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DIAGONALIZABILITY OF NON-HOMOGENEOUS QUANTUM MARKOV STATES AND ASSOCIATED VON NEUMANN ALGEBRAS

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Abstract. We give a constructive proof of the fact that any Markov state (even non-homogeneous) on $\overline{\bigotimes_{j\in\mathbf{Z}} M_{d_j}}^{C^*}$ is diagonalizable. However, due to the local entanglement effects, they are not necessarily of Ising type (Theorem 3.2). In addition, we prove that the underlying classical measure is Markov, and therefore, in the faithful case, it naturally defines a nearest neighbour Hamiltonian. In the translation invariant case, we prove that the spectrum of the two-point block of this Hamiltonian, in some cases, uniquely determines the type of the von Neumann factor generated by the Markov state (Theorem 5.3). In particular, we prove that, if all the quotients of the differences of two such eigenvalues are rational, then this factor is of type III_λ for some $\lambda\in(0, 1)$, and that, if this factor is of type III_1, then these quotients cannot be all rational. We conjecture that the converses of these statements are also true.

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1. INTRODUCTION

It is known that, in quantum statistical mechanics, concrete systems are identified with states on corresponding algebras. In many cases, the algebra is a quasi-local C^*-algebra of observables. The states satisfying the Kubo–Martin–Schwinger (KMS for short) boundary condition, as known, describe equilibrium states of the quantum system under consideration. On the other hand, for classical systems with finite radius of interaction, limiting Gibbs measures are known to be Markov random fields; see e.g. [13], [20], [25]. In connection with this, it is natural to address the problem of constructing quantum analogues of

Markov chains, the last arising from quantum statistical mechanics, or quantum field theory in a natural way. This problem was firstly explored in [1] by introducing the quantum Markov chains on the algebra of quasi-local observables. In the last decades, the investigation of quantum Markov processes had a considerable growth, in view of natural applications to quantum statistical mechanics, quantum field theory or quantum information theory as well. The reader is referred to [1], [3]–[8], [10], [16], and the references cited therein, for recent developments of the theory of quantum stochastic processes and their applications.

The investigation of a particular class of quantum Markov chains, called *quantum Markov states*, was pursued in [3], [4], [6], [7], where connections with properties of the modular operator of the states under consideration were established. This provides natural applications to temperature states arising from suitable quantum spin models, that is natural connections with the KMS boundary condition.[1]

In [3], the most general one-dimensional quantum Markov state has been considered. Among the other results concerning the structure of such states, the connection with classes of local Hamiltonians satisfying certain commutation relations and quantum Markov states has been obtained. The situation arising from quantum Markov states on the one-dimensional ordered chain describes some models of statistical mechanics with mutually commuting nearest neighbour interactions.

In the present paper, we clarify the meaning of diagonalizability of one-dimensional non-homogeneous quantum Markov states. Namely, in Section 3 we prove that, for each Markov state φ on the spin algebra

$$\mathfrak{M} := \overline{\bigotimes_{j\in\mathbf{Z}} M_{d_j}(\mathbf{C})}^{C^*},$$

there exist a suitable maximal Abelian subalgebra $\mathfrak{D} \subset \mathfrak{M}$ (called *diagonal* in the sequel), a Umegaki conditional expectation $\mathfrak{E}: \mathfrak{M} \mapsto \mathfrak{D}$ and a Markov measure μ on $\mathrm{spec}(\mathfrak{D})$ such that $\varphi = \varphi_\mu \circ \mathfrak{E}$, the Markov state φ_μ being the state on $\mathfrak{D}$ arising from the measure μ. This allows us also to clarify a question raised in Section 6 of [3], relative to the role played by the non-commuting boundary terms naturally arising from quantum Markov states, see Section 4 below.

Diagonal Markov states were considered in [27]. In [18], the diagonalizability of more general one-dimensional translation invariant quantum Markov states on the forward chain was proved, but not the Markovianity of the underlying classical measure. The proof in [18] of diagonalizability depends on the commuting square condition (3.10) for the increasing sequence of Umegaki conditional expectations. The proof of (3.10), omitted in [18], heavily depends

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[1] Most of the states arising from Markov processes considered in [16] describe ground states (i.e. states at zero temperature) of certain models of quantum spin chains.

on the fine structure of the local expected subalgebras and the corresponding potentials, first investigated in detail in [3].

Section 5 of this paper is devoted to determine the type of the von Neumann factors arising from the GNS representation of the quantum Markov states. This is done by using the explicit form of the nearest neighbour Hamiltonian associated with the quantum Markov state. We prove that the spectrum of the two-point block of this Hamiltonian, in some cases, uniquely determines the type of the von Neumann factor generated by the Markov state. In particular, we prove that, if all the quotients of the differences of two such eigenvalues are rational, then this factor is of type III_λ for some $\lambda \in (0, 1)$, and that, if this factor is of type III_1, then these quotients cannot be all rational. This classification result, in the form established in Theorem 5.3, is not known even for the Ising model, or for states arising from classical Markov chains, the last treated in some detail in Section 5. We conjecture that the converses of these statements are also true. At present, it is still an open problem.

We end by noticing that in the literature there are many examples of diagonal liftings of Markov states on factorizable Abelian algebras, e.g. the Ising model. However, the states considered here are diagonal liftings of classical Markov processes on non-factorizable Abelian subalgebras.[2] Thus, they provide concrete constructive examples of a situation abstractly considered in [31].

2. PRELIMINARIES

We start with recalling some well-known facts about inclusions of finite-dimensional C^*-algebras.

Let $N \subset M$ be an inclusion of finite-dimensional C^*-algebras. Consider the finite sets $\{p_i\}$, $\{q_j\}$ of all the minimal central projections of M, N, respectively. We symbolically write

$$\sum_j q_j N \subset \sum_i p_i M.$$

Let us set $M_i := M_{p_i}$, $N_j := N_{q_j}$, $M_{ij} := M_{p_i q_j}$, $N_{ij} := N_{p_i q_j}$. Then we have inclusions $N_{ij} \subset M_{ij}$ of finite-dimensional factors. Hence

$$M_{ij} \sim N_{ij} \otimes \bar{N}_{ij} \tag{2.1}$$

for other finite-dimensional factors $\bar{N}_{ij}$.[3]

[2] Other nontrivial quantum liftings of classical Markov chains are constructed and studied in [5].

[3] The square root of the dimension of $\bar{N}_{ij}$ is precisely the multiplicity of which the piece $q_j N \subset N$ appears into the piece $p_i M \subset M$.

