PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 24, Fasc. 1 (2004), pp. 145–164

A FAMILY OF SYMMETRIC STABLE-LIKE PROCESSES AND ITS GLOBAL PATH PROPERTIES

BY

YASUKI ISOZAKI (OSAKA) AND TOSHIHIRO UEMURA (KOBE)

Dedicated to Professor YŌICHI OSHIMA on the occasion of His sixtieth birthday

Abstract. We extend the idea of the symmetric stable-like processes so that it includes the Brownian motion as well as the symmetric α-stable processes. We exhibit some sufficient conditions for their recurrence and conservativeness relying on the notion of Dirichlet forms. A criterion for conservativeness (Lemma 3.2) is also shown in terms of general Dirichlet forms and the associated generators.

Mathematics Subject Classification: 60J75, 60G17.

Key words and phrases: Stable-like processes, recurrence, transience.

1. INTRODUCTION

Stable-like processes are introduced originally by Bass [2], [3] in terms of martingale problems as variants for α-stable processes; they have spatially inhomogeneous "index" $\alpha(x)$. Tsuchiya [10] defined the processes in terms of stochastic differential equations with jumps. Negoro [7] introduced the processes using pseudodifferential operators and studied the magnitude of local variation for their sample paths. Jacob and Schilling [6] considered more general symbols.

Symmetric stable-like processes are introduced by the second-named author [11] in terms of Dirichlet forms and they are accordingly symmetric Markov processes with respect to the Lebesgue measure.

The aim of this paper is to introduce, extending the idea in [11], a family of symmetric processes that contains in a natural way the symmetric α-stable processes and Brownian motions. Our definition of the processes is motivated by the fact that the Dirichlet form

$$\mathscr{E}^{(\alpha)}(u, v) = \iint_{\mathbf{R}^d \times \mathbf{R}^d} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{d+\alpha}} C(\alpha)\, dx dy$$

that corresponds to the symbol $|\xi|^\alpha$ (the symmetric α-stable processes on $\boldsymbol{R}^d$) converges, as α tends to 2, in a sense to

$$\mathscr{E}^{(2)}(u, v) = \int_{\boldsymbol{R}^d} \nabla u \cdot \nabla v dx$$

that corresponds to the symbol $|\xi|^2$ (the Brownian motion in $\boldsymbol{R}^d$). Here the function $C(\alpha)$ is defined by

$$C(\alpha) = \frac{\Gamma(1+\alpha/2)\,\Gamma((\alpha+d)/2)\sin((\pi\alpha)/2)}{2^{1-\alpha}\pi^{d/2+1}}.$$

See e.g. [1]. Note that if $C(\alpha) = \alpha(2-\alpha)\tilde{C}(\alpha)$, we have

$$(1.1) \qquad 0 < C_1 := \inf\{\tilde{C}(\alpha);\ 0 < \alpha < 2\} < \sup\{\tilde{C}(\alpha);\ 0 < \alpha < 2\} =: C_2 < \infty.$$

It may now be natural to let the index α vary spatially, namely, let $\alpha(x)$ be a $(0, 2]$-valued function and replace α in the definition of $\mathscr{E}^{(\alpha)}$ with $\alpha(x)$.

An interesting observation in this paper is the following. Let us say that α is in the *transience domain* if $\mathscr{E}^{(\alpha)}$ is transient. The *recurrence domain* is defined in a similar way. We will see in the last two sections that some $\alpha(x)$ taking values in the transience domain produce recurrent processes in one and two dimensions.

We exhibit various behaviours of the symmetric stable-like processes in Table 1. For transience obtained using a comparison argument, we refer the reader to [11] or [12].

This article is organized in the following way. In Section 2, we introduce precisely the notion of the symmetric stable-like processes. Proofs of the statements therein are given at the end of the section. We show the conservativeness under some conditions in Section 3. On our way we prove Lemma 3.2, a criterion for conservativeness. Although this lemma is an easy consequence of Theorem 1.6.6 in [5], we have not found it in the literature. In Sections 4 and 5, we give some sufficient conditions for recurrence, in one- and two-dimensional cases, respectively.

TABLE 1. Behaviours of $\mathscr{E}^\alpha$ (Cp means "a comparison argument" as in [11] or [12])

$\alpha(x)$	$\mathscr{E}^\alpha$ in $1D$	$\mathscr{E}^\alpha$ in $2D$	$\mathscr{E}^\alpha$ in $3D+$
$\alpha(x) \equiv 2$	BM, recurrent	BM, recurrent	BM, transient
$\alpha(x) \nearrow 2$ rapidly	recurrent by Cp	recurrent (Theorem 5.1)	unknown
$1 < \alpha(x) \equiv \alpha < 2$	reccurrent	transient	transient
$\alpha(x) \equiv 1$	Cauchy, recurrent	Cauchy, transient	Cauchy, transient
$\alpha(x) \nearrow 1$ rapidly	recurrent (Theorem 4.1)	transient by Cp	transient by Cp
$0 < \alpha(x) \equiv \alpha < 1$	transient	transient	transient
$\alpha(x) \searrow 0$ slowly	conservative under some condition (Theorem 3.1)		

2. DEFINITIONS AND PRELIMINARIES

Let $\alpha(x)$ be a $[0, 2]$-valued measurable function on $\boldsymbol{R}^d$. For a technical reason, we assume the set

$$A := \{x \in \boldsymbol{R}^d;\ \alpha(x) = 2\}$$

is the union of countable balls modulo a Lebesgue-null set: To be precise, let

$$B(x, r) = \{y \in \boldsymbol{R}^d;\ |y-x| < r\}$$

be the ball with centre x and radius r. Then, modulo a Lebesgue-null set,

$$A = \bigcup_{i \in \boldsymbol{N}} B(x_i, r_i).$$

We introduce a *bilinear form* $\mathscr{E}^\alpha$ on $L^2(\boldsymbol{R}^d; dx)$ by

DEFINITION 2.1.

$$\mathscr{D}[\mathscr{E}^\alpha] = \Big\{u \in L^2(\boldsymbol{R}^d; dx):\ u \text{ has the derivatives of the first order on } \bigcup_{i \in \boldsymbol{N}} B(x_i, r_i),$$

$$\iint_{A^c \times \boldsymbol{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha(x)}} C(\alpha(x))\, dx dy < \infty, \text{ and } \nabla u \in L^2(A; dx)\Big\},$$

$$\mathscr{E}^{\alpha,J}(u, v) = \iint_{A^c \times \boldsymbol{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha(x)}} C(\alpha(x))\, dx dy,$$

$$\mathscr{E}^\alpha(u, v) = (\nabla u, \nabla v)_{L^2(A;dx)} + \mathscr{E}^{\alpha,J}(u, v).$$

Moreover, we set

$$\mathscr{E}^\alpha_1(u, v) = \mathscr{E}^\alpha(u, v) + (u, v)_{L^2(\boldsymbol{R}^d, dx)}, \qquad u, v \in \mathscr{D}[\mathscr{E}^\alpha].$$

Note that the integrand in the definition of $\mathscr{E}^{\alpha,J}$ is non-zero only if $\alpha(x) \in (0, 2)$, so that the integral is equivalent if it is taken over the set $\{x \in \boldsymbol{R}^d;\ \alpha(x) \in (0, 2)\} \times \boldsymbol{R}^d$.

THEOREM 2.1. *The form $(\mathscr{E}^\alpha, \mathscr{D}[\mathscr{E}^\alpha])$ is a Dirichlet form on $L^2(\boldsymbol{R}^d; dx)$ in the wide sense.*

The theorems in this section are proved at the end of the section.

It is then natural to ask if the domain $\mathscr{D}[\mathscr{E}^\alpha]$ contains $\mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$, the space of all uniformly Lipschitz continuous functions with compact support. The following theorem provides us with a plausible sufficient condition.

THEOREM 2.2. *The domain $\mathscr{D}[\mathscr{E}^\alpha]$ contains $\mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$ if and only if*

$$\int_{\{|x|>1\}} |x|^{-d-\alpha(x)} \alpha(x)\, dx < \infty. \tag{2.1}$$

In the rest of this paper, we always assume that this condition (2.1) holds.

