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BOUNDS ON THE BIAS OF APPROXIMATION OF FRACTIONAL RECORD VALUES

BY

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Abstract. We present sharp bounds for an approximation of fractional kth record values by convex combinations of ordinary kth record values. The bounds are expressed in different scale units measured in pth central absolute moments of the underlying distribution. The distributions which attain the bounds are also specified. The bounds are derived by the projection method.

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1. INTRODUCTION

Let F be a continuous distribution function with the quantile function

$$F^{-1}(y) = \sup\{t\colon F(t) \leqslant y\}, \qquad y \in [0, 1),$$

and the hazard function $H_F(x) = -\log(1-F(x))$. The inverse function to H_F is

$$\psi_F(x) = F^{-1}(1-e^{-x}), \qquad x \geqslant 0.$$

For a given integer $k \in N$, let $\{Y_t^{(k)}, t \geqslant 0\}$ denote the kth record-values process for F defined by Bieniek and Szynal [2] as $Y_t^{(k)} = \psi_F(W_t^{(k)})$, $t \geqslant 0$, where $\{W_t^{(k)}, t \geqslant 0\}$ is the so-called kth exponential record-values process, i.e. the stochastic process starting from 0 with independent increments which are gamma distributed,

$$W_t^{(k)} - W_s^{(k)} \sim \Gamma(t-s, k), \qquad t > s \geqslant 0.$$

Here $\Gamma(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, denotes the gamma distribution with the density function

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0.$$

The random variables $Y_t^{(k)}$, $t \geqslant 0$, are called *fractional kth record values* since any finite-dimensional vector of fractional record values with integer indices t has the same distribution as the vector of kth record values of the sequence of i.i.d. random variables defined by Dziubdziela and Kopociński [6] as follows. For fixed $k \geqslant 1$ we define the kth record times $U_k(n)$, $n \geqslant 1$, of the sequence $\{X_n, n \geqslant 1\}$ as

$$U_k(1) = 1,$$
$$U_k(n+1) = \min\{j > U_k(n) \colon X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \quad n \geqslant 1,$$

and the kth record values as

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1} \quad \text{for } n \geqslant 1.$$

In the theory of record values it is well known that the kth record value $Y_n^{(k)}$, $n \in N$, can be considered as good approximation for $\psi_F(n/k)$ (see [1], p. 12, or [11]). But for $n/k \notin N$, a better approximation is obtained if the fractional record value is used instead of the ordinary one. Bieniek and Szynal [2] showed that for any fixed $u > 0$ the fractional record value $Y_{ku}^{(k)}$ is a good approximation for $\psi_F(u)$. However, fractional record values are a purely theoretical notion as they cannot be obtained from statistical data. Bieniek and Szynal [2] also stated that the fractional record $Y_t^{(k)}$ may be approximated by the convex combination $(1-\{t\})\, Y_{[t]}^{(k)} + \{t\}\, Y_{[t]+1}^{(k)}$ of neighboring kth record values, where $[t]$ and $\{t\}$ stand for the integer and fractional part of $t \in \boldsymbol{R}$, respectively. The aim of this paper is to derive sharp upper and lower bounds for the expectation of the random variable

$$\Delta_{n,h}^{(k)} = Y_{n+h}^{(k)} - (1-h)\, Y_n^{(k)} - h Y_{n+1}^{(k)},$$

where $k = 1, 2, \ldots$, $n = 1, 2, \ldots$, and $h \in (0, 1)$, i.e. bounds for the bias of approximation of fractional kth record values by kth record values. Also at the end of the paper we consider a different method of approximating of $Y_{n+h}^{(k)}$ by $Y_n^{(k)}$, where $h \in (0, 1)$. This approximation is obviously worse, especially if h is close to 1, but it does not require the value of $Y_{n+1}^{(k)}$. Therefore we also evaluate the bounds for the expectation of the increment

$$R_{t,h}^{(k)} = Y_{t+h}^{(k)} - Y_t^{(k)}$$

for $k = 1, 2, \ldots$, $t \geqslant 1$, and $h > 0$. Bounds for the expectation of $R_{t,h}^{(k)}$ with t, $h \in N$ can be found in [10] and [5], but they cannot be applied here since we are especially interested in the case $h \in (0, 1)$.

2. AUXILIARY RESULTS

From the above definition of fractional records and results of [2] one can easily obtain the representation

$$EY_t^{(k)} = \int_0^1 F^{-1}(x)\, f_t^{(k)}(x)\, dx,$$

where

$$f_t^{(k)}(x) = \frac{k^t}{\Gamma(t)}(1-x)^{k-1}\left(-\log(1-x)\right)^{t-1}, \quad x \in (0,\, 1), \tag{1}$$

is the density function of the fractional record value $U_t^{(k)}$ from the uniform $U(0,\, 1)$ distribution. Therefore

$$E\Delta_{n,h}^{(k)} = \int_0^1 F^{-1}(x)\, g_{n,h}^{(k)}(x)\, dx,$$

where for $k = 1, 2, \ldots,$ $n = 1, 2, \ldots,$ $h \in (0,\, 1)$

$$g_{n,h}^{(k)}(x) = f_{n+h}^{(k)}(x) - (1-h)\, f_n^{(k)}(x) - h f_{n+1}^{(k)}(x), \quad x \in (0,\, 1). \tag{2}$$

