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FUNCTIONAL LIMIT THEOREMS FOR PROBABILITY MEASURES ON HYPERGROUPS

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Abstract. Let K be a hypergroup with left Haar measure and (ν_n) a sequence of symmetric probability measures on K converging to ε_e. We will prove a functional limit theorem in the sense that convergence $\nu_n^{k_n} \to \mu \in \mathcal{M}^1(K)$ implies unique embeddability of μ into a symmetric convolution semigroup $(\mu_t)_{t \geqslant 0}$ and $\nu_n^{[k_n t]} \to \mu_t$ holds for all $t > 0$. This generalizes the corresponding result for hermitian hypergroups. Furthermore, by analogy with locally compact groups, it can be shown that for specific hypergroups similar results are available without symmetry assumptions.

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1. INTRODUCTION AND NOTATION

For triangular systems of identically distributed probability measures on $\boldsymbol{R}^d$ it is well known that weak convergence $\nu_n^{k_n} \to \mu$ implies that the limit measure μ is infinitely divisible, and hence uniquely embeddable into the continuous convolution semigroup $(\mu^t)_{t \geqslant 0}$. Further, the discrete semigroups $(\nu_n^{[k_n t]})_{t \geqslant 0}$ converge to $(\mu^t)_{t \geqslant 0}$, which means that

$$\nu_n^{[k_n t]} \xrightarrow[n \to \infty]{} \mu^t, \quad t \geqslant 0,$$

and the convergence is uniform on compact subsets of $\boldsymbol{R}_+$. This so-called *functional convergence* is essential for most nontrivial properties of limit laws $\mu = \lim_{n \to \infty} \nu_n^{k_n}$ including, for example, the concept of (semi) stability (see, e.g., [6] or [10]).

However, studying such convolution products on more general algebraic structures as, for example, hypergroups it is not clear whether the conclusion that simple convergence leads to functional convergence can be drawn. Following the notation of Nobel and Telöken ([15] and [18], see also [6]), who both treated the case of locally compact groups, statements concerning this problem are called *functional limit theorems*. Nobel was able to prove a functional limit theorem for strongly root compact groups which have no nontrivial compact subgroups (see [14] and [15]), Telöken generalized it to wider classes of locally compact groups, particularly to those admitting nontrivial compact subgroups (cf. [17] and [18]).

The purpose of this paper, which is drawn from the author's thesis [12], is to study functional limit theorems for probability measures on hypergroups K, where for standard definitions and relevant facts we use the notation in [2]. For this, two different approaches are chosen. On the one hand, a result on the embedding of infinitely divisible measures and a corresponding functional limit theorem for hermitian hypergroups shown in [12] are generalized to the case of symmetric probability measures on a hypergroup with left Haar measure. On the other hand, there are no difficulties in transferring Telöken's functional limit theorem from locally compact groups to hypergroups. After doing so, where a (suitably defined) assumption of root compactness occurs, we give examples for hypergroups satisfying the requirements of this general theorem.

Notation. For a locally compact Hausdorff space E let $\mathscr{B}(E)$ denote the Borel σ-algebra in E, $\mathscr{M}^1(E)$ the probability measures, and $\mathscr{M}_+^{(1)}(E)$ the positive contractive Radon measures on E. If for a sequence $(\mu_n)_{n\in N} \subseteq \mathscr{M}_+^{(1)}(E)$ the kind of convergence $\lim_{n\to\infty} \mu_n = \mu$ is not specified, then always weak convergence is meant. For $x \in E$, ε_x denotes the point measure in x. We write $C_c(E)$ for the vector space of continuous functions on E with compact support and $C^b(E)$ for the vector space of continuous and bounded functions on E. For a bounded function $f\colon E \to C$, $\|f\|_\infty$ is the sup-norm.

In the sequel we will make use of the following notation (see [2]): K always denotes a hypergroup, e the neutral element, and $K \ni x \mapsto x^-$ the involution of K. We always assume K to be second countable; then the vague topology of $\mathscr{M}_+^{(1)}(K)$ is metrizable. For bounded Radon measures μ, ν on K we denote by $\mu * \nu$ the convolution of μ and ν, and by μ^n, $n \in N$, the n-fold convolution product of μ. If K is commutative, then let $\hat{K}$ be the dual of K and $\hat{\mu}$ the Fourier transform of a bounded Radon measure μ on K. We write ω_K for a left Haar measure on K. For a nonvoid subset A of K, $[A]$ denotes the smallest subhypergroup of K containing A. If $(\mu_t)_{t\geqslant 0}$ is a continuous convolution semigroup in $\mathscr{M}_+^{(1)}(K)$, that means $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$ and $\lim_{t\searrow 0} \mu_t = \mu_0$ $(\neq 0)$ with respect to the weak topology, then necessarily $\mu_0 = \omega_H$, where H is a compact subhypergroup of K and ω_H its normed Haar measure ([2], 5.2.3).

For a hypergroup K admitting a left Haar measure ω_K we define, as usual, the Hilbert space $L^2(K) = L^2(K, \omega_K)$ with $\langle f, g\rangle = \int f\bar{g}\, d\omega_K$ for $f, g \in L^2(K)$. Let $L(L^2(K))$ denote the space of all linear and bounded operators $A\colon L^2(K) \to L^2(K)$ endowed with the operator norm. We write R_μ for the (left) convolution operator of $\mu \in \mathscr{M}_+^{(1)}(K)$, that means $R_\mu f = \mu * f$ for $f \in L^2(K)$, where the convolution of measures and functions is defined by

$$(\mu * f)(x) = \int\int f d(\varepsilon_{y^-} * \varepsilon_x) d\mu(y), \qquad x \in K.$$

If X is a topological space and M a subset of X, then $\overline{M}$ is the topological closure and $\mathrm{AP}(M)$ the set of all accumulation points of M.

Throughout this paper, $(k_n)_{n\in N}$ always denotes a sequence of natural numbers with $k_n \nearrow \infty$.

2. A FUNCTIONAL LIMIT THEOREM FOR SYMMETRIC PROBABILITY MEASURES

Remark 2.1. The following facts are known for hermitian hypergroups ([2], 5.3.11 and 5.3.4):

(a) Let $(\nu_n)_{n\in N} \subseteq \mathcal{M}^1(K)$ be infinitesimal, which means $\nu_n \xrightarrow[n\to\infty]{} \varepsilon_e$. If $(\nu_n^{k_n})_{n\in N}$ converges to a measure $\mu \in \mathcal{M}^1(K)$, then μ is infinitely divisible.

(b) If $\mu \in \mathcal{M}^1(K)$ is infinitely divisible, then there exists a unique continuous convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathcal{M}^1(K)$ such that $\mu_1 = \mu$.