COROLLARY 2.1. *Let $\mathscr{F}^\alpha$ be the closure of $\mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$ with respect to $\mathscr{E}^\alpha_1$. Then $(\mathscr{E}^\alpha, \mathscr{F}^\alpha)$ is a regular Dirichlet form on $L^2(\boldsymbol{R}^d)$.*

In particular, if

$$\liminf_{|x|\to\infty} \frac{|x|^{\alpha(x)}}{(\log|x|)^{1+\delta}\alpha(x)} > 0$$

with some $\delta > 0$ and the infimum of $\alpha(x)$ is positive on any compact set, then the conditions of the theorem are satisfied. To be more specific, if $\alpha(x) = \text{const}\,(\log(|x| \vee e))^{-\varepsilon}$ with $0 < \varepsilon < 1$ (see the Remark on page 85 of [11]) or $\varepsilon > 1$, then the condition (2.1) holds, but if $\alpha(x) = \text{const}\,(\log(|x| \vee e))^{-1}$, then (2.1) is violated. Note that the criterion in [11] is violated when $\varepsilon > 1$. Observe also that these examples of $\alpha(x)$ satisfy $\lim_{|x|\to\infty}\alpha(x) = 0$. Despite there is no 0-stable process, our $\mathscr{E}^\alpha$ can be defined.

Proof of Theorem 2.1. Let (u_l) be a sequence in $\mathscr{D}[\mathscr{E}^\alpha]$ such that

$$\mathscr{E}_1^\alpha(u_l - u_m, u_l - u_m) \to 0 \quad \text{as } l, m \to \infty.$$

It suffices to prove that there is a $u \in \mathscr{D}[\mathscr{E}^\alpha]$ and

$$\mathscr{E}_1^\alpha(u - u_m, u - u_m) \to 0 \quad \text{as } m \to \infty$$

since every normal contraction obviously operates on $\mathscr{D}[\mathscr{E}^\alpha]$. This will be done in a way similar to that in the proof of Theorem 2.1 in [11].

Because of closedness of the Sobolev space on each ball $B(x_i, r_i)$, there exists a function $u^i \in H^1(B(x_i, r_i))$ such that (u_l) converges to u^i in $H^1(B(x_i, r_i))$. By standard arguments, there are a subsequence $(\bar{u}_l)$ of (u_l) and a null set $N \subset \bigcup_i B(x_i, r_i)$ such that $(\bar{u}_l)$ and its derivatives converge to those of u at every point outside N. Moreover, since (u_l) is a Cauchy sequence in $L^2(\mathbf{R}^d; dx)$, u is extended over $\mathbf{R}^d$ as a function in $L^2(\mathbf{R}^d; dx)$ so that a subsequence $(\hat{u}_l) \subset (\bar{u}_l)$ converges at every point outside a null set in $\mathbf{R}^d$, say N'.

Let $\tilde{u}$ and $\tilde{u}_l$ be quasi-continuous versions of $1_{(N\cup N')^c}u$ and $1_{(N\cup N')^c}\hat{u}_l$, respectively. Then, by Fatou's lemma, we have

$$\begin{aligned}\mathscr{E}_1^\alpha(u - u_m, u - u_m) &= \mathscr{E}_1^\alpha(\tilde{u} - \tilde{u}_m, \tilde{u} - \tilde{u}_m) \\ &\leqslant \liminf_{l\to\infty} \mathscr{E}_1^\alpha(\tilde{u}_l - \tilde{u}_m, \tilde{u}_l - \tilde{u}_m) \to 0 \quad \text{as } m \to \infty. \ \blacksquare\end{aligned}$$

Proof of Theorem 2.2. Recall that the second-named author [11] has shown the following theorem. Let $\alpha(x)$ be a real-valued measurable function on $\mathbf{R}$ and let $\tilde{\mathscr{E}}^\alpha$ be defined by

$$\mathscr{D}[\tilde{\mathscr{E}}^\alpha] = \left\{u \in L^2(\mathbf{R}^d; dx)\colon \iint_{\mathbf{R}^d\times\mathbf{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha(x)}}\,dxdy < \infty\right\},$$

$$\tilde{\mathscr{E}}^\alpha(u, v) = \iint_{\mathbf{R}^d\times\mathbf{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha(x)}}\,dxdy. \tag{2.2}$$

Then the integral in the definition of $\tilde{\mathscr{E}}^{\alpha}(u, u)$ with u from $\mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$ is convergent if and only if the following conditions are satisfied:

(1) $0 < \alpha(x) < 2$ for a.e. x and $\dfrac{1}{2-\alpha(x)}, \dfrac{1}{\alpha(x)} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^d)$;

(2) there is a compact set K such that $\int_{\boldsymbol{R}^d \setminus K} |x|^{-d-\alpha(x)}\, dx < \infty$.

In our setting, the range of $\alpha(x)$ is confined to $[0, 2]$ and the Dirichlet integral on the set A is always convergent for $u \in \mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$. Moreover, the integrand in the definition of $\mathscr{E}^{\alpha}$ vanishes for x such that $\alpha(x) = 0$ since $C(0) = 0$. Hence quite the same argument in [11] leads us to conclude that the integral in the definition of $\mathscr{E}^{\alpha}(u, u)$ is convergent for all $u \in \mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^d)$ if and only if

(1′) $\dfrac{C(\alpha(x))}{2-\alpha(x)} 1_{\{x \in \boldsymbol{R}^d; 0 < \alpha(x) < 2\}}, \dfrac{C(\alpha(x))}{\alpha(x)} 1_{\{x \in \boldsymbol{R}^d; 0 < \alpha(x) < 2\}} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^d)$;

(2′) there is a compact set K such that

$$\int_{\{x \in \boldsymbol{R}^d; 0 < \alpha(x) < 2\} \setminus K} C(\alpha(x)) |x|^{-d-\alpha(x)}\, dx < \infty.$$

Since

$$C(\alpha(x)) \leqslant C_2\, \alpha(x)(2-\alpha(x))$$

and

$$\int_{\{x \in \boldsymbol{R}^d; |x| > 1, \alpha(x) > \delta\}} |x|^{-d-\alpha(x)}\, dx < \infty \quad \text{for any } \delta > 0,$$

we easily see the conditions (1′) and (2′) are equivalent to (2.1). ∎

3. CONSERVATIVENESS CRITERIA

The authors have a conjecture that our symmetric stable-like processes are conservative for practically any index function $\alpha(x)$. But this seems difficult to establish, so we write a weaker statement: We assume some smoothness of $\alpha(x)$ and write the generator explicitly to verify a sufficient condition for conservativeness.

In what follows we employ the notation

$$(3.1) \qquad M_1(x) = 1 \vee \sup_{\xi \in \boldsymbol{R}^d, |\xi| < 1} \left| \frac{\alpha(x+\xi) - \alpha(x)}{|\xi|} \right| \quad \text{for } x \in \boldsymbol{R}^d.$$

THEOREM 3.1. *Assume that $\alpha(x)$ is locally Lipschitz continuous, $\alpha(x) < 2$ for all $x \in \boldsymbol{R}^d$ and satisfies the condition* (2.1) *in Theorem* 2.2. *Then*:

(a) *$C_0^2(\boldsymbol{R}^d) \subseteq \mathscr{D}[\mathscr{E}^\alpha]$ and the generator A of the form $\mathscr{E}^\alpha$ has the following integral representation* (3.2) *on a subspace $C_0^2(\boldsymbol{R}^d) \subseteq \mathscr{D}[A]$: For $v \in C_0^2(\boldsymbol{R}^d)$,*

$$(3.2)\qquad Av(x) = \lim_{\varepsilon \to 0} \int_{y \in \boldsymbol{R}^d, |y-x| > \varepsilon} (v(y) - v(x)) \times \left(C(\alpha(x))|x-y|^{-d-\alpha(x)} + C(\alpha(y))|x-y|^{-d-\alpha(y)}\right) dy.$$

(b) *If we assume*

$$(3.3)\qquad \sup_{x \in \boldsymbol{R}^d} \int_{\{|y-x| > \max(1,|x|/2)\}} C(\alpha(y))|x-y|^{-d-\alpha(y)} dy < \infty,$$

$$(3.4)\qquad \sup_{n \in \boldsymbol{N}} n^{-1} \sup_{|x| \leqslant n} \frac{M_1(x) \exp(M_1(x)/e)}{2-\alpha(x)} < \infty,$$

$$(3.5)\qquad \sup_{n \in \boldsymbol{N}} n^{-1} \sup_{|x| \leqslant n} M_1(x)^2 \exp(M_1(x)/e) < \infty,$$

$$(3.6)\qquad \sup_{n \in \boldsymbol{N}} n^{-2} \sup_{|x| \leqslant n} \frac{2 - \inf\{\alpha(x+\xi);\ |\xi| \leqslant 1\}}{2 - \sup\{\alpha(x+\xi);\ |\xi| \leqslant 1\}} < \infty,$$

then the symmetric stable-like process is conservative.

Note that (3.3) implies the condition (2.1) in Theorem 2.2.