Moreover, if $\mu = \mu_F = \int_0^1 F^{-1}(x)\, dx$, then $E\Delta_{n,h}^{(k)}$ may be written as

$$E\Delta_{n,h}^{(k)} = \int_0^1 \left(F^{-1}(x) - \mu\right) g_{n,h}^{(k)}(x)\, dx. \tag{3}$$

If we used Hölder's inequality only, then the last equation would imply

$$E\Delta_{n,h}^{(k)} \leqslant \|g_{n,h}^{(k)}\|_q \sigma_p, \tag{4}$$

with $p,\, q \geqslant 1$ and $1/p + 1/q = 1$, where $\|g\|_q$ denotes the norm of a function g as an element of the Banach space $L^q([0,\, 1],\, dx)$ and

$$\sigma_p = \|F^{-1} - \mu\|_p = \Big(\int_0^1 |F^{-1}(x) - \mu|^p\, dx\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$

$$\sigma_\infty = \|F^{-1} - \mu\|_\infty = \sup_{0 \leqslant x \leqslant 1} |F^{-1}(x) - \mu|$$

denotes the pth central absolute moment of F (writing σ_p, $1 \leqslant p \leqslant \infty$, we tacitly assume that it is finite). But the equality in Hölder's inequality holds iff $F^{-1} - \mu$ is proportional to $g_{n,h}^{(k)}$, which is impossible since the former function is monotone and the latter in general is not. Therefore the bound (4) cannot be sharp, and to obtain sharp bounds we apply Hölder's inequality combined with Moriguti's [9] inequality, which is presented in the following lemma.

LEMMA 1 (Moriguti's inequality). *Let $g\colon [a,\, b] \mapsto \boldsymbol{R}$ be an integrable function. Let $\bar{g}$ be the right derivative of the greatest convex minorant $\bar{G}$ of the antiderivative $G(x) = \int_a^x g(t)\, dt$ of g. Then for every nondecreasing function $f\colon [a,\, b] \mapsto \boldsymbol{R}$*

$$\int_a^b f(x)\, g(x)\, dx \leqslant \int_a^b f(x)\, \bar{g}(x)\, dx. \tag{5}$$

Equality in (5) *holds iff f is constant on every interval where $G(x) > \bar{G}(x)$.*

The function $\bar{g}$ is the projection of g onto the convex cone $\mathscr{C}$ of nondecreasing functions in $L^2([0, 1])$. For detailed formal justification of this statement see [12], pp. 12–16.

Using Moriguti's inequality, by (3) we obtain

$$E\Delta_{n,h}^{(k)} \leqslant \int_0^1 (F^{-1}(x)-\mu)\,\bar{g}_{n,h}^{(k)}(x)\,dx = \int_0^1 (F^{-1}(x)-\mu)\big(\bar{g}_{n,h}^{(k)}(x)-c\big)\,dx \tag{6}$$

for arbitrary $c \in \boldsymbol{R}$. The last equality is true as $\int_0^1 (F^{-1}(x)-\mu)\,dx = 0$. To derive the lower bound on $E\Delta_{n,h}^{(k)}$ it is enough to note that

$$-E\Delta_{n,h}^{(k)} \leqslant \int_0^1 (F^{-1}(x)-\mu)\big(\overline{-g}_{n,h}^{(k)}(x)-c\big)\,dx, \tag{7}$$

where $\overline{-g}_{n,h}^{(k)}$ is the projection of $-g_{n,h}^{(k)}$ onto $\mathscr{C}$.

Similar considerations lead to

$$ER_{t,h}^{(k)} \leqslant \int_0^1 (F^{-1}(x)-\mu)\big(\bar{\varphi}_{t,h}^{(k)}(x)-c\big)\,dx, \tag{8}$$

where for $t \geqslant 1$, $h > 0$, and $k \geqslant 1$

$$\varphi_{t,h}^{(k)}(x) = f_{t+h}^{(k)}(x) - f_t^{(k)}(x), \qquad x \in (0, 1).$$

Therefore we need to determine the projections $\bar{g}_{n,h}^{(k)}$, $\overline{-g}_{n,h}^{(k)}$, and $\bar{\varphi}_{t,h}^{(k)}$. To this end, knowledge of the shapes of $g_{n,h}^{(k)}$ and $\varphi_{t,h}^{(k)}$ is crucial. This will be determined using the variation diminishing property (cf. [13]) of densities of fractional record values proved in [3].

For an arbitrary sequence $\boldsymbol{a} = (a_1, \ldots, a_n) \in \boldsymbol{R}^n$ let $S^-(\boldsymbol{a})$ denote the number of sign changes in the sequence $a_1, \ldots, a_n$ after deletion of zeros. For $k = 1, 2, \ldots$ and $t \in \boldsymbol{R}$ we write

$$u_t^{(k)}(x) = (1-x)^{k-1}\big(-\log(1-x)\big)^{t-1}, \qquad x \in (0, 1).$$

Moreover, for any function $f\colon [0, 1] \mapsto \boldsymbol{R}$ let $Z(f)$ denote the number of zeros of f in $(0, 1)$. The variation diminishing property of densities of fractional record values may be stated as follows.