In this situation, that means under the assumptions of (a), the relation $\hat{\mu}_t = (\hat{\mu})^t$, $t > 0$, holds for the Fourier transforms, and using the Lévy continuity theorem ([2], 4.2.2) one gets functional convergence

$$\nu_n^{[k_n t]} \xrightarrow[n\to\infty]{} \mu_t \text{ uniform on compact subsets of }]0, \infty[.$$

Note that for hermitian hypergroups the characters, and hence the Fourier transforms of probability measures are real-valued.

The proof of statement (b) can be done by showing that the functions $f_t := (\hat{\mu})^t \in C^b(\hat{K})$, $t > 0$, and $f_0 := 1_{\{\hat{\mu}>0\}}$ are Fourier transforms of probability measures $\mu_t \in \mathcal{M}^1(K)$ ([19], Theorem 4.3, or [2], Theorem 5.3.4). Considering hypergroups with left Haar measure ω_K, a similar conclusion can be drawn by making use of operators, provided the measures $\mu_n \in \mathcal{M}^1(K)$ with $\mu_n^n = \mu$ are symmetric in the sense that $\mu_n^- = \mu_n$ holds. The parts of the functions f_t are taken by operators $(T_t)_{t\geq 0}$ in $L^2(K)$, that can be constructed in an obvious way from the convolution operator R_μ via the spectral theorem for normal operators in Hilbert spaces. Based on these ideas it is possible to prove a functional limit theorem — in generalization to that in Remark 2.1 — for the case when $(\nu_n)_{n\in N}$ is a sequence of symmetric probability measures converging to ε_e.

Throughout this section let K always denote a hypergroup with left Haar measure ω_K and let the convolution operators act on $L^2(K)$. If $\mu \in \mathcal{M}^1(K)$ is symmetric, then the corresponding convolution operator R_μ is self-adjoint.

At first, the symmetric analogue of (b) in Remark 2.1 is proved.

THEOREM 2.2. *Let $\mu \in \mathcal{M}^1(K)$ be a symmetric infinitely divisible measure, which means that for every $n \in N$ there exists a symmetric measure $\mu_{(n)} \in \mathcal{M}^1(K)$*

with $\mu_{(n)}^n = \mu$. Then there exists a unique continuous convolution semigroup $(\mu_t)_{t \geq 0}$ in $\mathcal{M}^1(K)$ such that $\mu_1 = \mu$ and $\mu_t^- = \mu_t$ for all $t \geq 0$.

For the proof of Theorem 2.2 we will need the following lemma which is known for locally compact groups and can be transferred to hypergroups.

LEMMA 2.3. *Let the mapping Φ: $\mathcal{M}_+^{(1)}(K) \to L(L^2(K))$ be defined by*

$$\mathcal{M}_+^{(1)}(K) \ni \mu \mapsto \Phi(\mu) := R_\mu .$$

Then Φ is injective and continuous with respect to the vague topology, resp. the weak topology of operators in $L(L^2(K))$. Hence $\Phi(\mathcal{M}_+^{(1)}(K))$ is compact and Φ^{-1}: $\Phi(\mathcal{M}_+^{(1)}(K)) \to \mathcal{M}_+^{(1)}(K)$ is also continuous.

Proof. Theorem 6.2I in [9] yields the injectivity of Φ. If $(\mu_n)_{n \in N} \subseteq \mathcal{M}_+^{(1)}(K)$ is a sequence converging to $\mu \in \mathcal{M}_+^{(1)}(K)$ vaguely, then since for $\nu \in \mathcal{M}_+^{(1)}(K)$ and $f, g \in C_c(K)$

$$\langle R_\nu f, g\rangle = \int_K (f * g^{\sim})(y)\, d\nu^-(y)$$

holds and $f * g^{\sim}$ is in $C_c(K)$, we have

$$\langle R_{\mu_n} f, g\rangle \xrightarrow[n\to\infty]{} \langle R_\mu f, g\rangle \tag{1}$$

for all $f, g \in C_c(K)$. Here $g^{\sim}$ denotes the mapping $g^{\sim}(x) = \overline{g(x^-)}$ for $x \in K$. Because $C_c(K)$ is dense in $L^2(K)$, (1) is valid for all $f, g \in L^2(K)$, and hence Φ is continuous. The remaining part of the lemma follows by Theorem 8.12 in [16]. ∎

Proof of Theorem 2.2. Choose $\mu_{(2)} \in \mathcal{M}^1(K)$ such that $\mu_{(2)}^2 = \mu$ and $\mu_{(2)}^- = \mu_{(2)}$. Since $R_{\mu_{(2)}}$ is self-adjoint, the convolution operator R_μ is positive semidefinite (in short: positive), i.e.

$$\langle R_\mu f, f\rangle \geq 0 \quad \text{for all } f \in L^2(K).$$

The spectral theorem for normal operators in Hilbert spaces (e.g. [11], Theorem 18.10) implies the existence of a unique spectral measure E on the spectrum $\sigma(R_\mu) \subseteq [0, 1]$ such that

$$R_\mu = \int z\, dE,$$

where z denotes the identical mapping on $\sigma(R_\mu)$. Defining, for $t > 0$, h_t: $\sigma(R_\mu) \to \boldsymbol{R}_+$ by $h_t(x) := x^t$ and h_0 by $h_0 := 1_{\{x \in \sigma(R_\mu): x \neq 0\}}$, we see that the functions h_t are Borel-measurable and bounded, and they satisfy $h_{s+t} = h_s h_t$ for $s, t \geq 0$. If T_t for $t \geq 0$ is the operator

$$T_t := \int h_t\, dE, \tag{2}$$

then $(T_t)_{t \geq 0} \subseteq L(L^2(K))$ is a family of positive operators such that $T_1 = R_\mu$, $\|T_t\| \leq 1$ for all $t \geq 0$,

$$T_{s+t} = T_s T_t \quad \text{for all } s, t \geq 0, \tag{3}$$

and that $\boldsymbol{R}_+ \ni t \mapsto T_t$ is continuous with respect to the weak topology of operators. For $n \in N$, $T_{1/n}$ is the unique positive n-th root of the positive operator R_μ, and the family $(T_t)_{t \geq 0}$ is uniquely determined by the above properties.

Given $n \in N$ there exists $\mu_{(2n)} \in \mathscr{M}^1(K)$ such that $\mu_{(2n)}^{2n} = \mu$ and $\mu_{(2n)}^- = \mu_{(2n)}$. But then $R_{\mu_{(2n)}^2}$ is positive, and hence coincides with $T_{1/n}$. With the definition

$$\mu_r := \mu_{(2q)}^{2p}$$

for $r \in \boldsymbol{Q}_+^*$, $r = p/q$, where $p, q \in N$, we obtain

$$R_{\mu_r} = (R_{\mu_{(2q)}^2})^p = (T_{1/q})^p = T_{p/q} = T_r,$$

and thus $(\mu_r)_{r \in \boldsymbol{Q}_+^*}$ is a (symmetric) rational convolution semigroup.