If $\sup_x \alpha(x) < 2$ and $\alpha(x)$ is uniformly Lipschitz continuous, all the conditions are satisfied. A more remarkable example of $\alpha(x)$ satisfying these conditions is given by $\alpha(x) = 2 - c_0 \exp(-c_1(|x| \vee 1)^2)$ with $c_0, c_1 > 0$.

If $\sup_x \alpha(x) < 1$, a similar proof shows that $\mathscr{E}^\alpha$ is conservative without assumptions (3.4)–(3.6) on smoothness. The generator is still given by (3.2) with such $\alpha(x)$, but for each $v \in C_0^1(\boldsymbol{R}^d)$ not only $C_0^2(\boldsymbol{R}^d)$. This conclusion also holds with $\alpha(x)$ that satisfies (3.3), $\sup_{x \in K} \alpha(x) < 1$ with any compact set $K \subset \boldsymbol{R}^d$, and

$$\sup_{n \in \boldsymbol{N}} n^{-1} \frac{1}{1 - \sup\{\alpha(x);\ |x| \leqslant n\}} < \infty.$$

The second-named author defined $\tilde{\mathscr{E}}^\alpha$ in [11] as in (2.2), in the present manner but without the factor $C(\alpha(\cdot))$. The generator of $\tilde{\mathscr{E}}^\alpha$ is studied in [12] and the integral representation has almost the same form as our (3.2), accordingly without $C(\alpha(\cdot))$.

If the dimension d is one or two, there are some classes of $\alpha(x)$ without smoothness that provides us with recurrent (hence conservative) $\mathscr{E}^\alpha$, as is seen in the subsequent sections. Not all of these functions $\alpha(x)$, introduced in Theorems 4.1 and 5.1, satisfy the conditions in Theorem 3.1. A typical example in one dimension is $\alpha(x) = 2 - c_0 \exp(-c_1(|x| \vee 1)^3)$ with $c_0, c_1 > 0$. Unfortunately we have not arrived at a more plausible condition for conservativeness yet.

Remark 3.1. Bass [2], [3] considered the following martingale problem: Assume

$$0 < \inf_{x\in\mathbf{R}^d} \alpha(x) < \sup_{x\in\mathbf{R}^d} \alpha(x) < 2, \tag{3.7}$$

and let B be an operator on $C_b^2(\mathbf{R}^d)$:

$$Bf(x) := \int_{\mathbf{R}^d} \big(f(x+\xi)-f(x)-\xi\cdot\nabla f(x)\,1_{\{|\xi|<1\}}\big)\, C(\alpha(x))\,|\xi|^{-d-\alpha(x)}\,d\xi. \tag{3.8}$$

A probability measure P on $\Omega = D[0,\infty)$ is said to solve the *martingale problem* for B at $x_0 \in \mathbf{R}^d$ if for any $f \in C_b^2(\mathbf{R}^d)$

$$P[X_0 = x_0] = 1,$$

$$f(X_t)-f(X_0)-\int_0^t Bf(X_s)\,ds \text{ is a } P\text{-local martingale.}$$

Then Bass showed the continuity of $\alpha(x)$ implies the existence and the Dini continuity implies the uniqueness of the solution.

If we assume (3.7) and the global Lipschitz continuity of $\alpha(x)$, our generator A has the following relation (3.9) obtained by R. Schilling (private communication). Let B^* be the dual of B. Then for u and $v \in C_0^2(\mathbf{R}^d)$

$$\begin{aligned} \mathcal{E}^{\alpha}(u,v) &= -(Au, v)_{L^2(\mathbf{R}^d;dx)} \\ &= -\big((B+B^*)u, v\big)_{L^2(\mathbf{R}^d;dx)} + \int_{\mathbf{R}^d} B^*1(x)\,u(x)\,v(x)\,dx. \end{aligned} \tag{3.9}$$

Here 1 denotes the constant function on $\mathbf{R}^d$. Namely, the symmetrized part of Bass' generator plus some perturbation (creation or killing, depending on the sign of $B^*1(x)$) equals our generator.

If we assume in addition

$$0 < \inf_{x\in\mathbf{R}^d} \alpha(x) < \sup_{x\in\mathbf{R}^d} \alpha(x) < 1, \tag{3.10}$$

the dual operator B^* has the following integral representation:

For any $f \in C_0^1(\mathbf{R}^d)$,

$$B^*f(x) = \int_{\mathbf{R}^d} \left(\frac{f(y)\,C(\alpha(y))}{|x-y|^{d+\alpha(y)}} - \frac{f(x)\,C(\alpha(x))}{|x-y|^{d+\alpha(x)}}\right) dy. \tag{3.11}$$

The following equality also holds:

$$B^*1(x) = \int_{\mathbf{R}^d} \left(\frac{C(\alpha(y))}{|x-y|^{d+\alpha(y)}} - \frac{C(\alpha(x))}{|x-y|^{d+\alpha(x)}}\right) dy. \tag{3.12}$$

Proof of Theorem 3.1 (a). Put $\Delta_\varepsilon = \{(x, y): |x-y| \leqslant \varepsilon\}$ for $\varepsilon > 0$. If $u \in \mathscr{D}[\mathscr{E}^\alpha]$ and $v \in C_0^2(\boldsymbol{R}^d)$, then the integral in the definition of $\mathscr{E}^\alpha(u, v)$ converges by Theorem 2.2, and hence

$$\mathscr{E}^\alpha(u, v) = \lim_{\varepsilon \to 0} \iint_{\boldsymbol{R}^d \times \boldsymbol{R}^d \setminus \Delta_\varepsilon} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)|x-y|^{-d-\alpha(x)} C\big(\alpha(x)\big)\, dxdy.$$

The following argument is inspired by [8] (or [12]): Let

$$J(x, y) = |x-y|^{-d-\alpha(x)} C\big(\alpha(x)\big) + |x-y|^{-d-\alpha(y)} C\big(\alpha(y)\big)$$

and, by symmetry,

$$\begin{aligned}
\Psi_0 :&= \iint_{\{|x-y| \geqslant 1\}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)|x-y|^{-d-\alpha(x)} C\big(\alpha(x)\big)\, dxdy \\
&= \frac{1}{2} \iint_{\{|x-y| \geqslant 1\}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) J(x, y)\, dxdy, \\
\Psi_1(\varepsilon) :&= \iint_{\{\varepsilon < |x-y| < 1\}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)|x-y|^{-d-\alpha(x)} C\big(\alpha(x)\big)\, dxdy \\
&= \frac{1}{2} \iint_{\{\varepsilon < |x-y| < 1\}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) J(x, y)\, dxdy.
\end{aligned}$$

Then we have

$$\mathscr{E}^\alpha(u, v) = \lim_{\varepsilon \to 0} \big(\Psi_0 + \Psi_1(\varepsilon)\big).$$

Now let

$$A_0(v, x) := \int_{\{|x-y| \geqslant 1\}} \big(v(y)-v(x)\big) J(x, y)\, dy,$$

$$A_1(v, \varepsilon, x) := \int_{\{\varepsilon < |x-y| < 1\}} (y-x) \cdot \nabla v(x) J(x, y)\, dy,$$

$$A_2(v, \varepsilon, x) := \int_{\{\varepsilon < |x-y| < 1\}} \big(v(y)-v(x)-(y-x) \cdot \nabla v(x)\big) J(x, y)\, dy.$$

Then we easily see, by using $J(x, y) = J(y, x)$,

$$\Psi_0 = -\big(u(x), A_0(v, x)\big)_{L^2(\boldsymbol{R}^d; dx)}$$

and

$$\Psi_1(\varepsilon) = -\big(u(x), A_1(v, \varepsilon, x) + A_2(v, \varepsilon, x)\big)_{L^2(\boldsymbol{R}^d; dx)}.$$

It suffices to prove that $A_0(v, \cdot)$ is square-integrable and $A_1(v, \varepsilon, \cdot) + A_2(v, \varepsilon, \cdot)$ converges in $L^2(\boldsymbol{R}^d)$ as $\varepsilon \to 0$. Now we prove the following upper bound:

