{*n}(x) \quad \text{for all } x,$$

where $\tilde{G}$ is a Pareto distribution function with parameter (γ, α), where $\gamma = \max_{1 \leqslant i \leqslant m} \gamma_i$. Hence

$$E\Big(\sum_{j=1}^{n} \tilde{\vartheta}_{i_j,j}\Big)_+^{\delta} \leqslant E\Big(\sum_{j=1}^{n} \tilde{\vartheta}_j\Big)_+^{\delta},$$

where $\tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots$ are i.i.d. random variables with distribution function $\tilde{G}$.

Now, let $\vartheta_{i,j} = \tilde{\vartheta}_{i,j} - E\tilde{\vartheta}_{i,j}$, $\vartheta_j = \tilde{\vartheta}_j - E\tilde{\vartheta}_j$, let G_i be the distribution function of $\vartheta_{i,j}$, and G the distribution function of ϑ_j. Then

$$(32)\qquad E\Big(\sum_{j=1}^{n} \vartheta_{i_j,j}\Big)_+^{\delta} \leqslant E\Big(\sum_{j=1}^{n} \vartheta_j\Big)_+^{\delta}.$$

Now, notice that

$$(33)\quad E\Big(\sum_{j=1}^{n} \eta_{n,j}\Big)_+^{\delta} = \sum_{i_0,i_1,\ldots,i_{n-1}} E\Big(\Big(\sum_{j=1}^{n} \eta_{n,j}\Big)_+^{\delta} \,\Big|\, J_{n,0} = i_0, J_{n,1} = i_1, \ldots, J_{n,n-1} = i_{n-1}\Big)$$
$$\times P(J_{n,0} = i_0, J_{n,1} = i_1, \ldots, J_{n,n-1} = i_{n-1}).$$

Since random variables $\eta_{n,1}, \eta_{n,2}, \ldots, \eta_{n,n}$ are conditionally independent, given the condition $\{J_{n,0} = i_0, J_{n,1} = i_1, \ldots, J_{n,n-1} = i_{n-1}\}$, in view of the inequality $G_{n,i} \leqslant_c G_i$, (32), and (33), we get

$$E\Big(\sum_{j=1}^{n} \eta_{n,j}\Big)_+^{\delta} \leqslant \sum_{i_0,i_1,\ldots,i_{n-1}} E\Big(\sum_{j=1}^{n} \vartheta_{i_{j-1},j}\Big)_+^{\delta} P(J_{n,0} = i_0, J_{n,1} = i_1, \ldots, J_{n,n-1} = i_{n-1})$$
$$\leqslant E\Big(\sum_{j=1}^{n} \vartheta_j\Big)_+^{\delta}.$$

Since the Pareto distribution belongs to the domain of attraction of the α-stable distribution, we get

$$\limsup_{n\to\infty} E\left(\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} \vartheta_j\right)_+^{\delta} < \infty,$$

which concludes the proof of the proposition. ∎

The above idea of majorization can be refined to yield the following

PROPOSITION 8. *Let processes $X_n(t) = (1/c_n)\sum_{j=1}^{\lfloor nt\rfloor}\eta_{n,j}$, $t \geqslant 0$, $n \geqslant 1$, and normalizing constants c_n be such that $X_n \xrightarrow{\mathscr{D}} X$ and $c_{n\tau}/c_n \to \tau^{1/\alpha}$, $1 < \alpha < 2$, for an integer constant $\tau > 1$. Furthermore, let for any positive $x > 1$ the following conditions be satisfied:*

$$\limsup_{k\to\infty} \frac{P\big(\sup_{0\leqslant t\leqslant 1} (c_n/c_{n\tau^k}) X_n(t\tau^k) > x^k\big)}{P\big(\sup_{0\leqslant t\leqslant 1} (1/c_n) X_n(t) > x^k\big)} < \infty,$$

$$\limsup_{k\to\infty} \frac{P\big(\sup_{0\leqslant t\leqslant 1} X_n(t) > x^k\big)}{P\big(\sup_{0\leqslant t\leqslant 1} X(t) > x^k\big)} < \infty,$$

and

$$\limsup_{k\to\infty} \tau_2^k P\big(\sup_{0\leqslant t\leqslant 1} X(t) > \tau_0^k\big) < \infty$$

for some $\tau_2 > 1$ and $\tau_0 = \tau\tau_1 > 1$, where $\tau_1 < \tau^{-1/\alpha}$. Then the array $\{\eta_{n,k}, k \geqslant 1, n \geqslant 1\}$ satisfies condition (28).