Consider the canonical traces Tr_M and Tr_N, that is the traces which assign weight one to minimal projections. Notice that $\mathrm{Tr}_M = \mathrm{Tr}_M \circ E$, where E is the conditional expectation of M onto $\sum_{i,j} q_j (p_i M) q_j$ given by

$$E(x) = \sum_{i,j} q_j p_i x q_j.$$

Taking into account the identification (2.1) and the last considerations, one can write symbolically

$$\mathrm{Tr}_M = \bigoplus_{i,j} (\mathrm{Tr}_{N_{ij}} \otimes \mathrm{Tr}_{\bar{N}_{ij}}).$$

Furthermore, the completely positive $(\mathrm{Tr}_M, \mathrm{Tr}_N)$-preserving linear map E_N^M of M onto N is given by

$$E_N^M = \bigoplus_{i,j} (\mathrm{id}_{N_{ij}} \otimes \mathrm{Tr}_{\bar{N}_{ij}}). \tag{2.2}$$

Let φ be a positive functional on M, together with its restriction $\varphi\lceil_N$ to N. Consider the corresponding Radon–Nikodym derivatives T_M^φ and T_N^φ with respect to the canonical traces Tr_M and Tr_N, respectively. We get

$$T_N^\varphi = E_N^M (T_M^\varphi). \tag{2.3}$$

The starting point of our analysis is the C^*-infinite tensor product

$$\mathfrak{M} := \overline{\bigotimes_{j \in \boldsymbol{Z}} M_j}^{C^*},$$

where for $j \in \boldsymbol{Z}$

$$M_j = M_{d_j}(\boldsymbol{C}). \tag{2.4}$$

With abuse of the notation, we denote by the same symbols elements of local algebras and their canonical embeddings into bigger (local) algebras if this causes no confusion. For $k \leqslant l$, we denote by $M_{[k,l]}$ the local algebra relative to the segment $[k, l] \subset \boldsymbol{Z}$. Let $\mathscr{S}(\mathfrak{M})$ be the set of all states on $\mathfrak{M}$. The restriction of a state $\varphi \in \mathscr{S}(\mathfrak{M})$ to $M_{[k,l]}$ will be denoted by $\varphi_{[k,l]}$.

Suppose we have an increasing sequence $\{N_{[k,l]}\}_{k \leqslant l}$ of local algebras such that

$$N_{[k,k]} \subset M_{[k,k]} \equiv M_k, \qquad N_{[k,k+1]} \subset M_{[k,k+1]},$$
$$M_{[k,l]} \subset N_{[k-1,l+1]} \subset M_{[k-1,l+1]}, \qquad k \leqslant l.$$

Consider an increasing sequence of C^*-algebras $\{D_{[k,l]}\}_{k \leqslant l}$, where $D_{[k,l]}$ is maximal Abelian in $N_{[k,l]}$.

A *diagonal algebra* $\mathfrak{D} \subset \mathfrak{M}$ is the Abelian C^*-subalgebra of $\mathfrak{M}$ obtained as

$$\mathfrak{D} := \overline{(\varinjlim_{[k,l]\uparrow \mathbf{Z}} D_{[k,l]})}^{C^*}$$

for $D_{[k,l]}$ and $N_{[k,l]}$ as above.

We deal only with *locally faithful* states (i.e. states on $\mathfrak{M}$ with faithful restrictions to local subalgebras) even if most of the forthcoming analysis applies to non-faithful states as well. For $\varphi \in \mathscr{S}(\mathfrak{M})$, locally faithful, the *generalized conditional expectation*, or *φ-expectation*, $\varepsilon^{\varphi}_{k,l}$: $M_{[k,l+1]} \mapsto M_{[k,l]}$ is the completely positive φ-preserving linear map associated with the inclusion $M_{[k,l]} \subset M_{[k,l+1]}$ defined in [2]. We refer the reader to that paper for the precise definition and further details about the Accardi–Cecchini generalized conditional expectation.

3. DIAGONALIZABILITY OF MARKOV STATES

Let $\varphi \in \mathscr{S}(\mathfrak{M})$ be a locally faithful state.

DEFINITION 3.1. The state $\varphi \in \mathscr{S}(\mathfrak{M})$ is said to be a *Markov state* if, for $k, l \in \mathbf{Z}$, $k < l$, we have

$$\varepsilon^{\varphi}_{k,l}\lceil_{M_{[k,l-1]}} = \mathrm{id}_{M_{[k,l-1]}}.$$

Quantum Markov states were firstly studied in [1] and [6]. Among other potential applications, they are relevant in quantum statistical mechanics. The structure of quantum Markov states was intensively studied in [3] and [7], where most of their properties were understood. Here, we report some useful results relative to the structure of Markov states. We refer the reader to [3] for details and proofs.

After taking the ergodic limit of the φ-expectations $\varepsilon^{\varphi}_{k,l}$, and a decreasing martingale limit ([3], Section 5), it is possible to recover a sequence $\{\mathscr{E}^j\}_{j \in \mathbf{Z}}$ of transition expectations which are Umegaki conditional expectations

$$\mathscr{E}^j\colon M_j \otimes M_{j+1} \mapsto R_j \subset M_j$$

such that

$$\varphi_{[k,l]}(A_k \otimes \ldots \otimes A_l) = \varphi_{[k,k]}\big(\mathscr{E}^k(A_k \otimes \ldots \otimes \mathscr{E}^{l-1}(A_{l-1} \otimes A_l)\ldots)\big) \tag{3.1}$$

for every $k, l \in \mathbf{Z}$ with $k < l$, and $A_k \otimes \ldots \otimes A_{l-1} \otimes A_l$ any linear generator of $M_{[k,l]}$. Let $\{P^j_{\omega_j}\}_{\omega_j \in \Omega_j}$ be the set of all minimal central projections of the range $R_j = \mathscr{R}(\mathscr{E}^j)$ of $\mathscr{E}^j$. Put

$$B_j := \sum_{\omega_j \in \Omega_j} P^j_{\omega_j} M_j P^j_{\omega_j} \quad \text{and} \quad B_{[k,l]} := \bigotimes_{k \leq j \leq l} B_j.$$