LEMMA 3.1. *Let $R > 1$ be a number such that* $\operatorname{Supp}(v) \subset B(0, R)$ *and recall the notation $M_1(x)$ in* (3.1). *Then*

$$|A_0(v, x)| \leqslant \text{const}\, \|v\|_\infty R^d (|x|-R)^{-d} \quad \textit{for } |x| \geqslant 2R, \tag{3.13}$$

$$|A_0(v, x)| \leqslant \text{const}\, \|v\|_\infty \Big(1 + \int_{\{|y|>2R\}} |y|^{-d-\alpha(y)} \alpha(y)\, dy + R^d\Big) \textit{ for } |x| < 2R, \tag{3.14}$$

$$A_1(v, \varepsilon, x) = 0 \quad \textit{for } |x| \geqslant R, \tag{3.15}$$

$$\sup_{0<\varepsilon<1} |A_1(v, \varepsilon, x)| \tag{3.16}$$

$$\leqslant \text{const}\, \|\nabla v\|_\infty \left(\frac{\exp\big(M_1(x)/e\big) M_1(x)}{2-\alpha(x)} + \exp\big(M_1(x)/e\big) M_1(x)^2\right) \quad \textit{for } |x| \leqslant R,$$

$$A_2(v, \varepsilon, x) = 0 \quad \textit{for } |x| \geqslant R+1, \tag{3.17}$$

$$\sup_{0<\varepsilon<1} |A_2(v, \varepsilon, x)| \leqslant \text{const}\, \|\nabla^2 v\|_\infty \frac{2-\inf\{\alpha(x+\xi);\ |\xi| \leqslant 1\}}{2-\sup\{\alpha(x+\xi);\ |\xi| \leqslant 1\}} \tag{3.18}$$

$$\textit{for } |x| \leqslant R+1.$$

Here const *denotes a constant that depends only on d and varies from line to line.*

Proof of Lemma 3.1. If $|x| \geqslant 2R$,

$$A_0(v, x) = \int_{\{|y|<R\}} v(y) J(x, y)\, dy.$$

Hence

$$|A_0(v, x)| \leqslant \|v\|_\infty \int_{\{|y|<R\}} 4C_2 |x-y|^{-d} dy \leqslant \text{const}\, \|v\|_\infty R^d (|x|-R)^{-d},$$

which is (3.13).

If $|x| < 2R$ and $|x-y| > 3R$, then $v(y) = 0$ and $|x-y| > |y|/2$. Thus

$$|A_0(v, x)| \leqslant \int_{\{|x-y|>3R\}} |v(x)| J(x, y)\, dy + 2\|v\|_\infty \int_{\{1 \leqslant |x-y| \leqslant 3R\}} J(x, y)\, dy$$

$$< 2C_2 \alpha(x) \|v\|_\infty \int_{\{|\xi|>3R\}} |\xi|^{-d-\alpha(x)} d\xi$$

$$+ 2C_2 \|v\|_\infty \int_{\{|x-y|>3R\}} |y/2|^{-d-\alpha(y)} \alpha(y)\, dy + 4C_2 \|v\|_\infty \int_{\{1 \leqslant |\xi| \leqslant 3R\}} 1 d\xi$$

$$< 2C_2 \alpha(x) \|v\|_\infty \text{const} \int_{3R}^{\infty} t^{-1-\alpha(x)} dt$$

$$+ 2C_2 \|v\|_\infty \int_{\{|y|>2R\}} 2^{d+2} |y|^{-d-\alpha(y)} \alpha(y)\, dy + \text{const}\, \|v\|_\infty R^d,$$

where $\xi = x-y$ is a variable in $\boldsymbol{R}^d$ and t is a real variable for $|\xi|$. We have (3.14) if we observe that the second term in the rightmost side is finite by the condition (2.1) in Theorem 2.2.

We have (3.15) since $\nabla v(x) = 0$ for $|x| \geqslant R$.

We easily see that

$$A_1(v, \varepsilon, x) = \int_{\{\varepsilon < |\xi| < 1\}} \xi \cdot \nabla v(x) \left(|\xi|^{-d-\alpha(x)} C(\alpha(x)) + |\xi|^{-d-\alpha(x+\xi)} C(\alpha(x+\xi))\right) d\xi$$

$$= -\int_{\{\varepsilon < |\xi| < 1\}} \xi \cdot \nabla v(x) \left(|\xi|^{-d-\alpha(x)} C(\alpha(x)) + |\xi|^{-d-\alpha(x-\xi)} C(\alpha(x-\xi))\right) d\xi$$

and, by taking their mean, we obtain

$$A_1(v, \varepsilon, x) = \frac{1}{2} \int_{\{\varepsilon < |\xi| < 1\}} \xi \cdot \nabla v(x) \times \left(|\xi|^{-d-\alpha(x+\xi)} C(\alpha(x+\xi)) - |\xi|^{-d-\alpha(x-\xi)} C(\alpha(x-\xi))\right) d\xi .$$

We then prove

$$\text{(3.19)} \qquad \int_{\{0 < |\xi| < 1\}} |\xi|^{1-d-\alpha(x)} \left| |\xi|^{\alpha(x)-\alpha(x+\xi)} C(\alpha(x+\xi)) - |\xi|^{\alpha(x)-\alpha(x-\xi)} C(\alpha(x-\xi)) \right| d\xi$$

$$< \text{const} \left(\frac{\exp(M_1(x)/e) M_1(x)^2}{(3-\alpha(x))^2} + \frac{\exp(M_1(x)/e) M_1(x)}{2-\alpha(x)} \right).$$

To prove (3.19), note first that $|\alpha(x) - \alpha(x+\xi)| \leqslant M_1(x)|\xi|$ and $|\xi| \log(1/|\xi|) < 1/e$ for any $|\xi| < 1$. Then

$$\left|\log |\xi|^{\alpha(x)-\alpha(x-\xi)}\right| = |\alpha(x) - \alpha(x-\xi)| \log(1/|\xi|) \leqslant M_1(x)|\xi| \log(1/|\xi|) < M_1(x)/e.$$

If $t \in \boldsymbol{R}$ and $|t| < M_1(x)/e$, we easily see that $|e^t - 1| \leqslant \exp(M_1(x)/e)|t|$. Setting $t = \log(|\xi|^{\alpha(x)-\alpha(x-\xi)})$, we have

$$\left| |\xi|^{\alpha(x)-\alpha(x-\xi)} - 1 \right| \leqslant \exp(M_1(x)/e) \left| \log |\xi|^{\alpha(x)-\alpha(x-\xi)} \right|$$

$$\leqslant \exp(M_1(x)/e) M_1(x) |\xi| \log(1/|\xi|).$$

Then, since $C(\alpha)$ and $C'(\alpha)$ are bounded, we obtain

$$\begin{aligned}
&\left| |\xi|^{\alpha(x)-\alpha(x+\xi)} C(\alpha(x+\xi)) - C(\alpha(x)) \right| \\
&\leqslant \left| |\xi|^{\alpha(x)-\alpha(x+\xi)} C(\alpha(x+\xi)) - |\xi|^{\alpha(x)-\alpha(x+\xi)} C(\alpha(x)) \right| \\
&\quad + \left| |\xi|^{\alpha(x)-\alpha(x+\xi)} C(\alpha(x)) - C(\alpha(x)) \right| \\
&\leqslant \text{const}\, M_1(x) |\xi| \cdot |\xi|^{\alpha(x)-\alpha(x+\xi)} + \text{const} \exp(M_1(x)/e) M_1(x) |\xi| \log(1/|\xi|) \\
&\leqslant \text{const} \Big(M_1(x) |\xi| \left(1 + \exp(M_1(x)/e) M_1(x) |\xi| \log(1/|\xi|)\right) \\
&\quad + \exp(M_1(x)/e) M_1(x) |\xi| \log(1/|\xi|) \Big).
\end{aligned}$$

Using this twice, we get

$$\begin{aligned}&\left||\xi|^{\alpha(x)-\alpha(x+\xi)}C(\alpha(x+\xi))-|\xi|^{\alpha(x)-\alpha(x-\xi)}C(\alpha(x-\xi))\right|\\&\qquad< \mathrm{const}\big(M_1(x)|\xi|+\exp(M_1(x)/e)M_1(x)^2|\xi|^2\log(1/|\xi|)\\&\qquad\quad+\exp(M_1(x)/e)M_1(x)|\xi|\log(1/|\xi|)\big).\end{aligned}$$

Plugging this into the left-hand side of (3.19), we obtain the upper bound there by using

$$\int_0^1 t^{-1+\varepsilon}\log(1/t)\,dt = \Gamma(1)/\varepsilon^2,$$

which is valid for $\varepsilon > 0$. Now (3.16) follows immediately from (3.19) since $(3-\alpha(x))^2 \geqslant 1$.

If $|x| \geqslant R+1$ and $|\xi| < 1$, then x and $x+\xi$ are out of the support of $v(\cdot)$ and we have (3.17).