Proof. Let n_0 be an integer such that $c_n/c_{n\tau} \geqslant \tau_1$ for $n \geqslant n_0$. Then $c_n/c_{n\tau^k} \geqslant \tau_1^k$ for $n \geqslant n_0$, and

$$P\Big(\sup_{0 \leqslant t \leqslant \tau^k} X_n(t) > \tau^k\Big) = P\left(\sup_{0 \leqslant t \leqslant 1} \frac{c_n}{c_{n\tau^k}} X_n(t\tau^k) > \tau^k \frac{c_n}{c_{n\tau^k}}\right)$$
$$\leqslant P\left(\sup_{0 \leqslant t \leqslant 1} \frac{c_n}{c_{n\tau^k}} X_n(t\tau^k) > \tau_0^k\right).$$

This and the assumptions of the proposition give the following inequality:

$$\limsup_{k\to\infty} \tau_2^k P\Big(\sup_{0 \leqslant t \leqslant \tau^k} X_n(t) > \tau^k\Big) \leqslant \limsup_{k\to\infty} \frac{P\big(\sup_{0 \leqslant t \leqslant 1} (c_n/c_{n\tau^k}) X_n(t\tau^k) > \tau_0^k\big)}{P\big(\sup_{0 \leqslant t \leqslant 1} X_n(t) > \tau_0^k\big)}$$
$$\times \limsup_{k\to\infty} \frac{P\big(\sup_{0 \leqslant t \leqslant 1} X_n(t) > \tau_0^k\big)}{P\big(\sup_{0 \leqslant t \leqslant 1} X(t) > \tau_0^k\big)} \cdot \limsup_{k\to\infty} \tau_2^k P\Big(\sup_{0 \leqslant t \leqslant 1} X(t) > \tau_0^k\Big) < \infty.$$

This concludes the proof of the proposition. ∎

5. PROOFS

Proof of Theorem 1 and Corollary 1. The conditions of Theorem 1, Proposition 4 (iii), and Proposition 3 imply tightness of $\{(1/c_n)\omega_n\}$. This, jointly with the convergence $X_n \xrightarrow{\mathscr{D}} X$, $\beta_n \to \beta$ and the Heavy Traffic Invariance Principle, gives the first assertion of the theorem.

The assertion dealing with the form of the distribution of M in the case of a spectrally negative X follows from the fact that $X(t)-\beta t$ is also a spectrally negative Lévy process and $X(t)-\beta t \to -\infty$ a.s., as $t \to \infty$, because $\beta + b(r, v) - b_r > 0$. This, together with Proposition 5 (b) in Bingham (1975), implies that M has an exponential distribution with parameter λ, which is the largest root of the equation $\psi(\lambda) = 0$. As a matter of fact, there are only two roots and one of them is zero. Thus λ is the positive root of the equation $\psi(\lambda) = 0$.

The form of λ in the spectrally negative stable case follows from the special form of $\psi(u)$ in that case. More precisely, since $v(-\infty, x) = \gamma_1 |x|^{-\alpha}$ for $x < 0$, we have

$$\psi(u) = u\Big(b_r - \beta + \int_{-\infty}^{-r} x v(dx)\Big) + \int_{-\infty}^{-r} (e^{ux} - 1 - ux) v(dx) + \int_{-r}^{0-} (e^{ux} - 1 - ux) v(dx)$$
$$= -u\big(\beta + b(r, v) - b_r\big) + \int_{-\infty}^{0-} (e^{ux} - 1 - ux) v(dx)$$
$$= -u\beta + \gamma_1 \alpha \int_{-\infty}^{0-} (e^{ux} - 1 - ux) \frac{1}{|x|^{\alpha+1}} dx.$$