Consider the conditional expectation $E^j\colon M_j \mapsto B_j$ given by

$$E^j(A) := \sum_{\omega_j \in \Omega_j} P^j_{\omega_j} A P^j_{\omega_j}.$$

Define

$$E_{[k,l]} := \bigotimes_{k \leqslant j \leqslant l} E^j. \tag{3.2}$$

By (3.1), it is easy to show that

$$\varphi_{[k,l]} = \varphi_{[k,l]} \circ E_{[k,l]}.$$

After the identification $M_{jP^j_{\omega_j}} \cong P^j_{\omega_j} M_j P^j_{\omega_j}$ (i.e. the reduced algebra $M_{jP^j_{\omega_j}}$ acting on $P^j_{\omega_j} C^{d_j}$), we have

$$M_{jP^j_{\omega_j}} = N^j_{\omega_j} \otimes \bar{N}^j_{\omega_j}$$

for finite-dimensional factors $N^j_{\omega_j}$ and $\bar{N}^j_{\omega_j}$. Thus, we can write

$$B_{[k,l]} := \bigoplus_{\omega_k, \dots, \omega_l} (N^k_{\omega_k} \otimes \bar{N}^k_{\omega_k}) \otimes \dots \otimes (N^l_{\omega_l} \otimes \bar{N}^l_{\omega_l}). \tag{3.3}$$

Consider the potentials $\{h_{M_{[k,l]}}\}_{k \leqslant l}$ obtained by the formula

$$\varphi_{[k,l]} = \mathrm{Tr}_{M_{[k,l]}}(\exp(-h_{M_{[k,l]}}) \cdot).$$

Then $h_{M_{[k,l]}}$ has the nice decomposition

$$h_{M_{[k,l]}} = \bigoplus_{\omega_k, \dots, \omega_l} h^k_{\omega_k} \otimes h^k_{\omega_k, \omega_{k+1}} \otimes \dots \otimes h^{l-1}_{\omega_{l-1}, \omega_l} \otimes \hat{h}^l_{\omega_l} \tag{3.4}$$

for selfadjoint elements $h^j_{\omega_j}$, $\hat{h}^j_{\omega_j}$, $h^j_{\omega_j, \omega_{j+1}}$ localized in $N^j_{\omega_j}$, $\bar{N}^j_{\omega_j}$, $\bar{N}^j_{\omega_j} \otimes N^{j+1}_{\omega_{j+1}}$, respectively. After defining

$$H_j := \sum_{\omega_j} P^j_{\omega_j} (h^j_{\omega_j} \otimes I) P^j_{\omega_j}, \qquad \hat{H}_j := \sum_{\omega_j} P^j_{\omega_j} (I \otimes \hat{h}^j_{\omega_j}) P^j_{\omega_j},$$

$$H_{j,j+1} := \sum_{\omega_j, \omega_{j+1}} (P^j_{\omega_j} \otimes P^{j+1}_{\omega_{j+1}})(I \otimes h^j_{\omega_j, \omega_{j+1}} \otimes I)(P^j_{\omega_j} \otimes P^{j+1}_{\omega_{j+1}}),$$

we find sequences of selfadjoint operators $\{H_j\}_{j \in \mathbf{Z}}$, $\{\hat{H}_j\}_{j \in \mathbf{Z}}$ localized in $M_{[j,j]} \equiv M_j$, and $\{H_{j,j+1}\}_{j \in \mathbf{Z}}$ localized in $M_{[j,j+1]}$, respectively, satisfying the commutation relations

$$[H_j, H_{j,j+1}] = [H_{j,j+1}, \hat{H}_{j+1}] = [H_j, \hat{H}_j] = [H_{j,j+1}, H_{j+1,j+2}] = 0 \tag{3.5}$$

such that

$$h_{M_{[k,l]}} = H_k + \sum_{j=k}^{l-1} H_{j,j+1} + \hat{H}_l \tag{3.6}$$

for each $k \leqslant l$.

In Section 5 of [3] it is proved also the converse. Namely, if $\varphi \in \mathcal{S}(\mathfrak{M})$ is locally faithful, with potentials having the form (3.6), for addenda localized as above, and satisfying the commutation relations (3.5), then it is a Markov state.

We are ready to prove the diagonalizability result for quantum Markov states.

THEOREM 3.2. *Let $\varphi \in \mathcal{S}(\mathfrak{M})$ be a Markov state. Then there exist a diagonal algebra $\mathfrak{D} \subset \mathfrak{M}$, a classical Markov process with Markov measure μ on* $\mathrm{spec}(\mathfrak{D})$ *with respect to the same order localization of $\mathbf{Z}$, and a Umegaki conditional expectation $\mathfrak{E}\colon \mathfrak{M} \mapsto \mathfrak{D}$ such that $\varphi = \varphi_\mu \circ \mathfrak{E}$, where φ_μ is the state on $\mathfrak{D}$ corresponding to the measure μ.*

Proof. Let R_j be the range of the transition expectation $\mathcal{E}^j$ with relative commutant $R_j^c := R_j' \wedge M_j$. Define

$$N_{[k,k]} := Z(R_k), \qquad N_{[k,k+1]} := R_k^c \otimes R_{k+1},$$

$$N_{[k,l]} := R_k^c \otimes M_{[k+1,l-1]} \otimes R_l, \qquad k < l+1.$$

For each $k \leqslant j < l$, and $\omega_j \in \Omega_j$, choose a maximal Abelian subalgebra $D^j_{\omega_j,\omega_{j+1}}$ of $\bar{N}^j_{\omega_j} \otimes N^{j+1}_{\omega_{j+1}}$ containing $h^j_{\omega_j,\omega_{j+1}}$. Put

$$D_{[k,k]} := N_{[k,k]} \equiv Z(R_k),$$

$$D_{[k,l]} := \bigoplus_{\omega_k,\ldots,\omega_l} (D^k_{\omega_k,\omega_{k+1}} \otimes \ldots \otimes D^{l-1}_{\omega_{l-1},\omega_l}), \qquad k < l,$$

$$\mathfrak{D} := \overline{(\varinjlim_{[k,l]\uparrow \mathbf{Z}} D_{[k,l]})}^{C^*}.$$

According to our definition, $\mathfrak{D}$ is a diagonal algebra of $\mathfrak{M}$. Consider the potentials $h_{N_{[k,l]}}$ associated with the restrictions $\varphi\lceil_{N_{[k,l]}}$. We get, by (2.3),

$$\exp(-h_{N_{[k,l]}}) = E^{M_{[k,l]}}_{N_{[k,l]}}(\exp(-h_{M_{[k,l]}})).$$

Taking into account (2.2) and (3.4), we obtain

$$h_{N_{[k,l]}} = K_k + \sum_{j=k}^{l-1} H_{j,j+1} + \hat{K}_l \tag{3.7}$$

for

$$\begin{aligned} K_j &:= -\sum_{\omega_j} \ln\big(\mathrm{Tr}_{N^j_{\omega_j}} \exp(-h^j_{\omega_j})\big) P^j_{\omega_j}, \\ \hat{K}_j &:= -\sum_{\omega_j} \ln\big(\mathrm{Tr}_{\bar{N}^j_{\omega_j}} \exp(-\hat{h}^j_{\omega_j})\big) P^j_{\omega_j}. \end{aligned} \tag{3.8}$$

Summarizing, by restricting ourselves to the sequence $\{N_{[k,l]}\}_{k\leqslant l}$, we find a collection $\{h_{N_{[k,l]}}\}_{k\leqslant l}$ of mutually commuting potentials, with $h_{N_{[k,l]}} \in D_{[k,l]}$, arising from a nearest neighbour interaction; see (3.5), (3.7), (3.8). Namely, $\{h_{N_{[k,l]}}\}_{k\leq l}\subset\mathfrak{D}$.