LEMMA 2 (Bieniek and Szynal [3]). *Let $t_1 < t_2 < \ldots < t_n$ and for $\boldsymbol{a} \in \boldsymbol{R}^n$ let*

$$H_{\boldsymbol{a}}(x) = \sum_{i=1}^{n} a_i u_{t_i}(x), \qquad x \in (0, 1).$$

Then for all $\boldsymbol{a} \neq \boldsymbol{0} \in \boldsymbol{R}^n$ we have $Z(H_{\boldsymbol{a}}) \leqslant S^-(\boldsymbol{a})$. Moreover, the first and last signs of $H_{\boldsymbol{a}}$ are the same as the signs of the first and last nonzero elements of $\boldsymbol{a}$, respectively.

Putting $t_i = i \in \boldsymbol{N}$, $1 \leqslant i \leqslant n$, we obtain Lemma 5 of [7]. Our proof of Lemma 2 uses the following result.

LEMMA 3 (Karlin and Studden [8]). *Assume that $\alpha_1, \ldots, \alpha_n$ is a strictly increasing sequence of real numbers. Then for all $\boldsymbol{a} \in \boldsymbol{R}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$, the number of positive zeros of the polynomial $P_{\boldsymbol{a}}(x) = \sum_{i=1}^{n} a_i x^{\alpha_i}$ does not exceed $S^-(\boldsymbol{a})$.*

Proof of Lemma 2. We have

$$H_{\boldsymbol{a}}(x) = (1-x)^{k-1} \sum_{i=1}^{n} a_i \big(-\log(1-x)\big)^{t_i-1} = \frac{(1-x)^{k-1}}{-\log(1-x)} \sum_{i=1}^{n} a_i z^{t_i},$$

where $z = -\log(1-x)$. Obviously, $Z(H_{\boldsymbol{a}})$ is equal to the number of positive zeros of $\sum_{i=1}^{n} a_i z^{t_i}$ and Lemma 2 follows from Lemma 3. Furthermore, let a_r and a_s denote the first and the last nonzero element of $\boldsymbol{a}$. Then the latter statement follows from the equalities

$$\lim_{x\to 0^+} \frac{H_{\boldsymbol{a}}(x)}{\big(-\log(1-x)\big)^{t_r-1}} = a_r, \qquad \lim_{x\to 1^-} \frac{H_{\boldsymbol{a}}(x)}{(1-x)^{k-1}\big(-\log(1-x)\big)^{t_s-1}} = a_s.$$

This completes the proof. ∎

The shape of $g_{n,h}^{(k)}$ is determined in the following lemma.

LEMMA 4. *For $k \geqslant 2$, $n \geqslant 2$, there exist $0 < x_1 < x_2 < x_3 < 1$ such that $g_{n,h}^{(k)}$ is decreasing on the interval $(0, x_1)$ from 0 to $g_{n,h}^{(k)}(x_1) < 0$, increasing on (x_1, x_2) to $g_{n,h}^{(k)}(x_2) > 0$, decreasing on (x_2, x_3) to $g_{n,h}^{(k)}(x_3) < 0$, and increasing on $(x_3, 1)$ to 0. For $k = 1$, $n \geqslant 2$, $g_{n,h}^{(1)}$ is decreasing on $(0, x_1)$ from 0 to $g_{n,h}^{(1)}(x_1) < 0$, increasing on (x_1, x_2) to $g_{n,h}^{(1)}(x_2) > 0$, and decreasing on $(x_2, 1)$ to 0. For $k \geqslant 2$, $n = 1$, $g_{1,h}^{(k)}$ is increasing on $(0, x_1)$ from $-k(1-h)$ to $g_{1,h}^{(k)}(x_1) > 0$, decreasing on (x_1, x_2) to $g_{1,h}^{(k)}(x_2) < 0$, and increasing on $(x_2, 1)$ to 0. For $k = n = 1$, $g_{1,h}^{(1)}$ is increasing on $(0, x_1)$ from $-(1-h)$ to $g_{1,h}^{(1)}(x_1) > 0$ and decreasing on $(x_1, 1)$ to $-(1-h)$.*

Proof. We prove the first statement of this lemma only. The remaining statements are proved analogously.

First note that (1) and (2) imply that

$$f_t^{(k)}(0) = \begin{cases} 0 & \text{if } t > 1, \\ k & \text{if } t = 1, \\ +\infty & \text{if } t < 1, \end{cases} \qquad g_{n,h}^{(k)}(0) = \begin{cases} 0 & \text{if } n \geqslant 2, \\ -k(1-h) & \text{if } n = 1, \end{cases}$$

and $f_t^{(k)}(1) = 0$ except $f_1^{(1)}(1) = 1$, and $g_{n,h}^{(k)}(1) = 0$ except $g_{1,h}^{(1)} = -(1-h)$. Next, applying Lemma 2 to equation (2) written in the form

$$g_{n,h}^{(k)}(x) = -(1-h)\, f_n^{(k)}(x) + f_{n+h}^{(k)}(x) - h f_{n+1}^{(k)}(x),$$

we see that $g_{n,k}^{(k)}$ is first negative, then positive, and negative ($-+-$, for short) or negative on $(0, 1)$. But the latter case is impossible, since then $\int_0^1 g_{n,h}^{(k)}(x)\,dx = 0$ implies $g_{n,h}^{(k)} \equiv 0$, which is not true (see (2)). Moreover, using the identity

$$\left(f_t^{(k)}(x)\right)' = \frac{1}{1-x}\left(k f_{t-1}^{(k)}(x) - (k-1) f_t^{(k)}(x)\right),$$