Let now t be an element of $\boldsymbol{R}_+$ and $(r_n) \subseteq \boldsymbol{Q}_+^*$ a sequence converging to t. Then $(\mu_{r_n})_{n \in N}$ has a vague accumulation point $\lambda_t \in \mathscr{M}_+^{(1)}(K)$. Lemma 2.3 together with the continuity of $t \mapsto T_t$ and $R_{\mu_r} = T_r$ for all $r \in \boldsymbol{Q}_+^*$ implies

$$\langle T_t f, g \rangle = \langle R_{\lambda_t} f, g \rangle \quad \text{for all } f, g \in L^2(K),$$

and hence $T_t = R_{\lambda_t}$. In particular, λ_t is the only vague accumulation point of $(\mu_{r_n})_{n \in N}$, so that $\mu_{r_n} \to \lambda_t =: \mu_t$ holds with respect to the vague topology.

$(\mu_t)_{t \geq 0}$ is a convolution semigroup, and since for each $r \in \boldsymbol{Q}_+^*$ the measure μ_r is a probability measure, $(\mu_t)_{t \geq 0}$ is contained in $\mathscr{M}^1(K)$. The convolution semigroup is continuous by construction. ∎

The applied methods of functional analysis can also be used to prove the following functional limit theorem.

THEOREM 2.4. *Let $(\nu_n)_{n \in N} \subseteq \mathscr{M}^1(K)$ be a sequence of symmetric probability measures with $\nu_n \xrightarrow[n \to \infty]{} \varepsilon_e$. If $(\nu_n^{k_n})_{n \in N}$ converges to $\mu \in \mathscr{M}^1(K)$, then μ is uniquely embeddable into a continuous convolution semigroup $(\mu_t)_{t \geq 0} \subseteq \mathscr{M}^1(K)$ with the property $\mu_t^- = \mu_t$ for all $t \geq 0$ and we have functional convergence*

$$\nu_n^{[k_n t]} \xrightarrow[n \to \infty]{} \mu_t \text{ uniform on compact subsets of }]0, \infty[.$$

The proof will be performed in several steps. In one of them the following lemma, which is a special case of Theorem 2 in Section X.7 in [4], is needed.

LEMMA 2.5. *Let T and $(T_n)_{n \in N}$ be positive operators in a Hilbert space H with $\|T\| \leq 1$ and $\|T_n\| \leq 1$ for all $n \in N$. Then $T_n \xrightarrow[n \to \infty]{} T$ implies $\sqrt[N]{T_n} \xrightarrow[n \to \infty]{} \sqrt[N]{T}$ for every $N \in N$, convergence in each case with respect to the strong topology of operators, where $\sqrt[N]{T}$ and $\sqrt[N]{T_n}$, $n \in N$, denote the unique positive operators with $(\sqrt[N]{T})^N = T$, resp. $(\sqrt[N]{T_n})^N = T_n$.*

Proof of Theorem 2.4. Step 1. *R_μ is a positive operator.*

Since $(\nu_n^{k_n})$ converges to μ and (ν_n) to ε_e, we have

$$\langle \nu_n^{k_n} * f, f \rangle \xrightarrow[n \to \infty]{} \langle \mu * f, f \rangle \tag{4}$$

and

(5) $$\langle \nu_n^{k_n+1} * f, f \rangle \xrightarrow[n\to\infty]{} \langle \mu * f, f \rangle$$

for all $f \in L^2(K)$. If $(k_n)_{n\in N}$ admits a subsequence $(k_{n_l})_{l\in N}$ such that k_{n_l} is even for all $l \in N$, then the self-adjointness of $R_{\nu_{n_l}}$ and (4) imply

(6) $$\langle \mu * f, f \rangle \geqslant 0 \quad \text{for all } f \in L^2(K).$$

If there is no such subsequence, then $(k_n)_{n\in N}$ admits a subsequence with odd members only and (6) follows from (5).

Step 2. Let $(T_t)_{t\geqslant 0}$ be the family of positive operators constructed in the proof of Theorem 2.2; in particular, for each $N \in N$, $T_{1/N}$ is the unique positive N-th root of R_μ. Defining for $N \in N$

(7) $$\lambda_n^{(N)} := \nu_n^{[k_n/2N]} * \nu_n^{[k_n/2N]}, \qquad n \in N,$$

we can prove that the convolution operators $(R_{\lambda_n^{(N)}})_{n\in N}$ converge to $T_{1/N}$ in the strong topology of operators. Indeed, write for $n \in N$

$$k_n = \underbrace{\left[\frac{k_n}{2N}\right] + \ldots + \left[\frac{k_n}{2N}\right]}_{2N \text{ summands}} + \varepsilon_n$$

with $\varepsilon_n \in \{0, 1, \ldots, 2N-1\}$. Since $\nu_n \to \varepsilon_e$, we first obtain $\nu_n^{\varepsilon_n} \to \varepsilon_e$, and then because of $\nu_n^{k_n} \to \mu$ and the shift-compactness theorem ([2], Theorem 5.1.4) we get

$$(\lambda_n^{(N)})^N = \underbrace{\nu_n^{[k_n/2N]} * \ldots * \nu_n^{[k_n/2N]}}_{2N \text{ factors}} \xrightarrow[n\to\infty]{} \mu.$$

$R_{(\lambda_n^{(N)})^N}$ and $R_{\lambda_n^{(N)}}$ are positive operators so that Lemma 2.5 gives the above statement.

Step 3. *For every $N \in N$ there exists a symmetric $\mu_{1/N} \in \mathscr{M}^1(K)$ such that $T_{1/N} = R_{\mu_{1/N}}$.*

Indeed, for $N \in N$ fixed and a vague accumulation point $\varrho \in \mathscr{M}_+^{(1)}(K)$ of $(\lambda_n^{(N)})_{n\in N}$ Step 2 implies $R_\varrho = T_{1/N}$. In particular, ϱ is the only accumulation point. Since $R_\mu = R_{\varrho^N}$, $\mu_{1/N} := \varrho$ is a probability measure.

Step 4. Using Theorem 2.2 we obtain a unique continuous convolution semigroup $(\mu_t)_{t\geqslant 0} \subseteq \mathscr{M}^1(K)$ with $\mu_1 = \mu$ and $\mu_t^- = \mu_t$ for all $t \geqslant 0$, where $T_t = R_{\mu_t}$ holds for each $t \geqslant 0$.