By the Taylor expansion, $|v(x+\xi)-v(x)-\xi\cdot\nabla v(x)| \leqslant \mathrm{const}\,\|\nabla^2 v\|_\infty|\xi|^2$. Then (3.18) follows from

$$\begin{aligned}&\int_{\{0<|\xi|<1\}}\left(|\xi|^{2-d-\alpha(x)}C(\alpha(x))+|\xi|^{2-d-\alpha(x+\xi)}C(\alpha(x+\xi))\right)d\xi\\&\qquad< \mathrm{const}\int_0^1\left(t^{1-\alpha(x)}(2-\alpha(x))+t^{1-\sup\{\alpha(x+\xi);|\xi|<1\}}(2-\alpha(x+\xi))\right)dt\\&\qquad< \mathrm{const}\left(1+\frac{2-\inf\{\alpha(x+\xi);\ |\xi|\leqslant 1\}}{2-\sup\{\alpha(x+\xi);\ |\xi|\leqslant 1\}}\right),\end{aligned}$$

which is also bounded by this second term. ∎

Let us resume proving that $A_0(\cdot)\in L^2(\mathbf{R}^d;\,dx)$ and $A_1(\varepsilon,\cdot)+A_2(\varepsilon,\cdot)$ converge in $L^2(\mathbf{R}^d;\,dx)$ as $\varepsilon\to 0$. Since we assume $\alpha(x)<2$ and local Lipschitz continuity, (3.16) and (3.18) are bounded and supported on a compact set. Hence the bounds in (3.13)–(3.18) are square-integrable, and this is what we need concerning $A_0(v,\cdot)$. As for $A_1(v,\varepsilon,\cdot)$, the proof of (3.16) in fact shows

$$\begin{aligned}|A_1(v,\varepsilon,x)| &= \left|\int\int 1_{\{\varepsilon<|x-y|<1\}}(x-y)\cdot\nabla v(x)\,J(x,y)\,dy\right|\\&\leqslant \int\int \sup_{0<\varepsilon<1} 1_{\{\varepsilon<|x-y|<1\}}|(x-y)\cdot\nabla v(x)|\,J(x,y)\,dy,\end{aligned}$$

which is less than or equal to the right-hand side of (3.16). Hence $\lim_{\varepsilon\to 0}A_1(v,\varepsilon,x)$ exists for any x by the dominated convergence, yielding a bounded function with a compact support. This is also the case for $A_2(v,\varepsilon,\cdot)$. Now if we define

$$Av(x) := A_0(v,x)+\lim_{\varepsilon\to 0}A_1(v,\varepsilon,x)+\lim_{\varepsilon\to 0}A_2(v,\varepsilon,x),$$

we have $\mathscr{E}^\alpha(u,v) = -(u,Av)_{L^2(\mathbf{R}^d;dx)}$. ∎

We now assume (3.3)–(3.6) to prove the conservativeness for $\mathscr{E}^\alpha$. In fact, we verify the criterion in the lemma below.

This lemma is based on an idea found in Ethier and Kurtz [4], § 3.4, that states "Boundedness and pointwise convergence imply conservativeness," of which we have been informed by R. Schilling. We should point out that Lemma 2.1 in [9] is also reminiscent of this idea.

LEMMA 3.2. *Let $(\mathscr{E}, \mathscr{F})$ be a regular Dirichlet form on $L^2(X; m)$, where X is a locally compact separable metric space and m is a positive Radon measure on X with* supp $[m] = X$. *Suppose that a core $\mathscr{C}$ of $(\mathscr{E}, \mathscr{F})$ is contained in the domain of the generator A. Assume there exists a sequence of functions $(\varphi_n) \subset \mathscr{C}$ such that*

$$0 \leqslant \varphi_n(x) \leqslant 1 \text{ and } \lim_{n\to\infty} \varphi_n(x) = 1 \text{ for } x \in X, \qquad \sup_n \|A\varphi_n\|_{L^\infty(X;m)} < \infty$$

and

$$\lim_{n\to\infty} A\varphi_n(x) = 0 \qquad \text{for } m\text{-a.e. } x \in X.$$

Then $(\mathscr{E}, \mathscr{F})$ is conservative in the sense that, for every $t > 0$,

$$T_t 1(x) = 1 \qquad \text{for } m\text{-a.e. } x \in X.$$

Proof. Since $\varphi_n \in \mathscr{C}$, we have for any $v \in \mathscr{F} \cap L^1(X; m)$

$$\mathscr{E}(\varphi_n, v) = -\int_X A\varphi_n(x) v(x) m(dx).$$

Thus the assumption and Lebesgue's dominated convergence theorem imply

$$\lim_{n\to\infty} \mathscr{E}(\varphi_n, v) = -\lim_{n\to\infty} \int_X A\varphi_n(x) v(x) m(dx) = 0,$$

which is a criterion for conservativeness given in Theorem 1.6.6 in [5]. ∎

Proof of Theorem 3.1 (b). Take a smooth function $w(t)$ on $[0, \infty)$ that satisfies

$$0 \leqslant w(t) \leqslant 1 \qquad \text{for all } t,$$

$$w(t) = \begin{cases} 1 & \text{for } t \in [0, 1], \\ 0 & \text{for } t \in (2, \infty), \end{cases}$$

$$\sup_{t\in[0,\infty)} (|w'(t)| + |w''(t)|) < \infty.$$

Set $\varphi_n(x) = w(|x|/n)$ for $x \in \mathbf{R}^d$ so that $\varphi_n(x) \to w(0) = 1$ as $n \to \infty$ for any fixed x. Note that each φ_n is a smooth function on $\mathbf{R}^d$ supported on a compact subset and accordingly belongs to the domain of A.

Since $\varphi_n(x) - \varphi_n(y) \to 0$ as $n \to \infty$ for any x and y, $A_0(\varphi_n, x) \to 0$ follows by the dominated convergence if we prove

$$B_0(x) := \int_{\{|x-y|\geqslant 1\}} \sup_n |\varphi_n(x) - \varphi_n(y)| J(x, y) dy < \infty$$

for any fixed x. To bound B_0, we replace $\varphi_n(x) - \varphi_n(y)$ by 1 and obtain

$$B_0(x) < \int_{\{|x-y|\geqslant 1\}} J(x, y) dy < \infty$$

for each x by (3.3). On the other hand, we easily see that

$$\sup_{0<\varepsilon<1} |A_1(\varphi_n, \varepsilon, x)+A_2(\varphi_n, \varepsilon, x)| \to 0$$

by Lemma 3.1, $\|\nabla\varphi_n\|_\infty < \mathrm{const}/n$ and $\|\nabla^2\varphi_n\|_\infty < \mathrm{const}/n^2$.

To prove $\sup_{n,x}|A\varphi_n(x)| < \infty$, note that $\mathrm{supp}(\varphi_n) \subseteq \{x\in \boldsymbol{R}^d;\ |x| \leqslant 2n\}$. Then we have, by (3.18) in Lemma 3.1,

$$\sup_{n,x}|\lim_{\varepsilon\to 0} A_2(\varphi_n, \varepsilon, x)| \leqslant \sup_n \frac{\mathrm{const}}{n^2} \sup_{|x|\leqslant 2n+1} \frac{2-\inf\{\alpha(x+\xi);\ |\xi| \leqslant 1\}}{2-\sup\{\alpha(x+\xi); |\xi| \leqslant 1\}},$$

which is finite by (3.6). Similarly, $\sup_{n,x}|\lim_{\varepsilon\to 0} A_1(\varphi_n, \varepsilon, x)|$ is finite by (3.4), (3.5), (3.15) and (3.16).

To bound $\sup_{n,x}|A_0(\varphi_n, x)|$, we define

$$I(x) := \int_{\{|y-x|>\max(1,|x|/2)\}} J(x, y)\,dy.$$

Note that (3.3) implies $\sup_x I(x) < \infty$.

If $|x| < 2/3$, the integral defining $I(x)$ is taken over $\{|y-x| > 1\}$ and for any n we have $|A_0(\varphi_n, x)| < I(x)$ since $|\varphi_n(x)-\varphi_n(y)| \leqslant 1$. Otherwise, if $n > \frac{3}{2}|x|$ or $4n < |x|$, we have $\varphi_n(x)-\varphi_n(y) = 0$ for any y in $\{|y-x| \leqslant |x|/2\}$, and thus the integral defining $A_0(\varphi_n, x)$ can be taken over $\{y\in \boldsymbol{R}^d;\ |y-x| > \max(1, |x|/2)\}$ and $|A_0(\varphi_n, x)| < I(x)$. Now, it is enough to consider x such that $(2n)/3 \leqslant |x| \leqslant 4n$. But then

$$|\varphi_n(x)-\varphi_n(y)| \leqslant \mathrm{const}\frac{|x-y|}{n} \leqslant \mathrm{const}\frac{|x-y|}{|x|/4}.$$

Putting these together, we have, for all n,

$$|A_0(\varphi_n, x)| \leqslant I(x)+\int_{\{1<|y-x|\leqslant|x|/2\}} \frac{2|y-x|}{|x|} J(x, y)\,dy$$

$$\leqslant I(x)+|x|^{-1}\mathrm{const}\int_{\{1<|\xi|<|x|/2\}} |\xi|^{1-d-\inf\{\alpha(\eta);|\eta|\leqslant 3|x|/2\}}\,d\xi$$

$$= I(x)+|x|^{-1}\mathrm{const}\int_1^{|x|/2} t^{-\inf\alpha}\,dt \leqslant I(x)+|x|^{-1}\mathrm{const}\int_1^{|x|/2} 1\,dt \leqslant I(x)+\mathrm{const}.$$

Then $\sup_{n,x}|A_0(\varphi_n, x)| \leqslant \sup_x\big(I(x)+\mathrm{const}\big) < \infty$. ∎

4. ONE-DIMENSIONAL CASE

One-dimensional symmetric α-stable processes are *transient* if $\alpha < 1$ and *recurrent* if $\alpha \geqslant 1$.