which is valid for any $t \in \mathbf{R}$ adopting the convention $f_t \equiv 0$ for $t \in \{0, -1, -2, \ldots\}$, we see that

$$(g_{n,h}^{(k)})'(x) = \frac{1}{1-x}\Big(-k(1-h) f_{n-1}^{(k)}(x) + k f_{n+h-1}^{(k)}(x)$$

$$+\big((k-1)(1-h)-kh\big) f_n^{(k)}(x) - (k-1) f_{n+h}^{(k)}(x) + (k-1) f_{n+1}^{(k)}(x)\Big)$$

is either $-+-+$ or $-+$. Since $g_{n,h}^{(k)}(0) = g_{n,h}^{(k)}(1) = 0$, the second case implies that $g_{n,h}^{(k)}$ is negative on $(0, 1)$, which is impossible. Hence $g_{n,h}^{(k)}$ has the claimed shape. ∎

The shape of $\varphi_{t,h}^{(k)}$ is as follows.

LEMMA 5. *For $k \geqslant 2$ and $t > 1$ there exist points $0 < y_1 < y_2 < 1$ such that $\varphi_{t,h}^{(k)}$ is decreasing on $(0, y_1)$ from 0 to $\varphi_{t,h}^{(k)}(y_1) < 0$, increasing on (y_1, y_2) to $\varphi_{t,h}^{(k)}(y_2) > 0$, and decreasing on $(y_2, 1)$ to 0. For $k = 1$ and $t > 1$, $\varphi_{t,h}^{(1)}$ is decreasing on $(0, y_1)$ from 0 to $\varphi_{t,h}^{(1)}(y_1) < 0$, and increasing on $(y_1, 1)$ to $+\infty$. For $k \geqslant 2$ and $t = 1$, $\varphi_{1,h}^{(k)}$ is increasing on $(0, y_2)$ from $-k$ to $\varphi_{1,h}^{(k)}(y_2) > 0$ and decreasing on $(y_2, 1)$ to 0. For $k = t = 1$, $\varphi_{1,h}^{(1)}$ is increasing on $(0, 1)$ from -1 to $+\infty$.*

Proof. The proof is analogous to that of Lemma 4. It is based on analysis of the derivative

$$(\varphi_{t,h}^{(k)})'(x) = \frac{1}{1-x}\left(-k f_{t-1}^{(k)}(x) + k f_{t+h-1}^{(k)}(x) + (k-1) f_t^{(k)}(x) - (k-1) f_{t+h}^{(k)}(x)\right)$$

with the aid of Lemma 2. Note that since the coefficients of $f_{t+h-1}^{(k)}$ and $f_t^{(k)}$ are both nonnegative, it does not matter whether $t+h-1 < t$ or $t+h-1 > t$. ∎

Now, let $G_{n,h}^{(k)}$ be the antiderivative of $g_{n,h}^{(k)}$, namely $G_{n,h}^{(k)}(x) = \int_0^x g_{n,h}^{(k)}(t)\,dt$. Then by (2) we have $G_{n,h}^{(k)}(0) = 0 = G_{n,h}^{(k)}(1)$. By Lemma 4, for $k \geqslant 2$, $n \geqslant 2$, there exist exactly two zeros θ_1 and θ_2 of $g_{n,h}^{(k)}$ in $(0, 1)$ such that $0 < x_1 < \theta_1 < x_2 < \theta_2 < x_3 < 1$. Obviously, θ_1 and θ_2 are the points of global minimum and maximum, respectively, of $G_{n,h}^{(k)}$.

Let $l_u(x) = G_{n,h}^{(k)}(u) + g_{n,h}^{(k)}(u)(x-u)$, $x \in \mathbf{R}$, be the tangent to the graph of $G_{n,h}^{(k)}$ at $u \in (0, 1)$. We prove that, for $n \geqslant 2$, $k \geqslant 2$, there are exactly two tangent lines passing through each of the points $(0, 0)$ and $(1, 0)$.

LEMMA 6. *For $n \geqslant 2$, $k \geqslant 2$, the equation $l_u(0) = 0$ has exactly two solutions α_*, $\alpha^* \in (0, 1)$ such that $x_1 \leqslant \alpha_* \leqslant \theta_1$, $x_2 \leqslant \alpha^* \leqslant \theta_2$; the equation $l_u(1) = 0$ has exactly two solutions β_*, $\beta^* \in (0, 1)$ such that $\theta_1 \leqslant \beta_* \leqslant x_2$, $\theta_2 \leqslant \beta^* \leqslant x_3$. If $k = 1$, $n \geqslant 2$, then we have $\alpha^* = \beta^* = 1$. For $k \geqslant 2$, $n = 1$* we have $\alpha_* = 0$. *For $k = n = 1$ we may put $\alpha_* = 0$, $\beta^* = 1$.*

Proof. First we consider the function

$$l_u(0) = G_{n,h}^{(k)}(u) - u g_{n,h}^{(k)}(u).$$

We have

$$l_0(0) = l_1(0) = 0 \quad \text{and} \quad \frac{d}{du}(l_u(0)) = -u(g_{n,h}^{(k)})'(u).$$