Let $E_{k,l}$: $N_{[k,l]}\mapsto D_{[k,l]}$ be the canonical conditional expectation of $N_{[k,l]}$ onto the maximal Abelian subalgebra $D_{[k,l]}$.[4] We have

$$\varphi\lceil_{N_{[k,l]}}\equiv\mathrm{Tr}_{N_{[k,l]}}(\exp(-h_{N_{[k,l]}})\cdot)=\mathrm{Tr}_{N_{[k,l]}}(\exp(-h_{N_{[k,l]}})E_{k,l}(\cdot)). \tag{3.9}$$

Furthermore,

$$E_{k-1,l+1}\lceil_{N_{[k,l]}}=E_{k,l}. \tag{3.10}$$

Indeed, by projectivity,

$$E_{k,l}=E_{k,l}\circ E_{[k,l]}$$

with $E_{[k,l]}$ given in (3.2). The compatibility condition (3.10) immediately follows by (3.3).

Let $\varphi_\mu:=\varphi\lceil_{\mathfrak{D}}$, where μ is the probability measure on $\mathrm{spec}(\mathfrak{D})$ associated with $\varphi\lceil_{\mathfrak{D}}$. By (3.10),

$$\mathfrak{E}_0:=\varinjlim_{[k,l]\uparrow\boldsymbol{Z}}E_{k,l}$$

is well defined on $\bigcup_{k,l}N_{[k,l]}$ (which is a dense subalgebra of $\mathfrak{M}$), and extends by continuity to a Umegaki conditional expectation $\mathfrak{E}$ of $\mathfrak{M}$ onto $\mathfrak{D}$. Furthermore, by (3.9), $\varphi=\varphi\circ\mathfrak{E}_0\equiv\varphi_\mu\circ\mathfrak{E}_0$ on localized elements of $\mathfrak{M}$. By a standard continuity argument, we obtain $\varphi=\varphi_\mu\circ\mathfrak{E}$. The fact that μ is a Markov measure on $\mathrm{spec}(\mathfrak{D})$ with respect to the order localization of $\boldsymbol{Z}$ is checked in the Appendix. ∎

The diagonalizability result for homogeneous quantum Markov states on the forward chain is contained in [18] without any mention about the Markovianity of the underlying classical processes. As in our situation, the proof of the diagonalizability in Theorem 4.1 of [18] heavily depends on the commuting square condition (3.10). In the most general situation considered here (hence, including the case considered in [18]), (3.10) easily follows by a direct inspection of the structure of local expected subalgebras and potentials investigated in detail in [3], and reported in the present paper for the convenience of the reader.

We end by noticing that Theorem 3.2 can be proved for non-homogeneous processes on one-side (forward or backward) ordered chains. By looking at the

[4] Let $M\equiv\sum_i p_iM$ be a finite-dimensional C^*-algebra, $\{p_i\}$ being the set of its minimal central projections, and $D\subset M$ a maximal Abelian subalgebra. Then there exists a complete set of matrix units $\{e^i_{k_il_i}\}$ for M such that D is generated by the diagonal part $\{e^i_{k_ik_i}\}$. The canonical expectation E of M onto the diagonal algebra D is easily given by

$$E\Big(\sum_{i,k_i,l_i}a^i_{k_il_i}e^i_{k_il_i}\Big)=\sum_{i,k_i}a^i_{k_ik_i}e^i_{k_ik_i}.$$

support projections of the local restrictions of the states (or, equivalently, by defining the Markov property directly in terms of Umegaki transition expectations, see [3], Definition 2.1), it is straightforward to prove Theorem 3.2 even for general (not necessarily locally faithful) Markov states on ordered one-dimensional lattices.

4. THE STRUCTURE OF THE ASSOCIATED HAMILTONIAN: LOCAL ENTANGLEMENT

In standard models of statistical mechanics describing classical or quantum spin systems, one considers, on a quasi-local algebra $\mathfrak{A}$, local Hamiltonians $\{h_\Lambda\}_{\Lambda\subset \mathbf{Z}^d}$, Λ bounded, satisfying suitable conditions. Then one constructs the finite volume Gibbs states (to simplify matter, we reduce ourselves to the case with inverse temperature $\beta=1$)

$$\varphi_\Lambda := Z^{-1}\operatorname{Tr}_{\mathfrak{A}_\Lambda}(\exp(-h_\Lambda)\cdot), \tag{4.1}$$

Z being the partition function, see e.g. [11], [26], [28]. The local Hamiltonian h_Λ is usually based on an interaction term describing the mutual interaction of all spins in the volume Λ, and a boundary term arising from some fixed boundary conditions imposed on the spins surrounding the region Λ. After extending the φ_Λ to all of $\mathfrak{A}$, each $*$-weak limit $\lim_{\Lambda_n\uparrow \mathbf{Z}^d}\varphi_{\Lambda_n}$ of the net $\{\varphi_\Lambda\}_{\Lambda\subset \mathbf{Z}^d}$ is an infinite volume Gibbs state, or a Dobrushin–Lanford–Ruelle state (KMS state in quantum setting) for the system under consideration; see e.g. [14], [15], [19], [21].

In the classical case, it is established for finite range interactions that an infinite volume Gibbs state arises from a δ-Markov process and *vice versa*, δ being the range of the interaction, see e.g. [13], [20], [25]. For ordered unidimensional chains, a quantum analogue of that result is proved in [3], provided that the "leading" terms $\{H_{j,j+1}\}_{j\in\mathbf{Z}}$ commute with each other, see also [4] for connected results relative to the multidimensional case. In the quantum setting, it can happen that $\{h_\Lambda\}_{\Lambda\subset\mathbf{Z}^d}$ does not generate a commutative algebra due to the boundary effects (see [3], Section 6).