An interesting achievement in [11] is as follows: Let $\mathscr{E}^\alpha$ be as defined in (2.2). If the function $\alpha(x)$ takes its values in (0, 1), the domain of transience, but close to 1 for $|x|$ large, then the symmetric stable-like process associated with $\tilde{\mathscr{E}}^{\alpha}$ is recurrent! We revise this problem in the context of $\mathscr{E}^{\alpha}$ and observe that a comparison theorem holds.

Let $\mathscr{F}_e(\mathscr{E}^{\alpha})$ be the extended Dirichlet space of the Dirichlet form $\mathscr{E}^{\alpha}$ (see e.g. [5]).

THEOREM 4.1. (a) *Assume the condition* (2.1) *in Theorem* 2.2 *and the following are satisfied*:

(1) $\limsup_{n\to\infty} \int_{\{|x|<n\}} n^{-\alpha(x)}\,dx < \infty,$

(2) $\limsup_{n\to\infty} \int_{\{|y|<2n\}} dy \int_{\{|x|>3n\}} \dfrac{2-\alpha(x)}{|x-y|^{1+\alpha(x)}}\,dx < \infty.$

Then the space $\mathscr{F}_e(\mathscr{E}^{\alpha})$ *contains the constant function* 1, *and hence the symmetric stable-like process associated with* $(\mathscr{E}^{\alpha}, \mathscr{F}^{\alpha})$ *is recurrent.*

(b) *If in addition*

$$\alpha(x) \leqslant \beta(x) \leqslant 2 \text{ a.e. } x \in \boldsymbol{R}^d,$$

then the process associated with $(\mathscr{E}^{\beta}, \mathscr{F}^{\beta})$ *is also recurrent.*

Remark 4.1. (i) The following function $\alpha(x)$ is defined in Example 2 on page 90 of [11]: Let $\varepsilon \geqslant 1$, $c \in (0, 2)$, $a > e$, and

$$\alpha(x) = \begin{cases} 1-(\log|x|)^{-\varepsilon}, & |x| > a, \\ c, & \text{otherwise.} \end{cases}$$

Then the conditions in the above theorem can be verified by a similar argument to that in [11], which fact fills a column in Table 1. The recurrence of $\mathscr{E}^{\alpha}$ also follows from the equivalence argument concerning $\tilde{\mathscr{E}}^{\alpha}$ and $\mathscr{E}^{\alpha}$.

(ii) By Theorem 4.1 (b), the recurrence for $\mathscr{E}^{\beta}$ is concluded if $\varepsilon > 0$, $c \in (0, 2)$, $a > 1$, and

$$\beta(x) = \begin{cases} 2-|x|^{-\varepsilon}, & |x| > a, \\ c, & \text{otherwise.} \end{cases}$$

It is here that our specification of the Dirichlet form makes a real difference: This function $\beta(x)$ is defined in Example 3 (for the case $0 < \varepsilon \leqslant 1$) and Example 3′ (for $\varepsilon > 1$) on page 90 of [11] and the recurrence criterion on page 88 of [11] is not satisfied in the latter example.

Proof of Theorem 4.1. The statement (b) follows from (a) since the integrals in the conditions (1) and (2) are decreasing in $\alpha(x)$.

Before proving (a), note that the condition (2.1) in Theorem 2.2 is satisfied under the present assumptions, whence $\mathscr{D}[\mathscr{E}^{\alpha}] \supset \mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R})$.

For any $n \in N$, let $u_n(x)$ be the function on R defined by $u_n(x) = 1$ for $x \in [-n, n]$, $u_n(x) = 0$ for $|x| \geqslant 2n$ and $u_n(x) = 1-(|x|-n)/n$ for $n < |x| < 2n$. Then (u_n) is a sequence in $\mathscr{C}_0^{\mathrm{Lip}}(R)$ that tends to 1 everywhere as $n \to \infty$. In [11], the following estimate has been shown:

$$\begin{aligned}\tilde{\mathscr{E}}^{\alpha}(u_n, u_n) &:= \iint_{R\times R} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{1+\alpha(x)}} dxdy \\ &\leqslant \text{const} \int_{-2n}^{2n} \frac{n^{-\alpha(x)}}{\alpha(x)} dx + \text{const} \int_{\{|y|<2n\}} dy \int_{\{|x|>3n\}} \frac{1}{|x-y|^{1+\alpha(x)}} dx \\ &\quad + \text{const} \int_{-3n}^{3n} \frac{n^{-\alpha(x)}}{2-\alpha(x)} dx.\end{aligned}$$

We have, by the same argument,

$$\begin{aligned}\mathscr{E}^{\alpha,J}(u_n, u_n) \leqslant\ &\text{const} \int_{-2n}^{2n} \frac{n^{-\alpha(x)}}{\alpha(x)} C(\alpha(x))\, dx + \text{const} \int_{\{|y|<2n\}} dy \int_{\{|x|>3n\}} \frac{C(\alpha(x))}{|x-y|^{1+\alpha(x)}} dx \\ &+ \text{const} \int_{-3n}^{3n} \frac{n^{-\alpha(x)}}{2-\alpha(x)} C(\alpha(x))\, dx.\end{aligned}$$

Since $C(\alpha(x)) \leqslant C_2\,\alpha(x)(2-\alpha(x))$ by (1.1), the right-hand side remains bounded as $n \to \infty$ under the assumptions (1) and (2).

We reproduce from now on the proof of Theorem 3.2 in [11] and add a modification.

Relying on the standard techniques of the Hilbert space theory, we have a sequence of convex combinations

$$v_i = a(i, 1)\, u_1 + \ldots + a(i, i)\, u_i$$

of (u_n) such that $\lim_{i\to\infty} a(i, n) = 0$ for any n and (v_i) is an $\mathscr{E}^{\alpha,J}$-Cauchy sequence that converges to 1 for a.e. x. Hence $1 \in \mathscr{F}_e(\mathscr{E}^{\alpha,J})$. On the other hand, the form $\mathscr{E}^{\alpha,J}$ is represented by

$$\mathscr{E}^{\alpha,J}(u, v) = \iint (\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y)) \frac{C(\alpha(x))}{|x-y|^{1+\alpha(x)}} dxdy$$

for $u, v \in \mathscr{F}_e(\mathscr{E}^{\alpha,J})$, where $\tilde{u}$ and $\tilde{v}$ are quasi-continuous modifications of u and v, respectively. Then we have $\mathscr{E}^{\alpha,J}(1, 1) = 0$, and hence $\mathscr{E}^{\alpha,J}(v_i, v_i) \to 0$ as $i \to \infty$.

Since $(\nabla u_n, \nabla u_n) = O(1/n) \to 0$ as $n \to \infty$, we easily have $(\nabla v_i, \nabla v_i) \to 0$ as $i \to \infty$. Hence (v_i) is also an $\mathscr{E}^{\alpha}$-Cauchy sequence that satisfies $\mathscr{E}^{\alpha}(v_i, v_i) \to 0$. Finally, we have $1 \in \mathscr{F}_e(\mathscr{E}^{\alpha})$ and $\mathscr{E}^{\alpha}(1, 1) = 0$, which implies the recurrence. ∎

5. TWO-DIMENSIONAL CASE

The planar Brownian motion is a recurrent diffusion process while two-dimensional symmetric α-stable processes (where $0<\alpha<2$) are transient. In the following theorem, any value taken by $\alpha(x)$ is in the domain of transience. Nevertheless $(\mathscr{E}^{\alpha}, \mathscr{D}[\mathscr{E}^{\alpha}])$ is recurrent, and this fact fills the corresponding column in Table 1.

