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RANDOM SUMS STOPPED BY A RARE EVENT: A NEW APPROXIMATION

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Abstract. The convergence of a geometric sum of positive i.i.d. random variables to an exponential distribution is a well-known result. This convergence provided various and useful approximations in reliability, queueing or risk theory. However, for concrete applications, this exponential approximation is not sharp enough for small values of mission time. So, other approximations have been proposed (Bon and Pamphile (2001), Kalashnikov (1997)). In this paper we propose a new point of view where the exponential approximation appears as a first-order approximation. We consider more general random sums stopped by a rare event, where summands are no more assumed to be independent neither nonnegative. So we give a second-order approximation. As illustration we consider stopping time with negative binomial distribution. This approximation provides a new evaluation tool in reliability analysis of highly reliable systems. The accuracy of this approximation is studied numerically.

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1. INTRODUCTION

A geometric sum of nonnegative i.i.d. random variables (X_n) is defined as follows:

$$S_{N_\theta} = \sum_{n=1}^{N_\theta} X_n, \tag{1}$$

where N_θ is independent of the sequence (X_n) and has the geometric distribution

$$P(N_\theta = n) = (1-\theta)^{n-1}\theta, \quad n \geqslant 1.$$

Generally, an explicit computation of the distribution function of S_{N_θ} is not possible. So we have to resort to approximations. This sum usually modifies a discrete-time cumulative process stopped by a *rare event*. This means that we are in the asymptotic where θ goes to zero. In this case the following theorem is useful (see Kalashnikov (1997)):

THEOREM 1.1. *Assume that X_1 is integrable and let $\mu = \boldsymbol{E}[X_1]$. Then θS_{N_θ} converges in distribution to the exponential distribution with mean μ.*

This convergence theorem gives, for small θ, the following approximation formula:

$$\boldsymbol{P}(S_{N_\theta} > t/\theta) = \exp(-t/\mu) + o(1) \qquad (t \in \boldsymbol{R}). \tag{2}$$

In the special case where the summands in (1) have an exponential distribution (with mean μ), θS_{N_θ} has also, for any $\theta > 0$, the same exponential law. Hence we may think that if the common distribution of the random variables (X_n) is not far away from an exponential one, the approximation formula (2) should be sharp. This is why this first-order approximation has been widely used in various applied disciplines like reliability, queueing, insurance risk, storage and inventory (for applications see, for example, Gertsbakh (1984), Asmussen (1987), Kalashnikov (1997)) and various bounds have been obtained by using renewal technics (cf. Brown (1990) and Kalashnikov (1997)). This paper is motivated by the study of rare events in reliability models (see Section 3). Our first aim is to propose a tight approximation for situations where the summands are not independent and the random index is a stopping time. We choose a martingale framework, but other kind of dependence (Markovian or mixing) may also been studied.

The paper is organized as follows. Firstly we will extend Theorem 1.1 to a drifted stopped square-integrable martingale. This means that the summands in (1) are no more assumed to be independent neither nonnegative. Moreover, the random index and the summand are no more independent. Namely, the first-order approximation is that the normalized stopped drifted martingale shares the same asymptotic as the normalized stopping time. Further, assuming independence we give a second-order approximation (see Theorem 2.1). We point out that in the framework of i.i.d. summands this second-order approximation involved the generalized inverse Gaussian distribution introduced by Barndorff-Nielsen and Halgreen (see Seshadri (1993), p. 27). In Section 3 we illustrate our results, firstly considering stopping time with negative binomial distribution, and then discussing some examples of stopped martingales coming from reliability problems. The last section is devoted to some numerical experiments where the accuracy of the first-order and the second-order approximations are compared.

2. MAIN RESULTS

2.1. Model and assumptions. In what follows, all the limits are taken for θ going to 0^+. Hence $\xrightarrow{d}$ and $\xrightarrow{P}$ denote, respectively, the convergence in distribution and the convergence in probability for θ going to 0^+.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\mathscr{F} = (\mathscr{F}_n)_{n\in N}$ be a given filtration. In this probability space, let $\varepsilon = (\varepsilon_n,\ n \geqslant 1)$ be a real square-integrable martingale increments adapted to $\mathscr{F}$. That is, for any $n \in N$ we have

$$E[\varepsilon_{n+1} | \mathscr{F}_n] = 0 \quad \text{and} \quad E[\varepsilon_n^2] < +\infty.$$

Let $\mu > 0$; for any $n \geqslant 1$ we set $X_n = \mu + \varepsilon_n$. Let $(S_n)_{n\geqslant 1}$ be the partial sum process of the sequence $(X_n)_{n\geqslant 1}$. Notice that $(M_n)_{n\geqslant 1} = (S_n - n\mu)_{n\geqslant 1}$ is a martingale. We will consider the stopped sum

$$S_{N_\theta} = \sum_{n=1}^{N_\theta} X_n = \mu N_\theta + M_{N_\theta}, \tag{3}$$

where $(N_\theta)_{\theta\in]0,1[}$ is a family of stopping time with respect to the filtration $\mathscr{F}$. In order to establish a limit theorem for S_{N_θ}, we assume from now that N_θ converges in distribution:

$$\theta N_\theta \xrightarrow{d} \nu, \tag{4}$$

where ν is a nondegenerate (not concentrated at a single point) probability measure. Further, we assume that ν does not weight 0. This implies that N_θ diverges, almost surely (a.s.), towards infinity. For $a \in R^*$, we set

$$\nu_a(\cdot) = \nu\left(\frac{\cdot}{a}\right).$$

2.2. First-order approximation. We begin with the following

Remark. *If we assume*

$$\sum_{n\geqslant 1} n^{-2} E[\varepsilon_n^2 | \mathscr{F}_{n-1}] < +\infty \ \textit{a.s.}, \tag{5}$$

then θS_{N_θ} converges in distribution to ν_μ.