The last equality follows by Proposition 3 which asserts that, for a stable negative spectral case, $b_r = b(r, \nu)$. To find the positive root λ of the equation $\psi(u) = 0$ we need to solve the following equation:

$$\frac{\beta\lambda}{\alpha\gamma_1} = \int_{0+}^{\infty} (e^{-\lambda x} - 1 + \lambda x) \frac{1}{x^{\alpha+1}} dx.$$

But

$$\int_{0+}^{\infty} (e^{-\lambda x} - 1 + \lambda x) \frac{1}{x^{\alpha+1}} dx = \lambda^{\alpha} \int_{0+}^{\infty} (e^{-x} - 1 + x) \frac{1}{x^{\alpha+1}} dx = \lambda^{\alpha} \mu.$$

Hence

$$\lambda = \left(\frac{\beta}{\alpha\gamma_1 \mu}\right)^{1/(\alpha-1)},$$

which gives assertion (i).

To prove part (ii) we apply Theorem 4 in Szczotka and Woyczyński (2003), where the Lévy process $X(t) - \beta t$ has characteristic exponent $\psi_{b-\beta,\nu}(u)$. In our situation $X(t) - \beta t$ has characteristic exponent $\psi_{b_r-\beta,\nu}(u)$. Therefore, using the above-mentioned result with $b(r) = b_r$, we get assertion (ii).

Assertion (iii) follows from Theorem 8 in Szczotka and Woyczyński (2003). This completes the proof of the theorem. ∎

Proof of the Decomposition Theorem. Notice the following equalities:

$$\begin{aligned}\frac{1}{c_n}\omega_n &= \sup_{0 \leqslant t < \infty} \left(\frac{c_{n,1}}{c_n} V_n(t) + \frac{c_{n,2}}{c_n} U_n(t) - \beta_n(t)\right) \\ &= \sup_{0 \leqslant t < \infty} \left(\left(\frac{c_{n,1}}{c_n} V_n(t) - (1-p)\beta_n(t)\right) + \left(\frac{c_{n,2}}{c_n} U_n(t) - p\beta_n(t)\right)\right).\end{aligned}$$

Hence

$$\frac{1}{c_n}\omega_n \leqslant \sup_{0 \leqslant t < \infty} \left(\frac{c_{n,1}}{c_n} V_n(t) - (1-p)\beta_n(t)\right) + \sup_{0 \leqslant t < \infty} \left(\frac{c_{n,2}}{c_n} U_n(t) - p\beta_n(t)\right), \tag{34}$$

and

$$\frac{1}{c_n}\omega_n \geqslant \sup_{0 \leqslant t \leqslant c} \left(\frac{c_{n,1}}{c_n} V_n(t) - \beta_n(t) + \frac{c_{n,2}}{c_n} U_n(t)\right) \quad \text{for any } c > 0. \tag{35}$$

Let us consider case (i). Then, for sufficiently large n, we have $c_{n,2} \leqslant c_{n,1}$, so that, in view of (34), we get the following inequalities:

$$\begin{aligned}\frac{1}{c_n}\omega_n &\leqslant \sup_{0 \leqslant t < \infty} \left(V_n(t) - (1-p)\beta_n(t)\right) + \frac{c_{n,2}}{c_n} \sup_{0 \leqslant t < \infty} \left(U_n(t) - \frac{c_n}{c_{n,2}} p\beta_n(t)\right) \\ &\leqslant \frac{1}{c_{n,1}}\omega_n^{(1)} + \frac{c_{n,2}}{c_n}\frac{1}{c_{n,2}}\omega_n^{(2)}.\end{aligned} \tag{36}$$