In the present paper we have shown that, starting from a quantum Markov state on

$$\mathfrak{M} \equiv \overline{\bigotimes_{j\in\mathbf{Z}} M_{d_j}(\mathbf{C})}^{C^*},$$

we can recover a nontrivial filtration $\{N_{[k,l]}\}_{k\leq l}$ of $\mathfrak{M}$ and an increasing sequence $\{D_{[k,l]}\}_{k\leq l}$ of Abelian algebras with the $D_{[k,l]}$ nontrivial (i.e. not factorizable) maximal Abelian subalgebras of the $N_{[k,l]}$, such that φ is the lifting of $\varphi\lceil_{\mathfrak{D}}$, the last one being a classical Markov state on

$$\mathfrak{D} := \overline{(\varinjlim_{[k,l]\uparrow\mathbf{Z}} D_{[k,l]})}^{C^*},$$

constructed by the compatible sequence of Umegaki conditional expectations $E_{k,l}\colon N_{[k,l]} \mapsto D_{[k,l]}$ preserving the canonical trace $\mathrm{Tr}_{N_{[k,l]}}$.[5] This is possible as the nearest neighbour potentials $\{h_{N_{[k,l]}}\}_{k \leq l}$ generate a commutative subalgebra of $\mathfrak{D}$.

As straightforwardly seen, the converse is also true. Namely, one can start with any fixed filtration $\{N_{[k,l]}\}_{k \leq l}$ as above, together with a nearest neighbour interaction

$$h_{k,l} = \sum_{j=k}^{l-1} H_{j,j+1} \tag{4.2}$$

with $\{H_{j,j+1}\}_{j \in \boldsymbol{Z}}$ mutually commuting. By adding boundary terms K_k and $\hat{K}_l$ to (4.2) such that all addenda commute with each other, one can construct, for finite regions $\Lambda = [k, l]$, finite volume Gibbs states $\{\varphi_\Lambda\}_{\Lambda \subset \boldsymbol{Z}}$ as in (4.1), associated with the Hamiltonian

$$h_{[k,l]} = K_k + h_{k,l} + \hat{K}_l$$

having the same form as in (3.7).

Each $*$-weak limit point of the sequence $\{\varphi_\Lambda\}_{\Lambda \subset \boldsymbol{Z}}$ gives rise to a Markov state on $\mathfrak{M}$ which is the lifting of a classical Markov state on a suitable "diagonal" algebra, due to the commutativity of the $h_{[k,l]}$.

Now, the following remark is in order. In our generic situation, the spectral resolution of the two-point block of the Hamiltonian has the form

$$H_{n,n+1} = \sum_{i,j} \kappa_{ij}^{n,n+1} e_{(i,j)(i,j)}^{n,n+1}, \tag{4.3}$$

where $\{e_{(i,j)(k,l)}^{n,n+1}\} \subset M_{d_n}(\boldsymbol{C}) \otimes M_{d_{n+1}}(\boldsymbol{C})$ is a suitable system of matrix units for $M_{d_n}(\boldsymbol{C}) \otimes M_{d_{n+1}}(\boldsymbol{C})$. It it in general impossible, for any choice of the system of matrix units $\{e_{ik}^n\} \subset M_{d_n}(\boldsymbol{C})$, to write (4.3) as

$$H_{n,n+1} = \sum_{i,j} \kappa_{ij}^{n,n+1} e_{ii}^n \otimes e_{jj}^{n+1},$$

the last being the typical form of the interaction appearing in the Ising model, see (5.2). The generic case, when the spectral projections of two-point block of the Hamiltonian cannot be factorizable as above, has the meaning of a *local entanglement effect.*

Taking into account the above considerations, one can assert that each quantum Markov state on $\boldsymbol{Z}$ arises from some underlying (nontrivial) classical Markov process. But, due to this entanglement phenomenon, it is not of Ising type.[6]

[5] The restriction of $\mathrm{Tr}_{N_{[k,l]}}$ to $D_{[k,l]}$ is the uniform measure which assigns the same weight 1 to the minimal projections of $D_{[k,l]}$.

[6] For Markov states with multidimensional indices, where there is no canonical order (i.e. for the Markov fields considered in [4]), the appearance of non-diagonalizable examples is expected.

The quantum character of such states manifests itself in the following way. In order to construct (or recover) such states, one should take into account various nontrivial local filtrations of $\mathfrak{M}$, together with various (commuting) boundary terms. Conversely, if one chooses to investigate quantum Markov states by considering only the natural filtration $\{M_{[k,l]}\}_{k\leq l}$ of $\mathfrak{M}$, one obtains a leading term as that in (4.2). But non-commuting boundary terms could naturally arise in (3.6), see the examples in Section 6 of [3]. In the constructive approach, the appearance of such non-commuting boundary terms cannot be disregarded in order to obtain general infinite volume Gibbs states for a fixed nearest neighbour interaction. However, it should be noted that if the nearest neighbour model is translation invariant or periodic, then according to Theorem 1 of [9], the construction of quantum Markov states does not depend on boundary terms.

5. TYPES OF VON NEUMANN ALGEBRAS ASSOCIATED WITH QUANTUM MARKOV STATES

In this section we investigate the type of von Neumann factors generated by the GNS representation of the quantum Markov states.

Let us consider the C^*-algebra $\mathfrak{M}$ defined in Section 2. The shift automorphism of the algebra $\mathfrak{M}$ will be denoted by θ. A state $\varphi\in\mathcal{S}(\mathfrak{M})$ is called *l-periodic* if $\varphi(\theta^l(A))=\varphi(A)$ for all $A\in\mathfrak{M}$. If $l=1$, φ is *translation invariant.* Notice that, in order to have l-periodicity, it is necessary that $d_{j+l}=d_j$, $j\in\boldsymbol{Z}$, for the d_j in (2.4). We have, for the localized Hamiltonians (3.4) and their leading terms (4.2),

$$h_{M_{[j+l,k+l]}}=h_{M_{[j,k]}},\qquad h_{j+l,k+l}=h_{j,k}$$

for all $j,k\in\boldsymbol{Z}$. In order to avoid the trivial situation, we consider only non-tracial locally faithful translation invariant or l-periodic Markov states. This means that $h_{0,l}\neq CI$, that is $h_{0,l}$ is nontrivial.

We are going to connect the type of the von Neumann factor $\pi_\varphi(\mathfrak{M})''$ with properties of the spectrum $\sigma(h_{0,l})$ of the fundamental block $h_{0,l}$ of the leading term (4.2) of the Hamiltonian associated with φ.