Indeed, from (3) we obtain $\theta S_{N_\theta} = \theta N_\theta \mu + \theta M_{N_\theta}$. Now, by assumption, the first term in the last equality converges in distribution to ν_μ. Hence it remains to show that the second one converges in probability to 0. Now, we get

$$\theta M_{N_\theta} = \theta N_\theta \frac{M_{N_\theta}}{N_\theta} \mathbf{1}_{\{N_\theta \neq 0\}}.$$

On the one hand, using (5) and the Martingale Theorem 7.9.3, p. 243, of Feller (1971), M_n/n converges a.s. to 0. Further, as N_θ diverges a.s., M_{N_θ}/N_θ also converges a.s. to 0. On the other hand, as (θN_θ) converges, it is bounded in probability. Consequently, θM_{N_θ} converges to 0. ∎

Roughly speaking, this result states that if a law of large numbers is available for the martingale $S_n - n\mu$, then θS_{N_θ} and θN_θ share the same asymptotic.

In the case of Theorem 1.1, the convergence of θN_θ to an exponential distribution follows directly from the convergence of the binomial distribution to the Poisson law. The more general case of a gamma limit will be considered in Section 3.

For concrete applications, for example reliability analysis, the first-order approximation is not tight for mission time less than $E[S_{N_\theta}]$, see Section 5 for numerical examples. So it cannot be used for a highly reliable system, with very large MTTF, like nuclear plants.

2.3. Second-order approximation. In this section, we assume that the stopping time N_θ is independent of the process (ε_n). In this frame, a refined limit theorem can be derived:

THEOREM 2.1. *Assume that there exist positive real numbers μ_2 and δ such that*

$$\lim_{n\to\infty}\frac{1}{n}\langle M\rangle_n = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} E[\varepsilon_j^2\,|\,\mathscr{F}_{j-1}] = \mu_2 \text{ a.s.} \quad \text{and} \tag{6}$$

$$\sup_{n\in N} E[|\varepsilon_{n+1}|^{2+\delta}\,|\,\mathscr{F}_n] < +\infty.$$

Then $(\theta N_\theta, \sqrt{\theta}\, M_{N_\theta})$ converges in distribution to the law of (U, V), where U has the distribution ν and for $u \geqslant 0$ the conditional distribution of V, given $\{U = u\}$, is the centered Gaussian distribution with variance $u\mu_2$. In other words, for the Lebesgue measure λ on R the density of (U, V) with respect to $\nu(du)\otimes\lambda(dv)$ is

$$\frac{1}{\sqrt{2\pi u\mu_2}}\exp\left(-\frac{v^2}{2\mu_2 u}\right) \qquad ((u, v)\in R^{+*}\times R).$$

Proof. For a random variable Z let φ_Z denote its characteristic function. We shall prove that $\varphi_{(\theta N_\theta, \sqrt{\theta} M_{N_\theta})}$ converges to $\varphi_{(U,V)}$, which implies the result. First, using Fubini's theorem we have, for $(t, s)\in R^2$,

$$\begin{aligned}\varphi_{(U,V)}(t, s) &= \int_{R^{+*}\times R} \frac{\exp\left(i(tu+sv) - v^2/(2u\mu_2)\right)}{\sqrt{2\pi u\mu_2}}\,\nu(du)\,dv \\ &= \int_{R^{+*}} e^{itu}\exp\left(-\frac{\mu_2 u s^2}{2}\right)\nu(du) = E\left[\exp\left(U\left[it - \frac{\mu_2 s^2}{2}\right]\right)\right].\end{aligned}$$

Now, we write

$$\begin{aligned}&\varphi_{(\theta N_\theta, \sqrt{\theta} M_{N_\theta})}(t, s) - \varphi_{(U,V)}(t, s) \\ &\qquad = \varphi_{(\theta N_\theta, \sqrt{\theta} M_{N_\theta})}(t, s) - E\left[\exp\left(\theta N_\theta\left[it - \frac{\mu_2 s^2}{2}\right]\right)\right] \\ &\qquad\quad + E\left[\exp\left(\theta N_\theta\left[it - \frac{\mu_2 s^2}{2}\right]\right)\right] - E\left[\exp\left(U\left[it - \frac{\mu_2 s^2}{2}\right]\right)\right].\end{aligned}$$

As θN_θ converges in distribution to the law of U, the last difference goes to zero with θ. On the other hand, using the independence assumption we may write

$$(7)\quad \left|\varphi_{(\theta N_\theta, \sqrt{\theta} M_{N_\theta})}(t, s) - E\left[\exp\left(\theta N_\theta\left[it - \frac{\mu_2 s^2}{2}\right]\right)\right]\right| \leqslant \sum_{n=0}^{\infty}\left|\varphi_{M_n}(\sqrt{\theta}s) - \exp\left(-\frac{\mu_2 s^2 n\theta}{2}\right)\right| P(N_\theta = n).$$