The tightness of $\{(1/c_{n,1})\,\omega_n^{(1)}\}$ and $\{(1/c_{n,2})\,\omega_n^{(2)}\}$ yields the tightness of $\{(1/c_n)\,\omega_n\}$. Moreover,

$$\frac{c_{n,2}}{c_n}\frac{1}{c_{n,2}}\omega_n^{(2)} \xrightarrow{P} 0,$$

since $c_{n,2}/c_n \to 0$ and

$$\frac{1}{c_{n,1}}\omega_n^{(1)} \xrightarrow{\mathscr{D}} \sup_{0\leqslant t<\infty}\big(V(t)-(1-p)\,\beta t\big),$$

because of the Heavy Traffic Invariance Principle. Hence, for any $0<p<1$ and any $x>0$, being a continuity point of the distribution of $\sup_{0\leqslant t<\infty}\big(V(t)-(1-p)\,\beta t\big)$, we have the inequality

$$\limsup_{n\to\infty} \boldsymbol{P}\left(\frac{1}{c_n}\omega_n > x\right) \leqslant \boldsymbol{P}\big(\sup_{0\leqslant t<\infty}\big(V(t)-(1-p)\,\beta t\big) > x\big),$$

which, since the left-hand side is independent of p, implies that

$$\limsup_{n\to\infty} \boldsymbol{P}\left(\frac{1}{c_n}\omega_n > x\right) \leqslant \boldsymbol{P}\big(\sup_{0\leqslant t<\infty}\big(V(t)-\beta t\big) > x\big). \tag{37}$$

Now, by (35), we have

$$\boldsymbol{P}\left(\frac{1}{c_n}\omega_n > x\right) \geqslant \boldsymbol{P}\left(\sup_{0\leqslant t\leqslant c}\left(V_n(t)-\beta_n(t)+\frac{c_{n,2}}{c_n}\sup_{0\leqslant t\leqslant c}|U_n(t)|\right) > x\right)$$
$$\geqslant \boldsymbol{P}\left(\sup_{0\leqslant t\leqslant c}\left(V_n(t)-\beta_n(t)+\frac{c_{n,2}}{c_n}\sup_{0\leqslant t\leqslant c}|U_n(t)|\right) > x\right).$$

Taking the limit $\liminf_{n\to\infty}$ on both sides of the above inequality, using the convergences $V_n \xrightarrow{\mathscr{D}} V$, $\beta_n(t)\to\beta t$, and $(c_{n,2}/c_n)\sup_{0\leqslant t\leqslant c}|U_n(t)| \xrightarrow{P} 0$, and then applying the Continuous Mapping Theorem (see Theorem 5.1 in [1]) we get the following inequalities:

$$\liminf_{n\to\infty} \boldsymbol{P}\left(\frac{1}{c_n}\omega_n > x\right) \geqslant \liminf_{n\to\infty} \boldsymbol{P}\left(\sup_{0\leqslant t\leqslant c}\left(V_n(t)-\beta_n(t)+\frac{c_{n,2}}{c_n}\sup_{0\leqslant t\leqslant c}U_n(t)\right) > x\right)$$
$$\geqslant \boldsymbol{P}\big(\sup_{0\leqslant t\leqslant c}\big(V(t)-\beta t\big) > x\big).$$

Since the left-hand side above does not depend on c, we get

$$\liminf_{n\to\infty} \boldsymbol{P}\left(\frac{1}{c_n}\omega_n > x\right) \geqslant \boldsymbol{P}\big(\sup_{0\leqslant t\leqslant \infty}\big(V(t)-\beta t\big) > x\big), \tag{38}$$

which, together with (37), gives the convergence

$$\frac{1}{c_n}\omega_n \xrightarrow{\mathscr{D}} \sup_{0\leqslant t\leqslant\infty}\big(V(t)-\beta t\big).$$

To prove case (ii), where $c_{n,1}/c_n \to 0$, notice that the role of V_n and U_n is symmetric so that we can proceed here as in case (i). Thus the proof of the theorem is complete. ∎

Proof of Corollary 3. Since the array $\{-(1/c_n)(u_{n,-k}-\bar{u}_n),\ k \geqslant 1,\ n \geqslant 1\}$, with $c_n = c_{n,2} = n^{1/\alpha} h(n)$, satisfies the conditions of Corollary 7 from Section 4.1 (Prokhorov's result) with stable, spectrally negative, spectral measure ν, $\nu = (-\infty, x) = \gamma_1 |x|^{-\alpha}$, $x < 0$, we have $U_n \xrightarrow{\mathscr{D}} U$, where U is a spectrally negative Lévy process with spectral measure ν. Hence, also $\{\sup_{0 \leqslant t \leqslant c} |U_n(t)|,\ n \geqslant 1\}$ is tight for each $c > 0$.