Therefore, by Lemma 4, $l_u(0)$ is increasing on the interval $(0, x_1)$ from 0 to $l_{x_1}(0) > 0$, decreasing on (x_1, x_2) to $l_{x_2}(0) < 0$, increasing on (x_2, x_3) to $l_{x_3}(0) > 0$, and decreasing on $(x_3, 1)$ to 0, where x_1, x_2, x_3 are as in the statement of the first part of Lemma 4. Therefore $l_u(0)$ has unique roots α_* and α^* in each of the intervals (x_1, x_2) and (x_2, x_3), respectively. Moreover, since $l_{\theta_1}(0) = G_{n,h}^{(k)}(\theta_1) < 0$, $l_{\theta_2}(0) = G_{n,h}^{(k)}(\theta_2) > 0$ and $\theta_1 < x_2$, $\theta_2 < x_3$, we see that $x_1 \leqslant \alpha_* \leqslant \theta_1$ and $x_2 \leqslant \alpha^* \leqslant \theta_2$. Similarly, considering the function

$$l_u(1) = G_{n,h}^{(k)}(u) - (1-u) g_{n,h}^{(k)}(u)$$

with

$$l_0(1) = l_1(1) = 0, \quad \frac{d}{du}(l_u(1)) = (1-u)(g_{n,h}^{(k)})'(u), \quad l_{\theta_1}(1) < 0, \quad l_{\theta_2}(1) > 0,$$

we prove the second part of the lemma. ∎

The shapes of the projections of $g_{n,h}^{(k)}$ and $-g_{n,h}^{(k)}$ are given in the following lemma.

LEMMA 7. *For* $k \geqslant 1$, $n \geqslant 1$, *we have*

$$\bar{g}_{n,h}^{(k)}(x) = \begin{cases} g_{n,h}^{(k)}(\alpha_*), & 0 \leqslant x < \alpha_*, \\ g_{n,h}^{(k)}(x), & \alpha_* \leqslant x \leqslant \beta_*, \\ g_{n,h}^{(k)}(\beta_*), & \beta_* < x \leqslant 1, \end{cases}$$

$$\overline{-g}_{n,h}^{(k)}(x) = \begin{cases} -g_{n,h}^{(k)}(\alpha^*), & 0 \leqslant x < \alpha^*, \\ -g_{n,h}^{(k)}(x), & \alpha^* \leqslant x \leqslant \beta^*, \\ -g_{n,h}^{(k)}(\beta^*), & \beta^* < x \leqslant 1, \end{cases}$$

where $\alpha_* < \beta_*$, α_*, $\beta_* \in (0, 1)$, *are the smaller solutions of the equations*

$$G_{n,h}^{(k)}(\alpha) = \alpha g_{n,h}^{(k)}(\alpha), \tag{9}$$

$$G_{n,h}^{(k)}(\beta) + (1-\beta) g_{n,h}^{(k)}(\beta) = 0, \tag{10}$$

respectively, and $\alpha^* < \beta^*$, α^*, $\beta^* \in (0, 1)$, *are the greater solutions of these equations.*

Proof. First note that $G_{n,h}^{(k)}$ starts from the origin and by Lemma 4 it is concave decreasing on $(0, x_1)$, convex decreasing on (x_1, θ_1), convex increasing on (θ_1, x_2), concave increasing on (x_2, θ_2), concave decreasing on (θ_2, x_3), and convex decreasing on $(x_3, 1)$ to 0. Therefore the greatest convex minorant $\bar{G}_{n,h}^{(k)}$ of $G_{n,h}^{(k)}$ is given by

$$\bar{G}_{n,h}^{(k)}(x) = \begin{cases} G_{n,h}^{(k)}(\alpha_*) + g_{n,h}^{(k)}(\alpha_*)(x-\alpha_*), & 0 \leqslant x < \alpha_*, \\ G_{n,h}^{(k)}(x), & \alpha_* \leqslant x \leqslant \beta_*, \\ G_{n,h}^{(k)}(\beta_*) + g_{n,h}^{(k)}(\beta_*)(x-\beta_*), & \beta_* < x \leqslant 1, \end{cases}$$

where α_* and β_* are as in the statement of Lemma 6. Note that equations $G_{n,h}^{(k)}(\alpha) = \alpha g_{n,h}^{(k)}(\alpha)$ and $G_{n,h}^{(k)}(\beta) + (1-\beta) g_{n,h}^{(k)}(\beta) = 0$ are equivalent to $l_\alpha(0) = 0$ and $l_\beta(1) = 0$, respectively. Now differentiate $\bar{G}_{n,h}^{(k)}(x)$ with respect to x to obtain the assertion of the lemma. The statements concerning $-g_{n,h}^{(k)}$ are proved analogously after observing that the antiderivative of $-g_{n,h}^{(k)}$ is just $-G_{n,h}^{(k)}$. ∎

Remark 1. Note that the functions $g_{n,h}^{(k)}$ restricted to (α_*, β_*) and $-g_{n,h}^{(k)}$ restricted to (α^*, β^*) are strictly increasing. Therefore they have well-defined inverse functions $(g_{n,h}^{(k)})^{-1}$ and $(-g_{n,h}^{(k)})^{-1}$, respectively.

Let $\Phi_{t,h}^{(k)}$ denote the antiderivative of $\varphi_{t,h}^{(k)}$, i.e. $\Phi_{t,h}^{(k)}(x) = \int_0^x \varphi_{t,h}^{(k)}(u)\,du$. Then the projection of $\varphi_{t,h}^{(k)}$ is as follows.