Let $\varepsilon > 0$. As θN_θ is uniformly tight and its limit distribution ν does not weight 0, we may find an interval $I_\theta = [k_\varepsilon \theta^{-1}; K_\varepsilon \theta^{-1}]$ such that, for all sufficiently small θ, $P(N_\theta \notin I_\theta) < \varepsilon$. The right-hand side of (7) may be rewritten as

$$(8)\quad \sum_{n \notin I_\theta}\left|\varphi_{M_n}(\sqrt{\theta}s) - \exp\left(-\frac{\mu_2 s^2 n\theta}{2}\right)\right| P(N_\theta = n) + \sum_{n \in I_\theta}\left|\varphi_{M_n}(\sqrt{\theta}s) - \exp\left(-\frac{\mu_2 s^2 n\theta}{2}\right)\right| P(N_\theta = n).$$

Thus, the first sum in the last expression is bounded by 2ε. Now, by assumptions, as n goes to infinity, $M_n/\sqrt{n}$ converges in distribution to the centered Gaussian law with variance μ_2 (see, for example, Duflo (1997), p. 46). Therefore, its characteristic function $\varphi_{M_n/\sqrt{n}}(s)$ converges, for any s, to $\exp(-(\mu_2 s^2)/2)$. Moreover, by (6), the first moment of $|M_n|/\sqrt{n}$ is uniformly bounded. This implies, by Ascoli's theorem, that the convergence is uniform over the compact set $[k_\varepsilon, K_\varepsilon]$. Thus, for θ small enough, the second sum in (8) is bounded by ε. Hence

$$\limsup_{\theta \to 0^+} |\varphi_{(\theta N_\theta, \sqrt{\theta} M_{N_\theta})}(t, s) - \varphi_{(U,V)}(t, s)| \leqslant 3\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we may conclude that $\varphi_{(\theta N_\theta, \sqrt{\theta} M_{N_\theta})}$ converges everywhere to $\varphi_{(U,V)}$. ■

In the spirit of the Barndorff-Nielsen and Halgreen definition of the generalized inverse Gaussian distribution (see Seshadri (1993), p. 27), we introduce the following definition:

DEFINITION 2.2. Let ν be a probability measure on $\boldsymbol{R}^+$. The *generalized ν-inverse Gaussian distribution of parameters a, $b > 0$* is the probability measure $\nu^{a,b}$ defined by

$$(9)\quad \nu^{a,b}(du) = \left(Z(a, b)\sqrt{u}\right)^{-1} \exp(-a/u)\, \nu_b(du),$$

where $Z(a, b)$ is the partition function

$$Z(a, b) = \int_{\boldsymbol{R}^{+*}} \frac{1}{\sqrt{u}} \exp\left(-\frac{a}{u}\right) \nu_b(du).$$

In the framework of i.i.d. random variables the following result links the generalized ν-inverse Gaussian distribution with the asymptotic of $(\theta N_\theta, \sqrt{\theta} M_\theta)$.

COROLLARY 2.3. *Assume that the sequence (ε_n) is i.i.d. Suppose further that ε_1 has a finite third moment and that its characteristic function φ_{ε_1} lies, for some $\zeta > 1$, in $L^\zeta(\boldsymbol{R})$. Then, for any real v, the conditional law of θN_θ, given $\{\sqrt{\theta} M_\theta = v\}$, converges in distribution to the generalized ν-inverse Gaussian distribution of parameters $(v^2/(2\mu_2), 1)$.*

Proof. Without loss of generality we may assume that $\mu_2 = 1$. We only give the proof in the case where $\zeta = 1$. The case $\zeta > 1$ could be tackled similarly, but the formula (11) is heavier to write. As $\varphi_{\varepsilon_1} \in L^1(\boldsymbol{R})$, for $n \geqslant 1$ the distribution of $n^{-1/2} M_n$ has a density f_n. Moreover, using Theorem 8.2.1, p. 533, of Feller (1971), we may approximate uniformly over $\boldsymbol{R}$ this density by the Gaussian one:

$$\sup_{x \in \boldsymbol{R}} \left| f_n(x) - \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \right| = O\left(\frac{1}{\sqrt{n}}\right). \tag{10}$$

For $u \in \boldsymbol{R}$ and $\theta > 0$ small enough, the conditional distribution $\kappa_{\theta,v}(dy)$ of θN_θ, given $\{\sqrt{\theta} M_\theta = v\}$, is

$$\kappa_{\theta,v}(dy) = \frac{\sum_{n \geqslant 1} f_n(v/\sqrt{\theta})\, \boldsymbol{P}(N_\theta = n)\, \delta_{\theta,n}(dy)}{\sum_{n \geqslant 1} f_n(v/\sqrt{\theta})\, \boldsymbol{P}(N_\theta = n)}. \tag{11}$$

The characteristic function of this distribution is, for $t \in \boldsymbol{R}$,

$$\varphi_{\kappa_{\theta,v}}(t) = \frac{\sum_{n \geqslant 1} f_n(v/\sqrt{\theta})\, \boldsymbol{P}(N_\theta = n) \exp(in\theta t)}{\sum_{n \geqslant 1} f_n(v/\sqrt{\theta})\, \boldsymbol{P}(N_\theta = n)}.$$