THEOREM 5.1. (a) *Assume that*

$$0<\alpha(x)\leqslant 2 \text{ for a.e. } x \quad \text{and} \quad \alpha(x)=2-O\left(\frac{1}{\log|x|}\right) \text{ as } |x|\to\infty.$$

Then the condition (2.1) *in Theorem* 2.2 *is satisfied with this choice of* $\alpha(x)$ *and the symmetric stable-like process associated with* $(\mathscr{E}^{\alpha}, \mathscr{F}^{\alpha})$ *is recurrent.*

(b) *If in addition* $\alpha(x)\leqslant\beta(x)\leqslant 2$ *for a.e.* x, *the process associated with* $(\mathscr{E}^{\beta}, \mathscr{F}^{\beta})$ *is also recurrent.*

Proof of Theorem 5.1. This proof is done in almost the same way as that of Theorem 4.1.

Since $\alpha(x)\to 2$ as $|x|\to\infty$, the condition (2.1) in Theorem 2.2 is satisfied and $\mathscr{D}[\mathscr{E}^{\alpha}]$ contains $\mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^2)$.

For any $n\in\boldsymbol{N}$, let $u_n(x)$ be the function on $\boldsymbol{R}^2$ defined by $u_n(x)=1$ for $|x|\leqslant n$, $u_n(x)=0$ for $|x|\geqslant 2n$ and $u_n(x)=1-(|x|-n)/n$ for $n\leqslant|x|\leqslant 2n$. Then (u_n) is a sequence in $\mathscr{C}_0^{\mathrm{Lip}}(\boldsymbol{R}^2)$ that tends to 1 everywhere as $n\to\infty$.

We begin by showing that $\sup_n \mathscr{E}^{\alpha,J}(u_n, u_n)<\infty$ implies recurrence. Let $(\bar{u}_k)$ be the sequence of convex combinations of (u_n) defined by

$$\bar{u}_k=k^{-1}(u_1+u_2+u_4+\ldots+u_{2^{k-1}}).$$

Then

$$(\nabla\bar{u}_k, \nabla\bar{u}_k)_{L^2(A;dx)}\leqslant(\nabla\bar{u}_k, \nabla\bar{u}_k)_{L^2(\boldsymbol{R}^2;dx)}=O(1/k) \quad \text{as } k\to\infty$$

and $\sup_k \mathscr{E}^{\alpha,J}(\bar{u}_k, \bar{u}_k)<\infty$. By the same reasoning as in the proof of Theorem 4.1, there exists a sequence (v_i) in $\mathscr{D}[\mathscr{E}^{\alpha}]$ that converges a.e. to 1 and satisfies $\mathscr{E}^{\alpha}(v_i, v_i)\to 0$, which implies recurrence.

Now the statement (b) is an easy consequence of (a). Indeed, if we set $\xi=x-y$ in the first integral below, we have

$$\begin{aligned}\mathscr{E}^{\beta,J}(u_n, u_n)\leqslant &\iint_{\{|x-y|\leqslant 1\}}\frac{(u_n(x)-u_n(y))^2}{|x-y|^{2+\beta(x)}}C(\beta(x))\,dxdy\\ &+\iint_{\{|x-y|\geqslant 1\}}\frac{(u_n(x)-u_n(y))^2}{|x-y|^{2+\beta(x)}}C(\beta(x))\,dxdy\end{aligned}$$

$$\leqslant \frac{\text{const}}{n^2} \int_{\{|x| \leqslant 2n\}} \int_{\{|\xi| \leqslant 1\}} |\xi|^{-\beta(x)} (2-\beta(x))\, d\xi dx$$

$$+ \iint_{\{|x-y| \geqslant 1\}} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{2+\alpha(x)}} C(\alpha(x))\, dxdy$$

$$\leqslant \frac{\text{const}}{n^2} \operatorname{vol}(\overline{B(0, 2n)}) + \mathscr{E}^{\alpha, J}(u_n, u_n).$$

Here we have used $C(\alpha) \leqslant 2C_2(2-\alpha)$ and $|x-y|^{-\beta(x)} \leqslant |x-y|^{-\alpha(x)}$ for any x, y such that $|x-y| \geqslant 1$ and $\alpha(x) \leqslant \beta(x)$. Since $\operatorname{vol}(\overline{B(0, 2n)}) = O(n^2)$, $\sup_n \mathscr{E}^{\alpha, J}(u_n, u_n) < \infty$ implies $\sup_n \mathscr{E}^{\beta, J}(u_n, u_n) < \infty$.

By the same reasoning, we only have to show that $\sup_n \mathscr{E}^{\bar{\alpha}, J}(u_n, u_n) < \infty$ for some $\bar{\alpha}(x)$ such that $\bar{\alpha}(x) \leqslant \alpha(x)$ for a.e. x, $2-\bar{\alpha}(x) < \text{const}/(\log|x|)$ when $|x|$ is large and $\bar{\alpha}(x)$ is rotationally invariant, i.e., $\bar{\alpha}(x)$ is a function of $|x|$, which will be denoted by $\bar{\alpha}(|x|)$ with an abuse of the notation. Note also that $u_n(x) = w(x/n)$ by some function $w(\cdot)$ on $\boldsymbol{R}$. Then we have, by setting $y = \eta e^{\sqrt{-1}\theta}$ and $\eta = r|x|$,

$$\mathscr{E}^{\bar{\alpha}, J}(u_n, u_n) \leqslant \iint_{\boldsymbol{R}^2 \times \boldsymbol{R}^2} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{2+\bar{\alpha}(x)}} C(\bar{\alpha}(x))\, dxdy$$

$$\leqslant \int_{\boldsymbol{R}^2} C_2 \bar{\alpha}(|x|)(2-\bar{\alpha}(|x|))\, dx \int_{\boldsymbol{R}^2} \frac{(w(|x|/n)-w(|y|/n))^2}{|x-y|^{2+\bar{\alpha}(|x|)}}\, dy$$

$$= \int_{\boldsymbol{R}^2} C_2 \bar{\alpha}(|x|)(2-\bar{\alpha}(|x|))\, dx$$

$$\times \int_0^\infty (w(|x|/n)-w(\eta/n))^2 \left(\int_0^{2\pi} \frac{d\theta}{|x-\eta e^{\sqrt{-1}\theta}|^{2+\bar{\alpha}(|x|)}} \right) \eta d\eta$$

$$= \int_{\boldsymbol{R}^2} C_2 \bar{\alpha}(|x|)(2-\bar{\alpha}(|x|))\, dx$$

$$\times \int_0^\infty (w(|x|/n)-w(r|x|/n))^2 \left(\int_0^{2\pi} \frac{d\theta}{|x-r|x|e^{\sqrt{-1}\theta}|^{2+\bar{\alpha}(|x|)}} \right) |x|^2 r dr.$$

Let $f(\alpha, r)$ be a function defined by

$$f(\alpha, r) := \int_0^{2\pi} \frac{d\theta}{|1-re^{\sqrt{-1}\theta}|^{2+\alpha}}$$

for $0 < \alpha < 2$ and $r \in (0, 1) \cup (1, \infty)$. Using

$$\int_0^{2\pi} \frac{d\theta}{|x-r|x|e^{\sqrt{-1}\theta}|^{2+\bar{\alpha}(|x|)}} = |x|^{-2-\bar{\alpha}(|x|)} f(\bar{\alpha}(|x|), r)$$

and the rotation invariance of the integrand, we have, by setting $|x| = n\xi$,

$$\begin{aligned}\mathscr{E}^{\bar{\alpha},J}(u_n, u_n) &\leqslant \int_0^\infty 2\pi n\xi C_2\,\bar{\alpha}(n\xi)\big(2-\bar{\alpha}(n\xi)\big)\,d(n\xi)\\ &\quad\times (n\xi)^{-\bar{\alpha}(n\xi)}\int_0^\infty \big(w(\xi)-w(r\xi)\big)^2 f\big(\bar{\alpha}(n\xi), r\big)\,r\,dr\\ &\leqslant \int_0^\infty 2\pi n^{2-\bar{\alpha}(n\xi)}\,\xi^{1-\bar{\alpha}(n\xi)}\,C_2\,\bar{\alpha}(n\xi)\big(2-\bar{\alpha}(n\xi)\big)\,d\xi\\ &\quad\times \int_0^\infty \big(w(\xi)-w(r\xi)\big)^2 f\big(\bar{\alpha}(n\xi), r\big)\,r\,dr.\end{aligned}$$

We are going to prove that the right-hand side is bounded if

$$\sup_{n>1}\ \sup_{t\in[\sqrt{n},\infty)} n^{2-\bar{\alpha}(t)} < \infty. \tag{5.1}$$

Note that this condition is satisfied if $2-\bar{\alpha}(x) < \mathrm{const}/(\log|x|)$ for large $|x|$. Let

$$I(n, \xi) := \int_0^\infty \big(w(\xi)-w(r\xi)\big)^2 f\big(\bar{\alpha}(n\xi), r\big)\,r\,dr$$

for $n \in N$ and $\xi \geqslant 0$. Then we have the following estimates:

LEMMA 5.1. (a) $I(n, \xi) = O\big(1/(2-\bar{\alpha}(n\xi))\big)$ *for* $1/3 < \xi < 3$.
(b) $I(n, \xi) = O(\xi^{-2})$ *for* $\xi > 3$.
(c) $I(n, \xi) = O\big(\xi^{\bar{\alpha}(n\xi)}/\bar{\alpha}(n\xi)\big)$ *for* $\xi < 1/3$.