LEMMA 8. *For $k \geqslant 1$, $t \geqslant 1$, and $h > 0$*

$$\bar{\varphi}_{t,h}^{(k)}(x) = \begin{cases} \varphi_{n,h}^{(k)}(a_*), & 0 \leqslant x < a_*, \\ \varphi_{n,h}^{(k)}(x), & a_* \leqslant x \leqslant b_*, \\ \varphi_{n,h}^{(k)}(b_*), & b_* < x \leqslant 1, \end{cases}$$

where a_*, b_*, $0 \leqslant a_* < b_* \leqslant 1$, are the unique solutions of the equations

$$\Phi_{t,h}^{(k)}(a) = a\varphi_{t,h}^{(k)}(a), \qquad \Phi_{t,h}^{(k)}(b) = (b-1)\varphi_{t,h}^{(k)}(b),$$

respectively.

3. MAIN RESULTS

Applying Hölder's inequality to (6) we obtain

$$(11)\qquad E\Delta_{n,h}^{(k)} \leqslant \int_0^1 (F^{-1}(x)-\mu)(\bar{g}_{n,h}^{(k)}(x)-c)\,dx$$
$$\leqslant \|F^{-1}-\mu\|_p \|\bar{g}_{n,h}^{(k)}-c\|_q = \bar{B}_{n,h,p}^{(k)}(c)\,\sigma_p,$$

say, where $c \in \boldsymbol{R}$ is an arbitrary constant. Similarly, by (7), $E\Delta_{n,h}^{(k)} \geqslant -\underline{B}_{n,h,p}^{(k)}(c)\,\sigma_p$, where $\underline{B}_{n,h,p}^{(k)}(c) = \|\overline{-g}_{n,h}^{(k)} - c\|_q$, which together with (11) implies

$$(12)\qquad -\underline{B}_{n,h,p}^{(k)} \leqslant \frac{E\Delta_{n,h}^{(k)}}{\sigma_p} \leqslant \bar{B}_{n,h,p}^{(k)}$$

with $\overline{B}^{(k)}_{n,h,p} = \inf_{c\in \mathbf{R}} \overline{B}^{(k)}_{n,h,p}(c)$ and $\underline{B}^{(k)}_{n,h,p} = \inf_{c\in \mathbf{R}} \underline{B}^{(k)}_{n,h,p}(c)$. Therefore it suffices to specify values of c which minimize $\overline{B}^{(k)}_{n,h,p}(c)$ and $\underline{B}^{(k)}_{n,h,p}(c)$. We should also find distributions for which the bounds are attained.

In subsequent theorems we analyze the cases $1 < p < \infty$, $p = 1$, and $p = \infty$. Since the shape of the projection $\bar{g}^{(k)}_{n,h}$, given in Lemma 7, is the same as the shape of the projection obtained in [4] or [10], the proofs in these cases are similar to those in cited papers.

In the case $1 < p < \infty$ the result is as follows.

THEOREM 1. *Fix* $1 < p < \infty$ *and let* $q = p/(p-1)$. *For* $k \geqslant 1$, $n \geqslant 1$, $h \in (0, 1)$,

$$\overline{B}^{(k)}_{n,h,p} = \|\bar{g}^{(k)}_{n,h} - c_*\|_q \tag{13}$$

with $c_* = g^{(k)}_{n,h}(x_*)$, *where* x_* *is the unique solution of the equation*

$$\alpha_* \left(g^{(k)}_{n,h}(x) - g^{(k)}_{n,h}(\alpha_*)\right)^{q-1} + \int_{\alpha_*}^{x} \left(g^{(k)}_{n,h}(x) - g^{(k)}_{n,h}(u)\right)^{q-1} du \tag{14}$$

$$= \int_{x}^{\beta_*} \left(g^{(k)}_{n,h}(u) - g^{(k)}_{n,h}(x)\right)^{q-1} du + (1-\beta_*)\left(g^{(k)}_{n,h}(\beta_*) - g^{(k)}_{n,h}(x)\right)^{q-1}.$$

The bound (13) *is attained if*

$$F(x) = \begin{cases} 0, & \dfrac{x-\mu}{\sigma_p} < -\left(\dfrac{c_* - g^{(k)}_{n,h}(\alpha_*)}{\overline{B}^{(k)}_{n,h,p}}\right)^{p/q}, \\ (g^{(k)}_{n,h})^{-1}\left(c_* - \overline{B}^{(k)}_{n,h,p}\left(\dfrac{\mu - x}{\sigma_p}\right)^{p-1}\right), & -\left(\dfrac{c_* - g^{(k)}_{n,h}(\alpha_*)}{\overline{B}^{(k)}_{n,h,p}}\right)^{p/q} \leqslant \dfrac{x-\mu}{\sigma_p} < 0, \\ (g^{(k)}_{n,h})^{-1}\left(c_* + \overline{B}^{(k)}_{n,h,p}\left(\dfrac{x-\mu}{\sigma_p}\right)^{p-1}\right), & 0 \leqslant \dfrac{x-\mu}{\sigma_p} < \left(\dfrac{g^{(k)}_{n,h}(\beta_*) - c_*}{\overline{B}^{(k)}_{n,h,p}}\right)^{p/q}, \\ 1, & \dfrac{x-\mu}{\sigma_p} \geqslant \left(\dfrac{g^{(k)}_{n,h}(\beta_*) - c_*}{\overline{B}^{(k)}_{n,h,p}}\right)^{p/q}. \end{cases} \tag{15}$$