Now, using the uniform approximation given in (10), we may mimic the proof of Theorem 2.1 to evaluate both sums in the last equation. This leads to

$$\lim_{\theta \to 0^+} \varphi_{\kappa_{\theta,v}}(t) = \frac{\int_{\boldsymbol{R}^+} \exp(ity) \exp(-v^2/(2y))\, d\nu(y)}{\int_{\boldsymbol{R}^+} \exp(-v^2/(2y))\, d\nu(y)},$$

allowing to conclude the proof. ∎

Remarks. (i) Let Φ be the standard normal cumulative distribution function. From Theorem 2.1, using Fubini's theorem we have, for $t \in \boldsymbol{R}$ and $\theta > 0$,

$$\begin{aligned} \boldsymbol{P}(\mu U + \sqrt{\theta} V \leqslant t) &= \iint_{\mu u + \sqrt{\theta} v \leqslant t} \frac{1}{\sqrt{2\pi u \mu_2}} \exp\left(-\frac{v^2}{2\mu_2 u}\right) \nu(du)\, dv \\ &= \int_{\boldsymbol{R}^{+*}} \Phi\left(\frac{t - \mu u}{\sqrt{\theta \mu_2 u}}\right) \nu(du). \end{aligned}$$

Thus, we obtain the following "second-order approximation":

$$(12)\qquad P(\theta S_{N_\theta} \leqslant t) = \int_{R^{+*}} \Phi\left(\frac{t-\mu u}{\sqrt{\theta\mu_2 u}}\right) \nu(du) + o(1) \qquad (t \in R).$$

This approximation will be discussed and numerically evaluated in Section 5.

(ii) Theorem 2.1 also gives the asymptotic of the stopped martingale M_{N_θ}. Indeed, $\sqrt{\theta} M_{N_\theta}$ converges in distribution to the marginal law of V. In other words

$$(13)\qquad \lim_{\theta \to 0^+} P(\sqrt{\theta} M_{N_\theta} \leqslant t) = \int_{R^{+*}} \Phi\left(\frac{t}{\sqrt{\mu_2 u}}\right) \nu(du).$$

This result is a generalization of the i.i.d. case studied in Kruglov and Korolev (1991).

(iii) In the i.i.d. case, assuming that the family (N_θ) converges in probability, the same limit theorem holds for the random vector $(\theta N_\theta, \sqrt{\theta} M_{N_\theta})$ (see Dacunha-Castelle and Duflo (1983), p. 228, Proposition 7.4.30). The proof of this result relies on the Donsker theorem on functional convergence to a Brownian motion of the Donsker process (cf. Billingsley (1968)).

3. NEGATIVE BINOMIAL COMPOUND SUMS

In this section we consider special cases, widely used in reliability or risk theory, for which Theorem 2.1 leads to gamma approximation.

3.1. Negative binomial distribution. Here we take in stopping time N_θ, with a negative binomial distribution, for $\alpha \geqslant 1$ and $\theta \in]0, 1[$:

$$P(N_\theta = \alpha + n) = \binom{\alpha+n-1}{\alpha-1} (1-\theta)^n \theta^\alpha \quad \text{for } n \geqslant 0.$$

In the following subsections we shall see some concrete examples of negative binomial compound sums. For $\alpha, \mu > 0$, $\gamma(\alpha, 1/\mu)$ denotes the gamma distribution of parameters $(\alpha, 1/\mu)$. Recall that the characteristic function of this distribution is

$$\varphi_{\gamma(\alpha, 1/\mu)}(s) = \left[\frac{1}{1 - is\mu}\right]^\alpha \qquad (s \in R).$$

Let W and Z be two independent random variables both having the distribution $\gamma(\alpha, 1/\mu)$ $(\alpha, \mu > 0)$. We denote by $\tilde{\gamma}(\alpha, 1/\mu)$ the distribution of $W - Z$. Obviously, the corresponding characteristic function is

$$\varphi_{\tilde{\gamma}(\alpha, 1/\mu)}(s) = \left[\frac{1}{1 + s^2/\mu^2}\right]^\alpha \qquad (s \in R).$$

This distribution may also be obtained as a gamma mixture of centered Gaussian distributions (see the proof below). A very special case is when $\alpha = 1$; in this case the $\gamma(1, 1/\mu)$ distribution is the exponential law and the $\tilde{\gamma}(\alpha, 1/\mu)$ distribution is the Laplace distribution (double exponential). We have the following result:

COROLLARY 3.1. (i) *Assume that* (5) *is satisfied. Then* θS_{N_θ} *converges in distribution to the* $\gamma(\alpha, 1/\mu)$ *law.*

(ii) *If* (6) *is satisfied, then* $\sqrt{\theta} M_{N_\theta}$ *converges in distribution to the distribution* $\tilde{\gamma}(\alpha, \sqrt{2/\mu_2})$.