Due to commuting properties of the $h_{M_{[-n,n]}}$ (see (3.5)), the following strong limit

$$\sigma_t^\varphi(A)=\lim_{n\to\infty}\exp(ith_{M_{[-n,n]}})A\exp(-ith_{M_{[-n,n]}}),\qquad A\in\mathfrak{M},$$

exists. Furthermore, φ is a KMS state (at inverse temperature 1) for σ^φ. According to Theorem 1 of [9], it is the unique KMS state for σ^φ, and $\pi_\varphi(\mathfrak{M})''$ is a factor. Notice that we have also

$$\sigma_t^\varphi(A)=\lim_{n\to\infty}\exp(ith_{-n,n})A\exp(-ith_{-n,n}).$$

The extension to all of $\pi_\varphi(\mathfrak{M})''$, denoted also by σ^φ, is precisely the modular group associated with the normal extension of φ (denoted also by φ) to $\pi_\varphi(\mathfrak{M})''$.

Let $\mathrm{sp}(\tau)$ be the Arveson spectrum of the action τ of a locally compact group on a C^*-algebra.[7] Let us put $\sigma_t^n := \mathrm{ad}(\exp(ith_{-ln,ln}))$, where l is the period of the state under consideration.

LEMMA 5.1. *In the above situation, we have*

$$\mathrm{sp}(\sigma^\varphi) \subset \overline{\bigcup_n \left(\sigma(h_{-ln,ln}) - \sigma(h_{-ln,ln})\right)}.$$

Proof. By passing to the regrouped algebra, we can consider $l = 1$. Taking into account the commuting properties of the interaction, we have

$$\mathrm{sp}(\sigma^\varphi) = \overline{\bigcup_n \bigcup_{A \in M_{[-n,n]}} \mathrm{sp}^{\sigma^\varphi}(A)} = \overline{\bigcup_n \bigcup_{A \in M_{[-n,n]}} \mathrm{sp}^{\sigma^{n+1}}(A)}$$

$$\subset \overline{\bigcup_n \bigcup_{A \in M_{[-n-1,n+1]}} \mathrm{sp}^{\sigma^{n+1}}(A)} = \overline{\bigcup_n \mathrm{sp}(\sigma^{n+1} \lceil_{M_{[-n-1,n+1]}})}.$$

The proof follows by Proposition 14.13 of [30]. ∎

LEMMA 5.2. *Let* $\{x_1, \dots, x_n\} \subset \mathbf{R}\backslash\{0\}$ *be such that* $x_i/x_j \in \mathbf{Q}$ *for all* i, j. *Then*

$$\{x_1, \dots, x_n\} \subset \mathbf{Z} \ln \alpha \tag{5.1}$$

for some $\alpha \in (0, 1)$.[8]

Proof. By our assumptions we have

$$\frac{x_1}{x_i} = \frac{p_i}{q_i}, \qquad i = 2, \dots, n,$$

where $p_i \in \mathbf{N}\backslash\{0\}$, $q_i \in \mathbf{Z}\backslash\{0\}$. Define

$$\alpha := \exp\left(-\frac{|x_1|}{\prod_{j=2}^n p_j}\right).$$

Then

$$x_1 = -\mathrm{sign}(x_1)\Big(\prod_{j=2}^n p_j\Big) \ln \alpha,$$

$$x_i = -q_i\Big(\prod_{j=2, j\neq i}^n p_j\Big) \ln \alpha, \qquad i = 2, \dots, n. \ ∎$$

---

[7] For the definition of the Arveson spectrum $\mathrm{sp}(\tau)$, as well as $\mathrm{sp}^\tau(A)$, see e.g. [24].

[8] The best α in (5.1) is the minimum of the $\alpha \in (0, 1)$ such that (5.1) holds true.

Let $h_{0,l}$ be the fundamental block of the leading term of the canonical Hamiltonian associated with the locally faithful Markov state φ. Consider, for $h, k, h', k' \in \sigma(h_{0,l})$ with $h \neq k$, $h' \neq k'$, the following fractions: $(h-k)/(h'-k')$.

THEOREM 5.3. *Let $\varphi \in \mathcal{S}(\mathfrak{M})$ be a locally faithful Markov state. The following assertions hold true*:

(i) *If $\{(h-k)/(h'-k')\} \subset \boldsymbol{Q}$, then $\pi_\varphi(\mathfrak{M})''$ is a type III_λ factor for some $\lambda \in (0, 1)$.*

(ii) *If $\pi_\varphi(\mathfrak{M})''$ is a type III_1 factor, then $\{(h-k)/(h'-k')\} \nsubseteq \boldsymbol{Q}$.*

Proof. As before, we can consider only translation invariant Markov states. By applying Theorem 3.1 of [29], we get for the Connes invariant Γ (see [12]) $\Gamma(\pi_\varphi(\mathfrak{M})'') \equiv \Gamma(\sigma^\varphi) = \mathrm{sp}(\sigma^\varphi)$. Furthermore, this means also that $\pi_\varphi(\mathfrak{M})''$ is a type III_λ factor, $\lambda \in (0, 1]$, as we are considering non-tracial states. Then it is enough to prove the former, the latter being a direct consequence of the former.

Let $\{(h-k)/(h'-k')\} \subset \boldsymbol{Q}$ be satisfied. By Lemma 5.2,

$$\{h-k \mid h, k \in \sigma(H_{0,1})\} \subset \boldsymbol{Z} \ln \alpha$$

for some $\alpha \in (0, 1)$. From the simultaneous diagonalizability of the $H_{i,i+1}$ we infer that

$$\sigma(h_{-n,n}) \subset \Big\{ \sum_{i=-n}^{n-1} h_i \,\Big|\, h_i \in \sigma(H_{0,1}) \Big\}.$$

Then we have

$$\sigma(h_{-n,n}) - \sigma(h_{-n,n}) \subset \Big\{ \sum_{i=-n}^{n-1} (h_i - k_i) \,\Big|\, h_i, k_i \in \sigma(H_{0,1}) \Big\} \subset \boldsymbol{Z} \ln \alpha.$$

From Lemma 5.1 we infer that $\mathrm{sp}(\sigma^\varphi) \subset \boldsymbol{Z} \ln \alpha$, that is $\mathrm{sp}(\sigma^\varphi)$ is discrete. Hence there is a number $m \in \boldsymbol{N} \backslash \{0\}$ such that $\mathrm{sp}(\sigma^\varphi) = \boldsymbol{Z} \ln \lambda$ with $\lambda := \alpha^m$. Thus, $\pi_\varphi(\mathfrak{M})''$ is a type III_λ factor. ∎

Here, it should be noted that one might argue that the spectrum $\sigma(h_{0,l})$ of the fundamental block of the Hamiltonian associated with the l-periodic Markov state φ completely determines the type of $\pi_\varphi(\mathfrak{M})''$. Unfortunately, we are not able to prove the reverse statements in Theorem 5.3.