Step 5. It remains to show that $\nu_n^{[k_n t]} \to \mu_t$ for all $t > 0$ is satisfied. This is done by proving convergence of the corresponding convolution operators. In this step we assume that for each $n \in N$ the operator R_{ν_n} is positive, which also implies positivity of $R_{\nu_n^{k_n}}$.

For fixed $t > 0$ define the sequence $(r_n) \subseteq \boldsymbol{Q}_+^*$ by $r_n := [k_n t]/k_n$ for n sufficiently large. Let $E^{(n)}$ denote the spectral measure of the positive operator $R_{\nu_n^{k_n}}$. Then we have

$$\int h_{r_n} dE^{(n)} = \left(\int h_{1/k_n} dE^{(n)}\right)^{[k_n t]} = (R_{\nu_n})^{[k_n t]} = R_{\nu_n^{[k_n t]}} \tag{8}$$

for $n \in \boldsymbol{N}$, where the functions $(h_s)_{s>0}$ are defined as in the proof of Theorem 2.2, and so we have to verify that the operators $\left(\int h_{r_n} dE^{(n)}\right)$ converge to $R_{\mu_t} = \int h_t \, dE$.

Let $f \in L^2(K)$ and $\varepsilon > 0$ be given. Since (h_{r_n}) converges to h_t uniformly on $[0, 1]$, there exists $n_0 \in \boldsymbol{N}$ such that $\|h_{r_n} - h_{r_{n_0}}\|_\infty < \varepsilon$ for all $n \geqslant n_0$ and $\|h_{r_{n_0}} - h_t\|_\infty < \varepsilon$. Because of properties of the spectral measure and integrals with respect to it the estimations

$$\left\|\int h_{r_n} dE^{(n)} f - \int h_{r_{n_0}} dE^{(n)} f\right\|_2 \leqslant \varepsilon \|f\|_2 \quad \text{for all } n \geqslant n_0$$

and

$$\left\|\int h_{r_{n_0}} dE f - \int h_t \, dE f\right\|_2 \leqslant \varepsilon \|f\|_2$$

hold true. For $r = M/N \in \boldsymbol{Q}_+^*$ with $M, N \in \boldsymbol{N}$ convergence $R_{\nu_n^{k_n}} \to R_\mu$ and Lemma 2.5 imply

$$\int h_r \, dE^{(n)} = \left(\int h_{1/N} \, dE^{(n)}\right)^M \xrightarrow[n\to\infty]{} \left(\int h_{1/N} \, dE\right)^M = \int h_r \, dE$$

in the strong topology of operators. Thus we can find $n_1 \in \boldsymbol{N}$ with $n_1 \geqslant n_0$ such that

$$\left\|\int h_{r_{n_0}} dE^{(n)} f - \int h_{r_{n_0}} dE f\right\|_2 \leqslant \varepsilon \quad \text{for all } n \geqslant n_1.$$

The following inequality, valid for all $n \in \boldsymbol{N}$, completes the proof:

$$\begin{aligned} \left\|\int h_{r_n} dE^{(n)} f - \int h_t \, dE f\right\|_2 &\leqslant \left\|\int h_{r_n} dE^{(n)} f - \int h_{r_{n_0}} dE^{(n)} f\right\|_2 \\ &+ \left\|\int h_{r_{n_0}} dE^{(n)} f - \int h_{r_{n_0}} dE f\right\|_2 + \left\|\int h_{r_{n_0}} dE f - \int h_t \, dE f\right\|_2. \end{aligned}$$

With the same methods we can prove that the convergence is uniform on compact subsets of $]0, \infty[$. In fact, let $t > 0$; then defining $(r_n) \subseteq \boldsymbol{Q}_+^*$ by $r_n := [k_n t_n]/k_n$ for a sequence $(t_n) \subseteq \boldsymbol{R}_+^*$ converging to t, we conclude that

$$\nu_n^{[k_n t_n]} \xrightarrow[n\to\infty]{} \mu_t.$$

Step 6. Let $\delta_n := \nu_n * \nu_n$ for $n \in \boldsymbol{N}$ and $\varrho := \mu * \mu$. Then $(R_{\delta_n})_{n \in \boldsymbol{N}}$ are positive operators and Steps 1–5 imply that ϱ is uniquely embeddable into a continuous convolution semigroup $(\varrho_t)_{t \geqslant 0}$ with $\varrho_t^- = \varrho_t$ for all $t \geqslant 0$ and that functional convergence

$$\delta_n^{[k_n t]} \xrightarrow[n\to\infty]{} \varrho_t \text{ uniform on compact subsets of }]0, \infty[\tag{9}$$

holds true. Defining $\mu_t := \varrho_{t/2}$ for $t \geqslant 0$, we infer that $(\mu_t)_{t \geqslant 0}$ is the unique continuous convolution semigroup in $\mathcal{M}^1(K)$ satisfying $\mu_1 = \mu$ and $\mu_t^- = \mu_t$ for all $t \geqslant 0$. For fixed $t > 0$ and for a sequence $(t_n) \subseteq \boldsymbol{R}_+^*$ converging to t, (9) yields

$$\nu_n^{[k_n t_n]} = \nu_n^{[k_n t_n/2]+[k_n t_n/2]+\varepsilon_n} = \delta_n^{[k_n t_n/2]} * \nu_n^{\varepsilon_n} \xrightarrow[n\to\infty]{} \varrho_{t/2} = \mu_t,$$

where $\varepsilon_n \in \{0, 1\}$. ■

Remark 2.6. (a) Theorem 2.2 was formulated and proved first because the result on embedding is itself interesting, and furthermore, transferring the proof from the corresponding result for hermitian hypergroups in [12] to the level of operators, stresses the motivation for this functional analytic approach.

(b) Steps 1–3 yield the symmetric analogue to statement (a) in Remark 2.1 for hermitian hypergroups.

(c) With the methods used in this section we can also prove that for each symmetric convolution semigroup $(\mu_t)_{t>0}$ in $\mathcal{M}^1(K)$ the limit $\lim_{t\searrow 0}\mu_t$ exists with respect to the weak topology, and so $(\mu_t)_{t>0}$ is continuous. We only have to assume the existence of a left Haar measure on K. This generalizes the result in [2] (Proposition 5.2.7 (a)) for commutative hypergroups, which is verified by methods of Fourier transform.

3. A GENERAL FUNCTIONAL LIMIT THEOREM

In Section 2 a functional limit theorem was proved by making use of methods of functional analysis, which we were able to apply to the convolution operators because of symmetry of the probability measures (ν_n). We will now drop this assumption and prove a functional limit theorem that can be transferred almost verbatim from the case of locally compact groups treated in [17] and [18]. By analogy with this, the notion of infinitesimality is replaced by a more general concept.