By assumption (ii) of Corollary 3 the array $\{\eta_{n,k}^{(2)}\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition with $1 < \delta < \alpha$ and $c_n = c_{n,2}$, which, in view of Proposition 4 (ii), implies that $\{\eta_{n,k}^{(2)}\}$ satisfies condition (28) in Proposition 3 and that, in turn, implies that $\{(1/c_{n,2})\,\omega_n^{(2)}\}$ is tight.

On the other hand, applying Proposition 5 (iii) to the sequence $\{\eta_{n,k}^{(1)}\}$ in case (a) of Corollary 3 we infer that it satisfies the $(\delta, \{c_n\})$-boundedness condition with $\delta = \varepsilon$, $1 < \alpha \leqslant \varepsilon < 2$, and $c_n = c_{n,1} = n^{1/\varepsilon} h(n)$, which, in view of Proposition 4, implies that $\{\eta_{n,k}^{(1)}\}$ satisfies condition (28) of Proposition 3 and that, in turn, implies that $\{(1/c_{n,1})\,\omega_n^{(1)}\}$ is tight in case (a).

Similarly, applying Proposition 5 (iv) to the sequence $\{\eta_{n,k}^{(1)}\}$ in case (b) of Corollary 3 we infer that it satisfies the $(\delta, \{c_n\})$-boundedness condition with $\delta = 2$, $c_n = c_{n,1} = n^{1/2}$, which, by Proposition 4, implies that it satisfies condition (28) of Proposition 3 and that, in turn, implies that $\{(1/c_{n,1})\,\omega_n^{(1)}\}$ is tight in case (b).

Notice that, by Remark 1, the sequence $\{\sup_{0 \leqslant t \leqslant c} |V_n(t)|,\ n \geqslant 1\}$ is tight for each $c > 0$, in both cases (a) and (b).

Now, using the Decomposition Theorem in case (ii), we get the assertion of the corollary, which completes the proof. ∎

Proof of Corollary 4. Since the array $\{\eta_{n,k}^{(2)}\}$ satisfies the conditions of Corollary 8 with stable, spectrally negative, spectral measure ν, $\nu = \sum_{i=1}^{m} \pi_i \nu_i$, we have $U_n \xrightarrow{\mathscr{D}} U$, where U is a spectrally negative Lévy process with spectral measure ν. Hence, also $\{\sup_{0 \leqslant t \leqslant c} |U_n(t)|\}$ is tight for each $c > 0$.

From Proposition 7 it follows that the array $\{\eta_{n,k}^{(2)}\}$ satisfies the $(\delta, \{c_n\})$-boundedness condition with $c_n = c_{n,2} = n^{1/\alpha}$ and $1 < \delta < \alpha$, which, in turn, by Proposition 4 (iii), and then by Proposition 3, gives tightness of $\{(1/c_{n,2})\,\omega_n^{(2)}\}$.

Tightness of the sequence $\{(1/c_{n,1})\,\omega_n^{(1)}\}$ in cases (a) and (b) is shown in the proof of Corollary 3, where a proof of tightness of $\{\sup_{0 \leqslant t \leqslant c} |V_n(t)|,\ n \geqslant 1\}$, for each $c > 0$, is also given. Tightness of $\{(1/c_{n,1})\,\omega_n^{(1)}\}$ in case (c) follows from Remark 3 in [14], which provides sufficient conditions for tightness of $\{(1/c_{n,1})\,\omega_n^{(1)}\}$ under ϕ-mixing conditions. Also tightness of $\{\sup_{0 \leqslant t \leqslant c} |V_n(t)|,\ n \geqslant 1\}$, for each $c > 0$, follows from [14].

Using the Decomposition Theorem in case (i) completes the proof of the corollary. ∎

Proof of Corollary 5. Since the role of V_n and U_n is symmetric, the proof of the corollary is similar to the proof of Corollary 3 with $u_{n,k}$ and $v_{n,k}$ interchanged. ■

Proof of Corollary 6. With obvious adjustments, the proof follows the lines of the proof of Corollary 4, and is thus omitted. ■

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