We postpone its proof. Let K be a positive constant specified later and we divide the following integral into five parts:

$$\begin{aligned}&\int_0^\infty \xi^{1-\bar{\alpha}(n\xi)}\,n^{2-\bar{\alpha}(n\xi)}\,\bar{\alpha}(n\xi)\big(2-\bar{\alpha}(n\xi)\big)\,I(n, \xi)\,d\xi\\ &\qquad = \int_0^{K/n} + \int_{K/n}^{1/\sqrt{n}} + \int_{1/\sqrt{n}}^{1/3} + \int_{1/3}^{3} + \int_3^\infty := A_0 + A_1 + A_2 + A_3 + A_4.\end{aligned}$$

By Lemma 5.1 (c) and $\bar{\alpha}(n\xi) > 0$, A_0 is bounded as follows: by setting $\zeta = n\xi$, we obtain

$$A_0 < \mathrm{const}\int_0^{K/n} \xi n^{2-\bar{\alpha}(n\xi)}\,d\xi = \mathrm{const}\int_0^K \zeta n^{-\bar{\alpha}(\zeta)}\,d\zeta,$$

which tends to 0 by the dominated convergence.

To prove that A_1 is bounded, let K be a number such that $2-\bar{\alpha}(t) \leqslant 1$ for all $t \geqslant K$. Then

$$A_1 < \mathrm{const}\int_{K/n}^{1/\sqrt{n}} \xi n^{2-\bar{\alpha}(n\xi)}\,d\xi \leqslant \mathrm{const}\, n\int_{K/n}^{1/\sqrt{n}} \xi\,d\xi = O(1).$$

The rest is easy. For any $\delta > 0$,

$$A_2 < \Big(\sup_{n>1, t\geqslant\sqrt{n}} n^{2-\bar\alpha(t)}\Big) \int_{1/\sqrt{n}}^{1/3} \text{const}\, \xi d\xi,$$

$$A_3 < \Big(\sup_{n>1, t\geqslant\sqrt{n}} n^{2-\bar\alpha(t)}\Big) \int_{1/3}^{3} \text{const}\, (\xi \vee \xi^{-1}) d\xi,$$

$$A_4 < \Big(\sup_{n>1, t\geqslant\sqrt{n}} n^{2-\bar\alpha(t)}\Big) \int_{3}^{\infty} \text{const}\, \xi^{-1+\sup\{2-\bar\alpha(t); t\geqslant 3n\}} \xi^{-2} d\xi,$$

which are all bounded since $\inf_{n>1, t>\sqrt{n}} \bar\alpha(t) > 0$ and $\lim_{t\to\infty} 2-\bar\alpha(t) = 0$. ∎

Proof of Lemma 5.1. Note first that the function $r \mapsto f(\alpha, r)$ is easily seen to be continuous separately on the interval $0 \leqslant r < 1$ and on $1 < r < \infty$; its behaviour near the boundary is as follows:

$$f(\alpha, 0) = 2\pi,$$

$$f(\alpha, r) \sim |1-r|^{-1-\alpha} \left(\int_{-\infty}^{\infty} \frac{dt}{|t+\sqrt{-1}|^{2+\alpha}} \right) \quad \text{as } r \to 1,$$

$$f(\alpha, r) \sim 2\pi r^{-2-\alpha} \quad \text{as } r \to \infty.$$

The function

$$\int_{-\infty}^{\infty} \frac{dt}{|t+\sqrt{-1}|^{2+\alpha}}$$

of α is continuous on the interval $-1 < \alpha < \infty$, and hence bounded on $0 \leqslant \alpha \leqslant 2$.

Since $0 \leqslant w(\eta) \leqslant 1$, we have $|w(\xi) - w(r\xi)| \leqslant 1$. It is clear that w is Lipschitz continuous and $|w(\xi) - w(r\xi)| \leqslant \text{const}\, \xi |1-r|$ on any neighbourhood of $r = 1$.

If $1/3 < \xi < 3$ and $\delta > 0$, we have

$$I(n, \xi) < \left(\sup_{0\leqslant\alpha\leqslant 2} \int_{-\infty}^{\infty} \frac{dt}{|t+\sqrt{-1}|^{2+\alpha}} \right) \int_{1-\delta}^{1+\delta} 9\xi^2 |1-r|^2 \cdot |1-r|^{-1-\bar\alpha(n\xi)} r dr$$

$$+ \int_{0}^{1-\delta} \text{const}\, r dr + \int_{1+\delta}^{\infty} \text{const}\, r^{-2-\bar\alpha(n\xi)} dr$$

$$< \frac{\text{const}}{2-\bar\alpha(n\xi)} + \text{const}.$$

If $\xi > 3$, then $\varphi(\xi) = 0$ and $\varphi(r\xi) \neq 0$ only if $r < 2/\xi < 2/3$. Then

$$I(n, \xi) < \int_{0}^{2/\xi} 1 \cdot f\big(\bar\alpha(n\xi), r\big) r dr = O\big((2/\xi)^2\big)$$

since $f(\alpha, r)$ is continuous and bounded on $[0, 2/3]$.

If $\xi < 1/3$, $\varphi(\xi) = 1$, and hence $(\varphi(\xi) - \varphi(r\xi))^2 = 0$ for any $r \leqslant 1/\xi$. We have then

$$I(n, \xi) < \int_{1/\xi}^{\infty} 1 \cdot f(\bar{\alpha}(n\xi), r) r dr < \int_{1/\xi}^{\infty} \mathrm{const}\, r^{-1-\bar{\alpha}(n\xi)} dr = O(\xi^{\bar{\alpha}(n\xi)}/\bar{\alpha}(n\xi)). \blacksquare$$

REFERENCES

[1] N. Aronszajn and K. T. Smith, *Theory of Bessel potentials I*, Ann. Inst. Fourier (Grenoble) 11 (1961), pp. 385–475.

[2] R. F. Bass, *Uniqueness in law for pure jump Markov processes*, Probab. Theory Related Fields 79 (1988), pp. 271–287.

[3] R. F. Bass, *Occupation time densities for stable-like processes and other pure jump Markov processes*, Stochastic Process. Appl. 29 (1988), pp. 65–83.

[4] S. N. Ethier and T. G. Kurtz, *Markov Processes – Characterization and Convergence*, Wiley, New York 1986.

[5] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin 1994.

[6] N. Jacob and R. L. Schilling, *Lévy-type processes and pseudodifferential operators*, in: *Lévy Processes: Theory and Applications*, O. E. Barndorff-Nielsen et al. (Eds.), Birkhäuser, Princeton 2001, pp. 139–167.

[7] A. Negoro, *Stable-like processes: construction of the transition density and the behavior of sample paths near $t = 0$*, Osaka J. Math. 31 (1994), pp. 189–214.

[8] Y. Ogura, M. Tomisaki and M. Tsuchiya, *Convergence of local type Dirichlet forms to a non-local type one*, Ann. Inst. H. Poincaré Probab. Statist. 38 (2002), pp. 507–556.

[9] R. L. Schilling, *Conservativeness of semigroups generated by pseudo-differential operators*, Potential Anal. 9 (1998), pp. 91–104.

[10] M. Tsuchiya, *Lévy measure with generalized polar decomposition and the associated SDE with jumps*, Stochastics Stochastics Rep. 38 (1992), pp. 95–117.

[11] T. Uemura, *On some path properties of symmetric stable-like processes for one dimension*, Potential Anal. 16 (2002), pp. 79–91.

[12] T. Uemura, *On symmetric stable-like processes: Some path properties and generators*, J. Theoret. Probab. (to appear).

Yasuki Isozaki
Department of Mathematics
Graduate School of Science, Osaka University
Toyonaka, Osaka, 560-0043, Japan
E-mail: yasuki@math.sci.osaka-u.ac.jp

Toshihiro Uemura
Department of Management Science
Kobe University of Commerce
Nishi-ku, Kobe, 651-2197, Japan
E-mail: uemura@kobeuc.ac.jp

Received on 7.5.2003;
revised version on 19.4.2004