DEFINITION 3.1. Let K be a hypergroup. A sequence $(\nu_n)_{n\in\boldsymbol{N}} \subseteq \mathcal{M}^1(K)$ is called *infinitesimal* if

(i) $(\nu_n)_{n\in\boldsymbol{N}}$ converges, $\nu_n \xrightarrow[n\to\infty]{} \nu \in \mathcal{M}^1(K)$, and

(ii) $H := [\operatorname{supp}(\nu)]$ is compact and ν is not supported on a coset $\{x\} * G$ of any proper supernormal subhypergroup G of H.

The theorem of Kawada–Itô for compact hypergroups (Theorem 5.1.17 in [2]) gives the following characterization of infinitesimality.

PROPOSITION 3.2. *Let K be a hypergroup.*

(a) *If $(\nu_n)_{n\in\boldsymbol{N}} \subseteq \mathcal{M}^1(K)$ is infinitesimal, then $\nu^l \xrightarrow[l\to\infty]{} \omega_H$.*

(b) *For $(\nu_n)_{n\in\boldsymbol{N}} \subseteq \mathcal{M}^1(K)$ assume that $\nu_n \xrightarrow[n\to\infty]{} \nu \in \mathcal{M}^1(K)$ and $\nu^l \xrightarrow[l\to\infty]{} \omega_H$, where H is a compact subhypergroup of K. Then $(\nu_n)_{n\in\boldsymbol{N}}$ is infinitesimal and $[\operatorname{supp}(\nu)] = H$ holds true.*

Proof. (a) is clear by the Kawada–Itô theorem.

For (b) it has to be shown first that $[\operatorname{supp}(\nu)]$ is compact, and then the assertion also follows from the Kawada–Itô theorem. But the assumption $\nu^l \to \omega_H$, which leads to $\nu * \omega_H = \omega_H$, and Theorems 1.6.9 and 1.6.3 in [2] imply compactness. ■

Remark 3.3. (a) $(\nu_n)_{n\in N} \subseteq \mathcal{M}^1(K)$ with $\nu_n \to \varepsilon_e$ is infinitesimal, and so is $(\nu_n)_{n\in N} \subseteq \mathcal{M}^1(K)$ with $\nu_n \to \omega_H$, where H is a compact subhypergroup of K.

(b) If K admits no nontrivial compact subhypergroups, then $(\nu_n)_{n\in N}$ is infinitesimal iff (ν_n) converges to ε_e.

(c) It suffices to demand the existence of $\varrho := \lim_{l\to\infty} \nu^l$ in Proposition 3.2 (b).

(d) The result in Remark 2.1 (a) remains valid if the sequence is infinitesimal in the sense of the new definition and thus a functional limit theorem can easily be proved under this condition.

By analogy with the case of locally compact groups we define:

DEFINITION 3.4. A hypergroup K is called *aperiodic* if $\{e\}$ is its only compact subhypergroup.

THEOREM 3.5. *Let K be a hypergroup and $(\nu_n)_{n\in N} \subseteq \mathcal{M}^1(K)$ be infinitesimal with ν and H as in Definition 3.1. Assume that*

$$\mathcal{R} := \overline{\{\nu_n^l \colon n \in N,\, 0 \leqslant l \leqslant k_n\}}$$

is compact. If $(\nu_n^{k_n})_{n\in N}$ converges to a probability measure $\mu \in \mathcal{M}^1(K)$, then there exist a subsequence $\tilde{N} \subseteq N$ and a rational convolution semigroup $(\mu_r)_{r\in Q_+^*}$ such that

$$\nu_n^{[k_n r]} \to \mu_r, \qquad n \in \tilde{N},$$

and $\mu_r * \omega_H = \omega_H * \mu_r = \mu_r$, $\mu_r * \nu = \nu * \mu_r = \mu_r$ hold for all $r \in Q_+^*$. If furthermore $C \subseteq K$ is a compact subhypergroup such that

$$AP\{\nu_n^{r_n} \colon n \in N\} \subseteq \mathcal{M}^1(C) \tag{10}$$

holds for every sequence $(r_n)_{n\in N} \subseteq N$ with $r_n/k_n \xrightarrow[n\to\infty]{} 0$ as well as

$$\mu_r * \omega_C = \omega_C * \mu_r = \mu_r \quad \textit{for all } r \in Q_+^*, \tag{11}$$

then $(\mu_r)_{r\in Q_+^*}$ has a unique extension to a continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ with $\mu_0 = \omega_C$, and it follows that

$$\nu_n^{[k_n t]} \to \mu_t,\ n \in \tilde{N}, \quad \textit{for all } t > 0.$$

Moreover, the convergence is uniform on compact subsets of $]0, \infty[$ and uniform on compact subsets of R_+ iff $\nu_n \xrightarrow[n\to\infty]{} \omega_C$ (where $\nu_n^0 := \omega_C$).

Proof. We sketch the proof only in a very brief form because it can be directly transferred from that of Theorem 2.3 in [18].

If $L \in N$, then compactness of $\mathscr{R}$, the Tykhonov theorem and the continuity of the mapping $\mathscr{M}^1(K) \times \mathscr{M}^1(K) \ni (\mu, \nu) \mapsto \mu * \nu$ with respect to the weak topology guarantee that $\mathscr{R} * \ldots * \mathscr{R}$ (L factors) is compact. Since

$$\mathscr{R}_L := \overline{\{\nu_n^l \colon n \in N,\ 0 \leqslant l \leqslant k_n L\}}$$

is contained in $\mathscr{R} * \ldots * \mathscr{R}$, it is also compact. The latter property is important for the whole proof, which makes essentially use of subsequence arguments. The first part of the theorem can be verified analogously to Steps 1–3 in Theorem 2.3 in [18]. Since $(\mathscr{M}^1(K), *)$ is a topological semigroup, we can apply Lemma 3.4.4 and Theorem 3.4.6 in [8] in order to show that the semigroup homomorphism $Q_+ \ni r \mapsto \mu_r$, where $\mu_0 := \omega_C$, has a unique continuation to a continuous convolution semigroup $(\mu_t)_{t \geqslant 0}$. Using the methods of Step 4 in [18] we can prove that $Q_+ \ni r \mapsto \mu_r$ is continuous at 0. Note that at this point we make use of the fact that the weak topology on $\mathscr{M}^1(K)$ is metrizable (since K is second countable) and that the assumptions (10) and (11) are needed. The remaining parts of the theorem can be proved analogously to Steps 5–7 in [18], where again subsequence arguments are used and the conditions (10) and (11) are applied. ∎

Remark 3.6. (a) For the subhypergroups H and C the inclusion $H \subseteq C$ holds.