(iii) $(\theta S_{N_\theta}, \sqrt{\theta} M_{N_\theta})$ *converges in distribution to a random vector* $(\mu U, V)$ *having density*

$$\frac{1}{\sqrt{2\pi u \mu_2}} \exp\left(-\frac{v^2}{2\mu_2 u} - \frac{u}{\mu}\right) \frac{u^{\alpha-1}}{\Gamma(\alpha)\mu^\alpha} \mathbf{1}_{\mathbf{R}^{+*}}(u). \tag{14}$$

Proof. To prove the first point, we only have to show that θN_θ converges in law to the $\gamma(\alpha, 1)$ distribution. Let us write the characteristic function of θN_θ:

$$\varphi_{\theta N_\theta}(t) = \left(\frac{\theta}{1-(1-\theta)\exp(i\theta t)}\right)^\alpha = \left(\frac{1}{1-it+o(1)}\right)^\alpha \quad (t \in \mathbf{R}).$$

It converges to the characteristic function of the $\gamma(\alpha, 1)$ distribution.

To prove the second point, using Theorem 2.1 we obtain the limit characteristic function of $\sqrt{\theta} M_{N_\theta}$:

$$\lim_{\theta \to 0^+} \varphi_{\sqrt{\theta} M_{N_\theta}}(s) = \int_{\mathbf{R}^+} \frac{1}{\Gamma(\alpha-1)} u^{\alpha-1} e^{-u} \exp\left(-\frac{u\mu_2}{2} s^2\right) du = \left[\frac{1}{1+(\mu_2/2)s^2}\right]^\alpha.$$

The third point follows directly from Theorem 2.1. The distribution of the vector (U, V) is the gamma-Gaussian distribution (see Casalis (1996)). ■

The following theorem provides the rate of convergence of the first-order approximation.

THEOREM 3.2. *Let us denote by* $\gamma(\alpha, \mu)$ *the gamma distribution with parameters* α *and* μ. *Assume that there exists a positive real number* C *such that*

$$\sup_{n \in \mathbf{N}} \mathbf{E}[\varepsilon_{n+1}^2 \mid \mathscr{F}_n] \leqslant C.$$

Then the uniform distance between θS_{N_θ} *and* $\gamma(\alpha, \mu)$ *is bounded as follows*:

$$\left|\mathbf{P}(\theta S_{N_\theta} \leqslant t) - \mathbf{P}(\gamma(\alpha, \mu) \leqslant t)\right| \leqslant 3\left(\frac{\alpha}{2}\right)^{1/3} \left(\frac{C+\mu^2}{2\mu^2}\right)^{1/3} \theta^{1/3}.$$

Proof. Let $(Y_n;\ n \geqslant 1)$ be i.i.d. random variables with distribution $\gamma(1, \mu)$ and assume that $(Y_n;\ n \geqslant 1)$ are independent of $(\varepsilon_n;\ n \geqslant 1)$. Let us put

$$S_{N_\theta}^Y = \sum_{n=1}^{N_\theta} Y_n, \quad S_{N_\theta}^{X-Y} = \sum_{n=1}^{N_\theta} (X_n - Y_n).$$

Then

$$\theta S_{N_\theta} = \theta S_{N_\theta}^Y + \theta S_{N_\theta}^{X-Y}.$$

Furthermore, it follows from the divisible property of gamma distributions that

$$\theta S_{N_\theta}^Y \stackrel{d}{=} \gamma(\alpha, \mu).$$

Now, for $\varepsilon > 0$:

$$P(\theta S_{N_\theta} \leqslant t) \leqslant P(\theta S_{N_\theta}^Y \leqslant t+\varepsilon) + P(\theta S_{N_\theta}^{X-Y} < -\varepsilon), \tag{15}$$

$$P(\theta S_{N_\theta}^Y \leqslant t-\varepsilon) \leqslant P(\theta S_{N_\theta} \leqslant t) + P(\theta S_{N_\theta}^{X-Y} > \varepsilon). \tag{16}$$

Combining (15) and (16), we see that

$$\begin{aligned} \big|P(\theta S_{N_\theta} \leqslant t) - P\big(\gamma(\alpha, \mu) \leqslant t\big)\big| &= |P(\theta S_{N_\theta} \leqslant t) - P(\theta S_{N_\theta}^Y \leqslant t)| \\ &\leqslant \sup_t P(t < \theta S_{N_\theta}^Y \leqslant t+\varepsilon) + P(|\theta S_{N_\theta}^{X-Y}| > \varepsilon). \end{aligned}$$

The first term is the concentration of the gamma distribution,

$$\begin{aligned} P(t < \theta S_{N_\theta}^Y \leqslant t+\varepsilon) &= P\big(t < \gamma(\alpha, \mu) \leqslant t+\varepsilon\big) \\ &= P\big(\gamma(\alpha, \mu) > t\big) P\big(\gamma(1, \mu) \leqslant \varepsilon\big). \end{aligned}$$

Hence

$$\sup_t P(t < \theta S_{N_\theta}^Y \leqslant t+\varepsilon) \leqslant P\big(\gamma(1, \mu) \leqslant \varepsilon\big) \leqslant \varepsilon/\mu.$$

Next, from Tchebyshev's inequality,

$$P(|\theta S_{N_\theta}^{X-Y}| > \varepsilon) \leqslant \frac{\theta^2}{\varepsilon^2} E[|S_{N_\theta}^{X-Y}|^2].$$