The statements for $\underline{B}^{(k)}_{n,h,p}$ *are of the same form as* (13), (14) *and* (15) *with* $\overline{B}^{(k)}_{n,h,p}$, $g^{(k)}_{n,h}$, α_*, β_* *and* c_* *replaced with* $\underline{B}^{(k)}_{n,h,p}$, $-g^{(k)}_{n,h}$, α^*, β^* *and* c^*, *respectively.*

Proof. Note that the value of c_* that minimizes the norm $\|\bar{g}^{(k)}_{n,h} - c\|_q$ has to be of the form $c_* = g^{(k)}_{n,h}(x)$ for some $x \in [\alpha_*, \beta_*]$. Therefore we consider the function

$$D(x) = \|\bar{g}^{(k)}_{n,h} - g^{(k)}_{n,h}(x)\|^q_q$$

$$= \alpha_* \left(g^{(k)}_{n,h}(x) - g^{(k)}_{n,h}(\alpha_*)\right)^q + \int_{\alpha_*}^{x} \left(g^{(k)}_{n,h}(x) - g^{(k)}_{n,h}(u)\right)^q du$$

$$+ \int_{x}^{\beta_*} \left(g^{(k)}_{n,h}(u) - g^{(k)}_{n,h}(x)\right)^q du + (1-\beta_*)\left(g^{(k)}_{n,h}(\beta_*) - g^{(k)}_{n,h}(x)\right)^{q-1}$$

and we minimize it with respect to $x \in [\alpha_*, \beta_*]$. Differentiating the last equation we obtain $D'(x) = q(g_{n,h}^{(k)})'(x)(D_1(x) - D_2(x))$, where $D_1(x)$ and $D_2(x)$ stand for the left-hand and right-hand sides of (14), respectively. Note that $q > 0$, $(g_{n,h}^{(k)})'(x) > 0$ for $\alpha_* < x < \beta_*$ and $D_1(\alpha_*) = D_2(\beta_*) = 0$. Moreover,

$$D_1'(x) = (q-1)(g_{n,h}^{(k)})'(x)\{\alpha_*(g_{n,h}^{(k)}(x) - g_{n,h}^{(k)}(\alpha_*))^{q-2} + \int_{\alpha_*}^{x} (g_{n,h}^{(k)}(x) - g_{n,h}^{(k)}(u))^{q-2}\, du\} \geqslant 0$$

and similarly $D_2'(x) \leqslant 0$. This implies that D_1 increases from 0 to $D_1(\beta_*) > 0$ and D_2 decreases from $D_2(\alpha_*) > 0$ to 0. Therefore there exists x_* which is the unique solution of the equation $D_1(x) = D_2(x)$ or equivalently (14). Moreover, D' changes sign at x_* from negative to positive, which implies that x_* minimizes D on $[\alpha_*, \beta_*]$.

The equality in (12) is attained and the moment conditions

$$\int_0^1 F^{-1}(u)\, du = \mu, \qquad \int_0^1 |F^{-1}(u) - \mu|^p\, du = \sigma_p$$

are fulfilled if F^{-1} satisfies (cf. [12], p. 160)

$$\frac{F^{-1}(u) - \mu}{\sigma_p} = \frac{|\bar{g}_{n,h}^{(k)}(u) - c_*|^{q/p}}{\|\bar{g}_{n,h}^{(k)} - c_*\|_q^{q/p}} \operatorname{sgn}(\bar{g}_{n,h}^{(k)}(u) - c_*).$$

Equivalently,

$$\left|\frac{F^{-1}(u) - \mu}{\sigma_p}\right|^{p/q} \operatorname{sgn}(F^{-1}(u) - \mu) = \frac{1}{\bar{B}_{n,h,p}^{(k)}}(\bar{g}_{n,h}^{(k)}(u) - c_*),$$

which gives (15). Since the projections $\bar{g}_{n,h}^{(k)}$ and $\overline{-g_{n,h}^{(k)}}$ are of similar form, the proofs of the results for the lower bound are exactly the same as above with replacements as in the statement of the theorem. This completes the proof. ∎

Remark 2. In subsequent results we give precise expressions for upper bounds only. The lower bounds may be treated as in the case $1 < p < \infty$.

For $p = q = 2$, solving (14) we obtain $c_* = 0$, which implies the following corollary.

COROLLARY 1. *For* $k \geqslant 1$, $n \geqslant 1$, $h \in (0, 1)$,

$$\bar{B}_{n,h,2}^{(k)} = \{\alpha_*(g_{n,h}^{(k)}(\alpha_*))^2 + \int_{\alpha_*}^{\beta_*} (g_{n,h}^{(k)}(u))^2\, du + (1-\beta_*)(g_{n,h}^{(k)}(\beta_*))^2\}^{1/2}.$$

This bound is achieved if

$$F(x)=\begin{cases}0, & \dfrac{x-\mu}{\sigma_2}<\dfrac{g_{n,h}^{(k)}(\alpha_*)}{\bar{B}_{n,h,2}^{(k)}},\\ (g_{n,h}^{(k)})^{-1}\left(\bar{B}_{n,h,2}^{(k)}\left(\dfrac{x-\mu}{\sigma_p}\right)\right), & \dfrac{g_{n,h}^{(k)}(\alpha_*)}{\bar{B}_{n,h,2}^{(k)}}\leqslant\dfrac{x-\mu}{\sigma_2}<\dfrac{g_{n,h}^{(k)}(\beta_*)}{\bar{B}_{n,h,2}^{(k)}},\\ 1, & \dfrac{x-\mu}{\sigma_2}\geqslant\dfrac{g_{n,h}^{(k)}(\alpha_*)}{\bar{B}_{n,h,2}^{(k)}}.\end{cases}$$

In the case $p=1$ we have the following theorem.