(b) In [18] (Remark 2.4) examples on the torus are constructed, which show that the functional limit theorem is not valid if conditions (10) and (11) are dropped and that $H \subsetneqq C$ is possible.

4. A FUNCTIONAL LIMIT THEOREM FOR APERIODIC AND STRONGLY ROOT COMPACT HYPERGROUPS

In the functional limit Theorem 3.5, the compactness of $\mathscr{R}$ and the conditions (10) and (11) are — beside the infinitesimality of the sequence (ν_n) — the essential assumptions. In this section we will first introduce the notion of a strongly root compact hypergroup for which convergence $\nu_n^{k_n} \to \mu$ immediately implies that $\mathscr{R}$ is compact. Further, it is shown that conditions (10) and (11) are fulfilled for hypergroups which are aperiodic. Theorem 4.8 is an application of Theorem 3.5 for hypergroups satisfying the two properties just mentioned and corresponds to the functional limit theorem proved by Nobel for locally compact groups ([15], Theorem 1).

Definition 4.1. A hypergroup K is called *strongly root compact* if for every relatively compact subset $\mathscr{N} \subseteq \mathscr{M}^1(K)$ the root set

$$\mathscr{R}(\mathscr{N}) := \bigcup_{\mu \in \mathscr{N}} \mathscr{R}(\mu)$$

is also relatively compact, where

$$\mathscr{R}(\mu) := \bigcup_{n\in N} \{\nu^m \colon \nu \in \mathscr{M}^1(K) \text{ with } \nu^n = \mu,\ 1 \leqslant m \leqslant n\}$$

is the root set of μ.

Remark 4.2. If we define B-strongly root compact hypergroups by analogy with the case of locally compact groups ([8], Definition 3.1.10) strengthening the notion of root compactness in [1], these hypergroups are also strongly root compact in the sense of Definition 4.1. This can be shown in the same way as the implication (i)$\Rightarrow$(ii) in Theorem 4.4 in [1] is proved. (For the corresponding result for locally compact groups compare Theorem 3.1.13 in [8].)

A hypergroup K is called *B-strongly root compact* if for every compact subset C of K there exists a compact subset $C_0 \subseteq K$ with the property that for every $n\in N$ the finite sequences $\{x_1, \ldots, x_n\}$ of K with $x_n = e$ satisfying

$$\{x_i\} * C * \{x_j\} * C \cap \{x_{i+j}\} * C \neq \emptyset$$

for all $i+j \leqslant n$ are contained in C_0.

Since B-strong root compactness is a strong property that is difficult to handle and since for the following proposition only strong root compactness in the sense of Definition 4.1 is needed, we will use the latter notion, not only for hypergroups but also for locally compact groups.

PROPOSITION 4.3. *Let K be a hypergroup that is strongly root compact and $(\nu_n)_{n\in N} \subseteq \mathscr{M}^1(K)$. Assume $\nu_n^{k_n} \xrightarrow[n\to\infty]{} \mu \in \mathscr{M}^1(K)$. Then*

$$\mathscr{R} = \overline{\{\nu_n^l \colon n\in N,\ 0 \leqslant l \leqslant k_n\}}$$

is compact.

Proof. The assertion follows immediately from Definition 4.1. ∎

In order to prove that for aperiodic hypergroups the conditions (10) and (11) in the general functional limit Theorem 3.5 are satisfied we need the following proposition.

PROPOSITION 4.4. *Let K be an aperiodic hypergroup. If for $\varrho \in \mathscr{M}^1(K)$ the set $\{\varrho^k \colon k\in N\}$ is relatively compact, then $\varrho = \varepsilon_e$ holds true.*

Remark 4.5. The corresponding result for locally compact groups can be shown by using Theorem 2 in [13] (compare Lemma 2 in [15]). Since it is not known whether this theorem can be transferred to hypergroups, Proposition 4.4 is proved by applying a result on compact affine semigroups in [3].

Proof of Proposition 4.4. By assumption it follows that

$$\mathscr{A} := \overline{\{\varrho^k \colon k\in N\}}$$

is a compact semigroup with respect to $*$. We will show that the closure $\overline{\mathrm{co}(\mathscr{A})}$ of the convex hull of $\mathscr{A}$ is also compact. If this is the case, then $(\overline{\mathrm{co}(\mathscr{A})}, *)$ is a compact affine semigroup in the sense of the definition in [3]. For

$$\delta_n := \frac{1}{n}\sum_{k=1}^{n} \varrho^k \quad \text{for } n \in N$$

the sequence (δ_n) converges to an idempotent measure $\omega \in \overline{\mathrm{co}(\mathscr{A})}$ satisfying $\omega * \varrho = \varrho * \omega = \omega$ by Theorem 1 in [3]. Then the assumption on K implies $\omega = \varepsilon_e$ and hence $\varrho = \varepsilon_e$.

The following lemma shows that $\overline{\mathrm{co}(\mathscr{A})}$ is compact, which will complete the proof. ∎

LEMMA 4.6. *Let E be a (second countable) locally compact Hausdorff space and $\mathscr{A} \subseteq \mathscr{M}^1(E)$ be compact. Then $\overline{\mathrm{co}(\mathscr{A})}$ is compact.*

Proof. Defining for $\lambda \in \mathscr{M}^1(\mathscr{A})$

$$\sigma\colon\ C_c(E) \ni f \mapsto \int_{\mathscr{A}}\int_E f\,d\mu\,d\lambda(\mu),$$

we have $\sigma \in \mathscr{M}^1(E)$, and for $f \in C^b(E)$

$$\int_E f\,d\sigma = \int_{\mathscr{A}}\int_E f\,d\mu\,d\lambda(\mu).$$

Then the mapping

$$\Phi\colon\ \mathscr{M}^1(\mathscr{A}) \to \mathscr{M}^1(E),$$
$$\lambda \mapsto \int_{\mathscr{A}} \mu\,d\lambda(\mu)$$

is continuous (with respect to the weak topologies), and since $\mathscr{M}^1(\mathscr{A})$ is compact, so is $\Phi(\mathscr{M}^1(\mathscr{A}))$.