Even if one can construct, by the results in Section 4 of [3], a wide class of quantum Markov states to which the previous results apply, in order to explain some natural applications of Theorem 5.3 to pre-assigned models, we are going to consider some examples. We refer the reader to [17], [22], [23] for some results along the same line.

5.1. Ising model. In this situation,

$$\mathfrak{M} = \overline{\bigotimes_{\boldsymbol{Z}} M_2(\boldsymbol{C})}^{C^*}.$$

The Ising model on $\boldsymbol{Z}$ is defined by the following formal Hamiltonian:

$$H = -\sum_{j \in \boldsymbol{Z}} J_{j,j+1} \sigma_z^j \sigma_z^{j+1}, \tag{5.2}$$

where $J_{j,j+1} \in \boldsymbol{R}$ are coupling constants and σ_z^j is the Pauli matrix σ_z on the j-th site. Moreover, we suppose that the coupling constants are defined by

$$J_{j,j+1} = \begin{cases} J_1, & j \in 2\boldsymbol{Z}, \\ J_2, & j \in 2\boldsymbol{Z}+1, \end{cases}$$

where $J_1, J_2 \in \boldsymbol{R}$. It is known (see [9]) that for the given Hamiltonian there exists a unique Gibbs state φ on $\mathfrak{M}$ which is 2-periodic. In this case, the operators $H_{j,j+1}$ have the following form:

$$H_{j,j+1} = \begin{cases} \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & -J_1 & 0 & 0 \\ 0 & 0 & -J_1 & 0 \\ 0 & 0 & 0 & J_1 \end{pmatrix}, & j \in 2\boldsymbol{Z}, \\ \\ \begin{pmatrix} J_2 & 0 & 0 & 0 \\ 0 & -J_2 & 0 & 0 \\ 0 & 0 & -J_2 & 0 \\ 0 & 0 & 0 & J_2 \end{pmatrix}, & j \in 2\boldsymbol{Z}+1. \end{cases}$$

The spectrum of $H_{j,j+1}$ is $\{J_1, -J_1\}$ if $j \in 2\boldsymbol{Z}$, $\{J_2, -J_2\}$ if $j \in 2\boldsymbol{Z}+1$, respectively. Now, if J_1/J_2 is rational, the rationality condition of Theorem 5.3 is satisfied, and consequently the von Neumann factor $\pi_\varphi(\mathfrak{M})''$ is of type III_λ for some $\lambda \in (0, 1)$.

5.2. Markov chain. Consider a Markov chain with the state space $\boldsymbol{d} := \{1, \ldots, d\}$ and the transition probabilities defined by the stochastic matrix $P = (p_{ij})_{i,j=1}^d$ with (not all equal) $p_{ij} > 0$ for all i, j. Consider the canonical inclusion

$$\mathfrak{D} = \overline{\bigotimes_{\boldsymbol{Z}} \boldsymbol{C}_d}^{C^*} \subset \mathfrak{M} = \overline{\bigotimes_{\boldsymbol{Z}} M_d(\boldsymbol{C})}^{C^*}.$$

Here, $\mathfrak{D} \sim C(\Omega)$, where $\Omega = \prod_{\boldsymbol{Z}} \boldsymbol{d}$. Let μ_P be the translation invariant Markov measure on Ω determined by the transition matrix P. Define the diagonal lifting of the classical process associated with P as

$$\varphi(A) := \int_\Omega \mathfrak{E}(A)(\omega)\, \mu_P(d\omega),$$

where $\mathfrak{E}$ is the canonical Umegaki conditional expectation of $\mathfrak{M}$ onto the Abelian algebra $\mathfrak{D}$.

It is not hard to check that the corresponding $H_{j,j+1}$ operator has the form

$$H_{j,j+1} = \begin{bmatrix} B^{(1)} & 0 & . & . & . & 0 \\ 0 & B^{(2)} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & B^{(d)} \end{bmatrix},$$

where $B^{(k)} = (b_{ij,k})_{i,j=1}^{d}$, $k = 1, \ldots, d$, are $d \times d$ diagonal matrices such that

$$b_{ij,k} = \begin{cases} -\ln p_{k,i}, & i = j,\ i = 1, \ldots, d, \\ 0, & i \neq j. \end{cases}$$

If there exist integers m_{ij}, $i, j \in \{1, \ldots, d\}$, and some number $\alpha \in (0, 1)$ such that $p_{11}/p_{i,j} = \alpha^{m_{ij}}$, then we easily see that the rationality condition of Theorem 5.3 is satisfied, which means that the von Neumann factor $\pi_\varphi(\mathfrak{M})''$ is of type III_λ for some $\lambda \in (0, 1)$. This extends a result of [17].

6. APPENDIX

For convenience of the reader we verify that the measure μ on $\mathfrak{D}$ associated with $\varphi\lceil_{\mathfrak{D}}$ is a Markov measure on $\operatorname{spec}(\mathfrak{D})$ with respect to the order localization of $\boldsymbol{Z}$.

For our purpose, it suffices to verify that for every $k \leqslant n \leqslant l$ in $\boldsymbol{Z}$ and $B \in \operatorname{spec}(D_{[n,l]})$ we have, for the conditional probability,

$$P(B \mid \bar{\omega}_k, \ldots, \bar{\omega}_n) = P(B \mid \bar{\omega}_n).$$

Here, $\bar{\omega}_k, \ldots, \bar{\omega}_n$ are fixed elements in $\operatorname{spec}(Z(R_k)), \ldots, \operatorname{spec}(Z(R_n))$, respectively, and $\operatorname{spec}(Z(R_j))$ is isomorphic to Ω_j. In order to make computations, we should see the past algebra $D_{[k,n]}$, the present algebra $D_{[n,n]} \equiv Z(R_n)$, and the future algebra $D_{[n,l]}$ inside the ambient algebra $D_{[k,l]}$. In such a situation, $\operatorname{spec}(D_{[k,l]})$ is given by the disjoint union

$$\operatorname{spec}(D_{[k,l]}) = \bigcup_{\omega_k, \ldots, \omega_l} S^k_{\omega_k, \omega_{k+1}} \times \ldots \times S^{l-1}_{\omega_{l-1}, \omega_l}.$$

Using formulae (3.4) and (3.7), for

$$f := \sum_{\omega_k, \ldots, \omega_l} \mathcal{X}_{S^k_{\omega_k, \omega_{k+1}} \times \ldots \times S^{l-1}_{\omega_{l-1}, \omega_l}} f^k_{\omega_k, \omega_{k+1}} \otimes \ldots \otimes f^{l-1}_{\omega_{l-1}, \omega_l},$$

we calculate

$$(6.1)\quad \varphi(f) = \sum_{\omega_k,\dots,\omega_l} \Big(\int_{S^k_{\omega_k,\omega_{k+1}}} T^k_{\omega_k,\omega_{k+1}} f^k_{\omega_k,\omega_{k+1}}\Big) \times \dots \times \Big(\int_{S^{l-1}_{\omega_{l-1},\omega_l}} T^{l-1}_{\omega_{l-1},\omega_l} f^{l-1}_{\omega_{l-1},\omega_l}\Big),$$

where the densities T are positive functions, and $\int$ assigns weight 1 to atoms (see footnote 5).