Since $X_n - Y_n = \varepsilon_n - (Y_n - \mu)$, $(S_n^{X-Y};\ n \geqslant 0)$ is a martingale adapted to $\mathscr{F}$. Consequently, using the Wald identity,

$$\begin{aligned} E[|S_{N_\theta}^{X-Y}|^2] &= E\Big[\sum_{j=1}^{N_\theta} E\big[(\varepsilon_j - (Y_j - \mu))^2 \,\big|\, \mathscr{F}_{j-1}\big]\Big] \\ &\leqslant \big(C + \mathrm{Var}(Y_1)\big) E[N_\theta] = (C+\mu^2)\frac{\alpha}{\theta}. \end{aligned}$$

Thus

$$P(|\theta M_{N_\theta}| > \varepsilon) \leqslant (C+\mu^2)\,\alpha\frac{\theta}{\varepsilon^2},$$

and finally

$$\left|P(\theta S_{N_\theta} \leqslant t) - P\big(\gamma(\alpha, \mu) \leqslant t\big)\right| \leqslant \frac{\varepsilon}{\mu} + (C+\mu^2)\,\alpha\frac{\theta}{\varepsilon^2}.$$

Minimizing the right term in ε, we complete the proof. ■

3.2. Exponential family. Now we assume that the stopping time belongs to a fairly regular exponential model:

Let π be a probability measure on the set N. Using π, we build the natural exponential family π_τ by setting

$$\pi_\tau(dt) = \exp\big(\tau x - \psi(\tau)\big)\,\pi(dt), \tag{17}$$

where the real number τ is such that the log-partition function

$$\psi(\tau) = \log \int \exp(\tau x)\,\pi(dt) \tag{18}$$

is well defined. The distribution of N_τ is π_τ. We assume that the exponential family π_τ has a breaking point $\tau_0 > 0$ such that

$$\lim_{\tau \to \tau_0^-} E[N_\tau] = \lim_{\tau \to \tau_0^-} \psi'(\tau) = +\infty. \tag{19}$$

Consequently, the following theorem gives us the behaviour of the stopped martingale

$$S_\tau = \sum_{n=1}^{N_\tau} X_n$$

near of the breaking point τ_0.

THEOREM 3.3. *If the exponential model is fairly regular, namely if there exist positive constants A and α such that*

$$\pi(]t, +\infty[) \sim A\exp(-\tau_0 t)\,t^{\alpha-1} \quad \text{as } t \to +\infty, \tag{20}$$

then when τ approaches τ_0, either the following (i) *or* (ii) *holds*:

(i) *the X_n are centered; then $\sqrt{\tau_0 - \tau} S_\tau$ converges in distribution to the Laplace law*;

(ii) $(\tau_0 - \tau) S_\tau$ *converges in distribution to a gamma law.*

Using Theorem 3.2 and Corollary 3.1, we obtain this theorem directly from the following lemma:

LEMMA 3.4. $(\tau_0 - \tau) N_\tau$ *converges in distribution to the gamma law* $\gamma(\alpha, 1)$.

Proof of Lemma 3.4. Let $\tau < \tau_0$. We may write

$$\begin{aligned} E\left[\exp\big(s(\tau_0-\tau)N_\tau\big)\right] &= \int \exp\big(\tau t - \psi(\tau)\big)\exp\big(s(\tau_0-\tau)t\big)\,\pi(dt) \\ &= \exp\big(\psi(\tau + s(\tau_0-\tau)) - \psi(\tau)\big). \end{aligned}$$

Let us recall that $\exp(\psi(\tau))$ denotes the Laplace transform of π evaluated at τ. Integrating by parts we obtain

$$\exp(\psi(\tau)) = 1+\tau \int_0^{+\infty} \exp(\tau t)\, Q_\pi(t)\, dt.$$

From Theorem 34.4, p. 233, of Doetsch (1974), under the assumption (20) the last integral is equivalent to $A\Gamma(\alpha)(\tau_0-\tau)^{-\alpha}$; this allows us to conclude that

$$E\left[\exp\left(s(\tau_0-\tau)N_\tau\right)\right] \sim \frac{\tau_0 A\Gamma(\alpha)\left((\tau_0-\tau)(1-s)\right)^{-\alpha}}{\tau_0 A\Gamma(\alpha)(\tau_0-\tau)^{-\alpha}} = \frac{1}{(1-s)^\alpha}.$$

Since the last expression is the Laplace transform of the gamma law, we may conclude the lemma. ∎

4. SOME APPLICATIONS

In this section we give some applications.

4.1. Reliability examples. Random sums are widely used in reliability theory; actually, the behaviour of a reparable system of components can be described as a succession of operating periods following by periods with maintenance procedures. Maintenance procedures are achieved by a complete repair of the system and a new operating period starts again. But, few and far between, system failure takes place before a complete repair. More formally, let the operating period start at $T_0 = 0 < T_1 < \ldots$, and denote by N_θ the number of operating periods before system failure:

- if $X_n = T_n - T_{n-1}$ is the length of the nth operating period, then the compound sum S_{N_θ} is the system lifetime and $P(S_{N_\theta} > t)$ the reliability of the system;
- if $X_n = \phi_n(T_0, \ldots, T_n)$ is a reward (positive or negative) during the nth operating period, then the compound sum S_{N_θ} is the total return before system failure.