Now, to prove the assertion it is enough to show that the inclusion $\mathrm{co}(\mathscr{A}) \subseteq \Phi(\mathscr{M}^1(\mathscr{A}))$ holds true. If $\nu \in \mathscr{A}$, then $\nu \in \Phi(\mathscr{M}^1(\mathscr{A}))$ because of $\Phi(\varepsilon_\nu) = \nu$. For

$$\nu = \sum_{i=1}^{m} \alpha_i \nu_i \in \mathrm{co}(\mathscr{A}),$$

where $m \in N$, $\nu_i \in \mathscr{A}$, $\alpha_i \geqslant 0$ and $\sum_{i=1}^m \alpha_i = 1$, define

$$\lambda := \sum_{i=1}^{m} \alpha_i \varepsilon_{\nu_i} \in \mathscr{M}^1(\mathscr{A}).$$

Then $\Phi(\lambda) = \nu$ is fulfilled, and hence $\nu \in \Phi(\mathscr{M}^1(\mathscr{A}))$. ∎

COROLLARY 4.7. *Let K be an aperiodic hypergroup and $(\nu_n)_{n\in N} \subseteq \mathscr{M}^1(K)$. Assume that*

$$\mathscr{R} = \overline{\{\nu_n^l\colon\ n \in N,\ 0 \leqslant l \leqslant k_n\}}$$

is compact. Then $\nu_n^{r_n} \xrightarrow[n\to\infty]{} \varepsilon_e$ *holds for every sequence* $(r_n)_{n\in N} \subseteq N$ *with* $\lim_{n\to\infty} r_n/k_n = 0$.

Proof. If $\varrho = \lim_{n\in\tilde{N}} \nu_n^{r_n}$ is an accumulation point of the relatively compact sequence $(\nu_n^{r_n})$, then $\{\varrho^k\colon k\in N\} \subseteq \mathscr{R}$ is relatively compact, and Proposition 4.4 gives $\varrho = \varepsilon_e$. ∎

Theorem 3.5 implies the following functional limit theorem.

THEOREM 4.8. *Let K be a hypergroup that is strongly root compact and aperiodic and* $(\nu_n)_{n\in N} \subseteq \mathscr{M}^1(K)$. *If* $(\nu_n^{k_n})_{n\in N}$ *converges to a probability measure* $\mu \in \mathscr{M}^1(K)$, *then there exist a subsequence* $\tilde{N} \subseteq N$ *and a continuous convolution semigroup* $(\mu_t)_{t\geq 0} \subseteq \mathscr{M}^1(K)$ *such that*

$$\nu_n^{[k_n t]} \to \mu_t,\ n\in\tilde{N},\ \textit{uniformly on compact subsets of }]0, \infty[.$$

Proof. Using Proposition 4.3 and Corollary 4.7, which in particular implies that (ν_n) is infinitesimal (that means $\nu_n \to \varepsilon_e$ since $\{e\}$ is the only compact subhypergroup of K), we see that the assumptions of Theorem 3.5 are satisfied. ∎

5. EXAMPLES

In this section we give examples for hypergroups satisfying the assumptions of the general functional limit theorem.

Remark 5.1. Note that for the requirement of "root compactness" in the functional limit Theorem 3.5 we only need the following condition:

$$(*)\qquad \nu_n^{k_n} \xrightarrow[n\to\infty]{} \mu \Rightarrow \mathscr{R}_1(\{\nu_n\colon n\in N\}) \text{ relatively compact},$$

where we define

$$\mathscr{R}_1(\{\lambda_n\colon n\in N\}) := \{\lambda_n^l\colon n\in N,\ 0\leqslant l\leqslant k_n\} \subseteq \mathscr{R}(\{\lambda_n\colon n\in N\})$$

for a fixed sequence $(k_n)_{n\in N} \subseteq N$ with $k_n \nearrow \infty$ and a sequence $(\lambda_n)_{n\in N} \subseteq \mathscr{M}^1(K)$ of probability measures.

The condition $(*)$ is satisfied for hermitian Godement hypergroups (see [2], 2.5.3 for the definition) provided that $(\nu_n)_{n\in N}$ is relatively compact. This can easily be shown by using Proposition 5.1.10 in [2]. But since there already exists a functional limit theorem for hermitian hypergroups (cf. Remark 2.1), the example just mentioned is of less interest in this context.

We will now show that examples for strongly root compact and aperiodic hypergroups are given by orbit hypergroups G^H (see the following definition) arising from locally compact groups satisfying these two properties.

DEFINITION 5.2. Let G be a locally compact Hausdorff group, $\mathrm{Aut}(G)$ the group of topological automorphisms of G furnished with the topology described in Definition 26.3 in [7], and $H \subseteq \mathrm{Aut}(G)$ a compact subgroup with normed Haar measure ω_H. If for $x \in G$

$$x^H = \{x^h\colon\ h \in H\} = \{a(x)\colon\ a \in H\}$$

denotes the orbit of x under H, then the orbit space

$$G^H = \{x^H\colon\ x \in G\}$$

is a hypergroup with the convolution

$$\varepsilon_{x^H} * \varepsilon_{y^H} := \int_H \varepsilon_{(a(x)y)^H}\, d\omega_H(a) = \int_H \varepsilon_{(xb(y))^H}\, d\omega_H(b)$$

for $x^H, y^H \in G^H$. In fact, the orbit space G^H is a decomposition of G into compact subsets and a locally compact Hausdorff space with respect to the quotient topology. The neutral element of this hypergroup is $e^H = \{e\}$, and the involution is given by $(x^H)^- = (x^{-1})^H$ for $x \in G$.

Remark 5.3. If $(G^H, *)$ is an orbit hypergroup and

$$\mathscr{M}_H^1(G) := \{\mu \in \mathscr{M}^1(G)\colon\ \mu \text{ is } \tau\text{-invariant for all } \tau \in H\},$$

then the mapping

$$q_H\colon \mathscr{M}_H^1(G) \to \mathscr{M}^1(G^H),$$
$$\mu \mapsto q_H(\mu) := \mu_q,$$

where $\mu_q \in \mathscr{M}^1(G^H)$ denotes the image measure of μ under the canonical mapping $q\colon G \to G^H$, has the following properties (see the proof of Theorem 1.1.7 in [2]): q_H is bijective and for $\mu, \nu \in \mathscr{M}_H^1(G)$

$$\mu * \nu \in \mathscr{M}_H^1(G) \quad \text{and} \quad q_H(\mu * \nu) = q_H(\mu) * q_H(\nu)$$

hold. Further, q_H and q_H^{-1} are continuous.