Inside $\mathrm{spec}(D_{[k,l]})$, we have for the sets $\Gamma_{\bar\omega_k,\dots,\bar\omega_n} \subset \mathrm{spec}(D_{[k,n]})$, $\Gamma_{\bar\omega_n} \subset \mathrm{spec}(D_{[n,n]})$, describing the collection of points $\bar\omega_k, \dots, \bar\omega_n$ and the point $\bar\omega_n$, respectively,

$$(6.2)\quad \Gamma_{\bar\omega_k,\dots,\bar\omega_n} = \bigcup_{\omega_{n+1},\dots,\omega_l} S^k_{\bar\omega_k,\bar\omega_{k+1}} \times \dots \times S^{n-1}_{\bar\omega_{n-1},\bar\omega_n} \times S^n_{\bar\omega_n,\omega_{n+1}} \times \dots \times S^{l-1}_{\omega_{l-1},\omega_l},$$

$$\Gamma_{\bar\omega_n} = \bigcup_{\substack{\omega_k,\dots,\omega_{n-1},\\ \omega_{n+1},\dots,\omega_l}} S^k_{\omega_k,\omega_{k+1}} \times \dots \times S^{n-1}_{\omega_{n-1},\bar\omega_n} \times S^n_{\bar\omega_n,\omega_{n+1}} \times \dots \times S^{l-1}_{\omega_{l-1},\omega_l}.$$

Furthermore, the generic points of $B \in \mathrm{spec}(D_{[n,l]})$ have the form

$$b = b^n_{\omega_n(b),\omega_{n+1}(b)} \times \dots \times b^{l-1}_{\omega_{l-1}(b),\omega_l(b)}.$$

Define $\lambda(b) := \omega_n(b)$. Taking into account (6.1) and (6.2), we have

$$\varphi(\mathscr{X}_{\Gamma_{\bar\omega_k,\dots,\bar\omega_n}}) = \Big(\int_{S^k_{\bar\omega_k,\bar\omega_{k+1}}} T^k_{\bar\omega_k,\bar\omega_{k+1}}\Big) \times \dots \times \Big(\int_{S^{n-1}_{\bar\omega_{n-1},\bar\omega_n}} T^{n-1}_{\bar\omega_{n-1},\bar\omega_n}\Big)$$
$$\times \sum_{\omega_{n+1},\dots,\omega_l} \Big(\int_{S^n_{\bar\omega_n,\omega_{n+1}}} T^n_{\bar\omega_n,\omega_{n+1}}\Big) \times \dots \times \Big(\int_{S^{l-1}_{\omega_{l-1},\omega_l}} T^{l-1}_{\omega_{l-1},\omega_l}\Big),$$

$$\varphi(\mathscr{X}_{\Gamma_{\bar\omega_n}}) = \sum_{\omega_k,\dots,\omega_{n-1}} \Big(\int_{S^k_{\omega_k,\omega_{k+1}}} T^k_{\omega_k,\omega_{k+1}}\Big) \times \dots \times \Big(\int_{S^{n-1}_{\omega_{n-1},\bar\omega_n}} T^{n-1}_{\omega_{n-1},\bar\omega_n}\Big)$$
$$\times \sum_{\omega_{n+1},\dots,\omega_l} \Big(\int_{S^n_{\bar\omega_n,\omega_{n+1}}} T^n_{\bar\omega_n,\omega_{n+1}}\Big) \times \dots \times \Big(\int_{S^{l-1}_{\omega_{l-1},\omega_l}} T^{l-1}_{\omega_{l-1},\omega_l}\Big),$$

$$\varphi(\mathscr{X}_{\Gamma_{\bar\omega_k,\dots,\bar\omega_n}} \mathscr{X}_B) = \Big(\int_{S^k_{\bar\omega_k,\bar\omega_{k+1}}} T^k_{\bar\omega_k,\bar\omega_{k+1}}\Big) \times \dots \times \Big(\int_{S^{n-1}_{\bar\omega_{n-1},\bar\omega_n}} T^{n-1}_{\bar\omega_{n-1},\bar\omega_n}\Big)$$
$$\times \sum_{b\in B\,|\,\lambda(b)=\bar\omega_n} T^n_{\bar\omega_n,\omega_{n+1}(b)}\big(b^n_{\bar\omega_n,\omega_{n+1}(b)}\big) \times \dots \times T^{l-1}_{\omega_{l-1}(b),\omega_l(b)}\big(b^{l-1}_{\omega_{l-1}(b),\omega_l(b)}\big),$$

$$\varphi(\mathscr{X}_{\Gamma_{\bar\omega_n}} \mathscr{X}_B) = \sum_{\omega_k,\dots,\omega_{n-1}} \Big(\int_{S^k_{\omega_k,\omega_{k+1}}} T^k_{\omega_k,\omega_{k+1}}\Big) \times \dots \times \Big(\int_{S^{n-1}_{\omega_{n-1},\bar\omega_n}} T^{n-1}_{\omega_{n-1},\bar\omega_n}\Big)$$
$$\times \sum_{b\in B\,|\,\lambda(b)=\bar\omega_n} T^n_{\bar\omega_n,\omega_{n+1}(b)}\big(b^n_{\bar\omega_n,\omega_{n+1}(b)}\big) \times \dots \times T^{l-1}_{\omega_{l-1}(b),\omega_l(b)}\big(b^{l-1}_{\omega_{l-1}(b),\omega_l(b)}\big).$$

Collecting together the last computations, we get

$$P(B \mid \bar{\omega}_k, \ldots, \bar{\omega}_n) \equiv \frac{\varphi(\mathscr{X}_{\Gamma_{\bar{\omega}_k, \ldots, \bar{\omega}_n}} \mathscr{X}_B)}{\varphi(\mathscr{X}_{\Gamma_{\bar{\omega}_k, \ldots, \bar{\omega}_n}})} = \frac{\varphi(\mathscr{X}_{\Gamma_{\bar{\omega}_n}} \mathscr{X}_B)}{\varphi(\mathscr{X}_{\Gamma_{\bar{\omega}_n}})} \equiv P(B \mid \bar{\omega}_n),$$

which is the assertion.

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