As a concrete example, let us consider a machine that operators as long as a crucial part is operational. At the nth use, the machine produces a market value $X_n = \mu + \varepsilon_n$, where (ε_n) are fluctuations. During the nth use, the part has a probability θ of failure; if the part failed, it is replaced by a spare part for the next use. The system fails when α parts have failed: N_θ, the number of operating periods before the system failure, is a "shifted" negative binomial random variable. For highly reliable parts, failures are rare events, so we assume that θ vanishes. Then we want to evaluate the distribution of S_{N_θ}, the total amount produced until system failure. In the framework of the first-order approximation, i.e. (ε_n) i.i.d. and $\alpha = 1$, accurate approximations have been given in Kalashnikov (1997). More generally, in the framework of the second-order approximation:

$$P(\theta S_{N_\theta} > x) = \int_x^{+\infty} \frac{u^{\alpha-1}}{\Gamma(\alpha-1)\,\mu^\alpha} e^{-u/\mu}\, du + o(1)$$
$$= \int_{R^+} \Phi\left(\frac{x-u}{\sqrt{\theta\mu_2 u}}\right) \frac{u^{\alpha-1}}{\Gamma(\alpha-1)\,\mu^\alpha} e^{-u/\mu}\, du + o(1).$$

4.2. Randomized Poisson process. Compound Poisson process is another class of random sums widely used in reliability or risk theory. A compound Poisson process is defined as follows:

$$S(t) = \sum_{n=1}^{N(t)} X_n,$$

where $(N(t),\, t \geqslant 0)$ is a Poisson process with intensity Λ and $X_n = \mu + \varepsilon_n$. For example:

- $N(t)$ is the number of shocks that have occurred in a system up to times t, X_n the damage caused by the nth shock. Then $S(t)$ is the cumulative damage up to t.
- $N(t)$ is the number of share capital traded up to times t, X_n the financial profit obtained in the nth transaction. Then $S(t)$ is the total profit up to t.

For concrete applications, a stochastic intensity Λ is a more adequate model rather than a homogeneous intensity; when (X_n) is an i.i.d. sequence of random variables, the general compound Poisson process $S(t)$ has been studied in Grandell (1997).

If Λ is a random variable gamma distributed with parameters α and λ, then $N(t)$ is a negative binomial random variable with parameters α and $\lambda/(\lambda+t)$ (cf. Engel and Zijlstra (1980)). Therefore, if (ε_n) are martingale differences, then from Theorem 3.2 when $t/\lambda \to +\infty$ we obtain

$$P\left(\frac{\lambda}{\lambda+t} S(t) > x\right) = \int_x^{+\infty} \frac{u^{\alpha-1}}{\Gamma(\alpha-1)\,\mu^\alpha} e^{-u/\mu}\, du + o(1).$$

Furthermore, when Λ has any distribution, under the assumption of Corollary 2.3, we get the conditional law of $N(t)/t$, given $S(t)/t$, when $t \to +\infty$.

5. NUMERICAL EXPERIMENTS

Here we consider a production facility with several operating machines and three others on standby. During an operating period, machines are subject to breakdown; the maintenance policy consists in the replacement for a faulty machine a standby machine if the one is available, either the production is halted. Various types of random sums and their approximations have been studied: with the notation of the previous sections, let us denote by N_θ the number of production periods,

$$R(x) = P(S_{N_\theta} \leqslant x), \qquad \nu_\mu(x) = P(\gamma(\alpha, 1/\mu) \leqslant x),$$

$$R_1(x) = \nu_\mu(x\theta), \qquad R_2(x) = \int_{R^+} \Phi\left(\frac{x\theta - u\mu}{\sqrt{\theta\mu_2 u}}\right) \nu(du).$$

• **Weibull:** The random variables $(X_n;\, n \geqslant 1)$ are nonnegative and independent, with common Weibull distribution

$$P(X_n \leqslant t) = \int_0^t abx^{b-1} \exp(-ax^b)\, dx.$$

Here X_n is a running time between two renewal procedures and R is the reliability of the system.

(**N.A.**: $\theta = 0.01$, $\alpha = 3$, $a = 1$, $b = 0.5$; see Figure 1.)

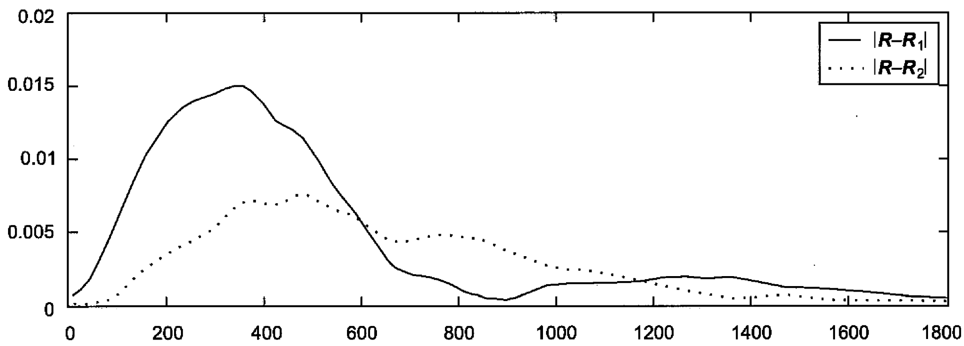

FIGURE 1. Weibull distribution

• **Normal:** $(X_n;\, n \geqslant 1)$ is i.i.d. with common distribution

$$X_n = \mu + \varepsilon_n \qquad \text{with } \varepsilon_n \sim \mathcal{N}(0, \sigma^2) \ (\sigma > 0).$$

Here (ε_n) are Gaussian fluctuations. Typically, X_n is the performance of the machines between two renewal procedures. Then S_{N_θ} is the cumulated performance before halt.