Proposition 5.4. *Let $K = G^H$ be an orbit hypergroup, where G is strongly root compact. Then K is also strongly root compact.*

Proof. Suppose that $\mathscr{N} \subseteq \mathscr{M}^1(G^H)$ is relatively compact. Since q_H^{-1} is continuous, $q_H^{-1}(\mathscr{N})$ is also relatively compact, and strong root compactness of G implies that the root set $\mathscr{R}(q_H^{-1}(\mathscr{N}))$ is relatively compact. q_H^{-1} is a homomorphism with respect to convolution, and thus the inclusion

$$q_H^{-1}(\mathscr{R}(\mathscr{N})) \subseteq \mathscr{R}(q_H^{-1}(\mathscr{N}))$$

holds. Therefore $q_H^{-1}(\mathscr{R}(\mathscr{N}))$ is relatively compact, and so is $\mathscr{R}(\mathscr{N})$. ∎

Proposition 5.5. *Let $K = G^H$ be an orbit hypergroup, where G is aperiodic. Then K is also aperiodic.*

Proof. Let C be a compact subhypergroup of G^H. Then $\mathscr{M}^1(C)$, regarded as a subspace of $\mathscr{M}^1(K)$, is compact, and so is $q_H^{-1}(\mathscr{M}^1(C)) \subseteq \mathscr{M}^1(G)$. If

$\varrho \in \mathscr{M}^1(C)$, then $\{(q_H^{-1}(\varrho))^k\colon k \in N\}$ is relatively compact because it is contained in $q_H^{-1}(\mathscr{M}^1(C))$. This yields $q_H^{-1}(\varrho) = \varepsilon_e$, which implies $\varrho = \varepsilon_{e^H}$, and hence $C = \{e^H\}$. ∎

Remark 5.6. Applying methods similar to those used in the proofs of Propositions 5.4 and 5.5 we can show the following statements:

(a) If K is a strongly root compact hypergroup and $H \subseteq K$ a compact subhypergroup, then the corresponding double coset hypergroup $K//H$ (see 1.5.13 in [2] for the definition) is also strongly root compact.

(b) If K_1 and K_2 are strongly root compact hypergroups, then the product hypergroup $K_1 \times K_2$ (defined as in 1.5.28 in [2]) is strongly root compact.

(c) If K_1 and K_2 are aperiodic hypergroups, then the product hypergroup $K_1 \times K_2$ is also aperiodic.

Concrete examples for – nonhermitian – hypergroups which are strongly root compact and aperiodic, and hence satisfy the assumptions of the functional limit Theorem 4.8, can be constructed as orbit hypergroups G^H, where the underlying locally compact group G is the Heisenberg group.

EXAMPLE 5.7. (a) Let $G = \boldsymbol{H}_n$ be the $(2n+1)$-dimensional Heisenberg group ($n \in N$), that means $\boldsymbol{H}_n = \boldsymbol{C}^n \times \boldsymbol{R}$ with the composition

$$(z, t)(z', t') = (z+z', t+t'-\tfrac{1}{2}\operatorname{Im}\langle z, z'\rangle)$$

for $(z, t), (z', t') \in \boldsymbol{H}_n$. If $U(n)$ denotes the group of unitary $n \times n$-matrices, then for $A \in U(n)$ the mapping

$$F_A\colon \boldsymbol{H}_n \to \boldsymbol{H}_n,$$
$$(z, t) \mapsto (Az, t)$$

is an automorphism of $\boldsymbol{H}_n$, and $H := \{F_A\colon A \in U(n)\}$ is a compact subgroup of $\operatorname{Aut}(\boldsymbol{H}_n)$. Since the Heisenberg group is simply connected and nilpotent, it is strongly root compact and aperiodic. By Propositions 5.4 and 5.5 the corresponding properties hold for the – nonhermitian – orbit hypergroup G^H.

(b) In particular, for the 3-dimensional Heisenberg group $G = \boldsymbol{H}_1 = \boldsymbol{R}^2 \times \boldsymbol{R}$ with

$$((x, y), s)((a, b), t) = ((x+a, y+b), s+t+\tfrac{1}{2}(xb-ya))$$

for $((x, y), s), ((a, b), t) \in \boldsymbol{H}_1$, $\mathrm{SO}(2) = \{A \in \boldsymbol{R}^{2\times 2}\colon A^T A = I, \det A = 1\} \simeq U(1)$, $H = \{F_A\colon A \in \mathrm{SO}(2)\} \subseteq \operatorname{Aut}(\boldsymbol{H}_1)$, and F_A defined as above, the orbit hypergroup G^H has the desired properties.

The spaces G^H and $\boldsymbol{R}_+ \times \boldsymbol{R}$ are homeomorphic by the mappings

$$\Phi\colon \boldsymbol{R}_+ \times \boldsymbol{R} \to G^H,$$
$$(x, t) \mapsto ((x, 0), t)^H$$

and

$$\Phi^{-1}: G^H \to \boldsymbol{R}_+ \times \boldsymbol{R},$$

$$(x, t)^H \mapsto (\|x\|_2, t)$$

(well-defined and continuous). Carrying the convolution structure from G^H to $\boldsymbol{R}_+ \times \boldsymbol{R}$ one gets

$$\varepsilon_{(x,s)} * \varepsilon_{(y,t)} = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon_{(\sqrt{x^2+y^2+2xy\cos\phi}, s+t+(1/2)xy\sin\phi)} \, d\phi$$

for $(x, s), (y, t) \in \boldsymbol{R}_+ \times \boldsymbol{R}$, and with this convolution the group $\boldsymbol{R}_+ \times \boldsymbol{R}$ is the so-called Laguerre hypergroup (with parameter 0).

(c) A further example of a (nonhermitian) hypergroup having the two properties is the orbit hypergroup G^H, where $G = \boldsymbol{H}_1$ and $H = \{F_A\colon A \in M\}$ with

$$M := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Remark 5.8. For Gelfand pairs (G, H) the corresponding space $G//H$ of double cosets is a commutative hypergroup. In the paper [5] conditions on Gelfand pairs are given such that $K = G//H$ has certain root compactness properties. For example (compare [5], Proposition 5.2), if (G, H) is a symmetric pair and G is 2-root compact (see [8] for the definition), then for any compact set $C \subseteq \mathcal{M}^1(K)$ the factor set

$$\mathscr{F}(C) = \bigcup_{\mu \in C} \mathscr{F}(\mu) = \bigcup_{\mu \in C} \{\nu \in \mathcal{M}^1(K)\colon \exists\, \lambda \in \mathcal{M}^1(K),\ \mu = \nu * \lambda\}$$

is also compact. Since

$$\mathscr{R}_1(\{\nu_n\colon n \in \boldsymbol{N}\}) \subseteq \bigcup_{n \in \boldsymbol{N}} \mathscr{F}(\nu_n^{k_n}),$$

the condition (*) in Remark 5.1 is satisfied.

Concluding Remark 5.9. The hypergroups above are all aperiodic, and therefore the results correspond to those of Nobel for locally compact groups. But since there are many examples of non-aperiodic locally compact groups for which the generalized functional limit theorem of Telöken can be applied (see [17] and [18]), more general examples fulfilling the assumptions of the general functional limit Theorem 3.5 can be constructed also within the framework of hypergroups.

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