(**N.A.**: $\theta = 0.01$, $\alpha = 3$, $\mu = 2$, $\sigma = \sqrt{20}$; see Figure 2.)

• **Martingale:** Let (ξ_n) be independent random variables having the common distribution $\mathcal{N}(0, b^2)$ $(b > 0)$, and $|a| < 1$. We set $Y_0 = 0$ and, for $n \geqslant 1$,

$$X_n = \mu + \varepsilon_n \qquad \text{with } \varepsilon_n = Y_{n-1}\xi_n, \ Y_n = aY_{n-1} + \xi_n.$$

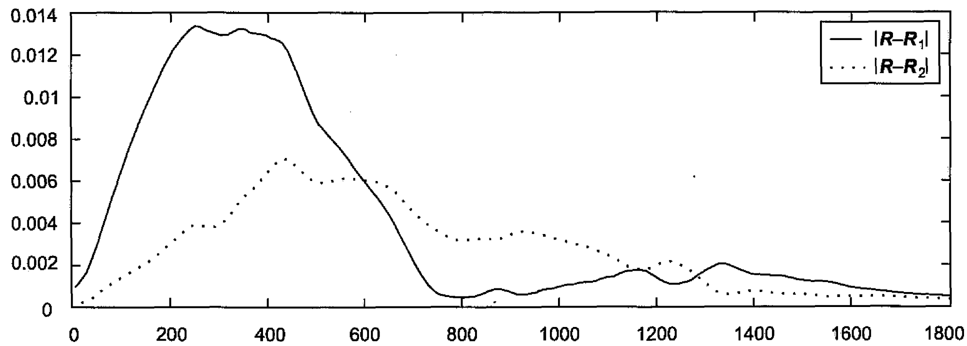

FIGURE 2. Normal distribution

Here, the perturbation sequence (ε_n) is defined by an autoregressive sequence (Y_n), (ε_n) are square martingale differences, satisfying the assumption of Theorem 2.1.

(N.A.: $\theta = 0.01$, $\alpha = 3$, $\mu = 2$, $a = 0.5$, $\sigma = \sqrt{20}$; see Figure 3.)

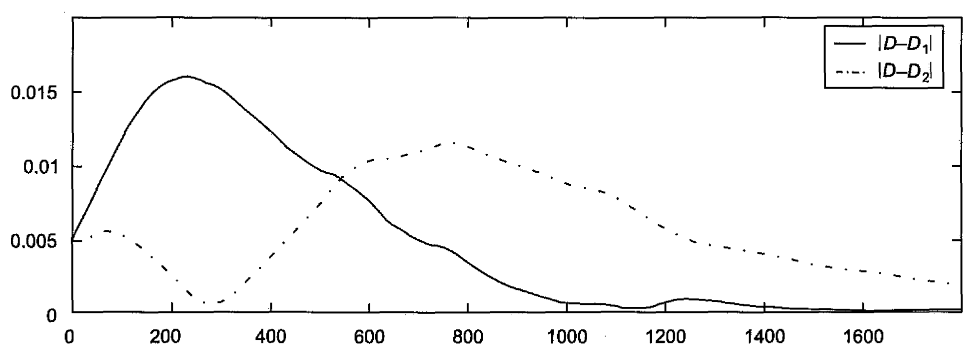

FIGURE 3. Martingale

Exact evaluations of $\boldsymbol{R}$ are obtained empirically by the Monte Carlo simulations. Numerical integration is used for the second-order approximation. In every case, $\boldsymbol{E}[S_{N_\theta}] = 600$.

Remarks. Analytical expansion of the rate of convergence of the second-order approximation will be done in a forthcoming paper. Nevertheless, from numerical calculations some conjectures may be done: The second-order approximation $\boldsymbol{R}_2$ seems to be better for $x \leqslant \boldsymbol{E}[S_{N_\theta}]$ (useful times of the system in concrete applications). Actually, in the first-order approximation, invoking the law of large numbers, the summands are replaced by their "mean". Consequently, the first-order approximation is enough for x of the order of $\boldsymbol{E}[S_{N_\theta}]$.

However, for small x, the law of large numbers has few effects. In the second-order approximation, the variance of the summands has been taken into account. That is why it is better for x on that scale.

Furthermore, for the first-order approximation, the uniform distance is maximum for small t. Thus the second-order approximation $\boldsymbol{R}_2$ is better for the uniform distance, even for the classical scheme of positive and i.i.d. summands.

6. CONCLUSIONS

In conclusions, we show that exponential approximations, as Theorem 1.1, can be generalized for random sums with summands which are neither nonnegative nor i.i.d. and a random index which is a stopping time. Those approximations appear as the first-order ones. Consequently, we propose a second-order approximation for random sums with independent index. In this paper we have chosen a martingale frame for the summands, but another kind of dependence (Markovian or mixing) will be studied in a forthcoming paper.

Further, in our numerical experiments, the uniform error of the second-order approximation is possibly overestimated. Indeed, the numerical evaluation of the integral involved in the second-order approximation is bad for large values. In a forthcoming paper we will give some expansions of this integral.

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