THEOREM 2. *Suppose that α_* and β_* are as in Lemma 7. For $k\geqslant 1$, $n\geqslant 1$, $h\in(0,1)$*

$$\bar{B}_{n,h,1}^{(k)}=\tfrac{1}{2}\big(g_{n,h}^{(k)}(\beta_*)-g_{n,h}^{(k)}(\alpha_*)\big). \tag{16}$$

The bound (16) is attained for the distribution concentrated at the points $\mu-\sigma_1/(2\alpha_*)$, μ and $\mu+\sigma_1/2(1-\beta_*)$ with probabilities α_*, $\beta_*-\alpha_*$ and $1-\beta_*$, respectively.

Proof. As in the case $p>1$, we start with determining the value of c_* which minimizes $\|\bar{g}_{n,h}^{(k)}-c\|_\infty$. We have

$$\inf_{c\in\mathbf{R}}\|\bar{g}_{n,h}^{(k)}-c\|_\infty=\inf_{c\in\mathbf{R}}\sup_{0\leqslant u\leqslant 1}|\bar{g}_{n,h}^{(k)}(u)-c|$$
$$=\inf_{c\in\mathbf{R}}\max\big(c-g_{n,h}^{(k)}(\alpha_*),\,g_{n,h}^{(k)}(\beta_*)-c\big)=\tfrac{1}{2}\big(g_{n,h}^{(k)}(\beta_*)-g_{n,h}^{(k)}(\alpha_*)\big),$$

which is attained for $c=\frac{1}{2}\big(g_{n,h}^{(k)}(\alpha_*)+g_{n,h}^{(k)}(\beta_*)\big)$.

The bound (16) is attained if $F^{-1}-\mu$ is constant on $[0,\alpha_*)$ and $(\beta_*,1]$ (cf. Lemma 1) and is equal to 0 on (α_*,β_*) (by Hölder's inequality). Therefore,

$$F^{-1}(u)-\mu=\begin{cases}A, & 0\leqslant x<\alpha_*,\\ 0, & \alpha_*<x<\beta_*,\\ B, & \beta_*<x\leqslant 1,\end{cases}$$

and the moment conditions $\int_0^1(F^{-1}(u)-\mu)\,du=0$ and $\int_0^1|F^{-1}(u)-\mu|\,du=\sigma_1$ imply $A=-\sigma_1/(2\alpha_*)$ and $B=1/2(1-\beta_*)$. ∎

When $p=\infty$, we obtain the following statement.

THEOREM 3. *Suppose that α_* and β_* are as in Lemma 7, F is a distribution function concentrated on $[\mu-\sigma_\infty,\mu+\sigma_\infty]$ and $k\geqslant 1$, $n\geqslant 1$, $h\in(0,1)$. If $\alpha_*\geqslant\frac{1}{2}$, then $\bar{B}_{n,h,\infty}^{(k)}=-g_{n,h}^{(k)}(\alpha_*)$ and the bound is attained for the distribution concentrated at the points $\mu-\sigma_\infty\cdot(1-\alpha_*)/\alpha_*$ and $\mu+\sigma_\infty$ with probabilities α_* and $1-\alpha_*$, respectively. If $\beta_*\leqslant\frac{1}{2}$, then $\bar{B}_{n,h,\infty}^{(k)}=g_{n,h}^{(k)}(\beta_*)$ and the distribution which attains the bound is concentrated at the points $\mu-\sigma_\infty$ and $\mu+\sigma_\infty\cdot(1-\beta_*)/\beta_*$ with* *probabilities β_* and $1-\beta_*$, respectively. Finally, if $\alpha_* < \frac{1}{2} < \beta_*$, then $\bar{B}^{(k)}_{n,h,\infty} = -2G^{(k)}_{n,h}(\frac{1}{2})$ and the bound is achieved for the distribution concentrated at the points $\mu-\sigma_\infty$ and $\mu+\sigma_\infty$, each with probability $\frac{1}{2}$.*

Proof. Again, first we determine $c_* = g^{(k)}_{n,h}(x_*)$, for some $x_* \in [\alpha_*, \beta_*]$, which minimizes $D(x) = \|\bar{g}^{(k)}_{n,h} - g^{(k)}_{n,h}(x)\|_1$. Using (9) and (10) we see that D may be written as $D(x) = (2x-1)g^{(k)}_{n,h}(x) - 2G^{(k)}_{n,h}(x)$ with derivative $D'(x) = (2x-1)(g^{(k)}_{n,h})'(x)$. Moreover, $\bar{B}^{(k)}_{n,h,\infty}$ is attained if $F^{-1}(u)-\mu = \sigma_\infty(\bar{g}^{(k)}_{n,h}(u)-c)$ (Hölder's inequality) as well as $F^{-1}-\mu$ is constant on each of the intervals where $\bar{g}^{(k)}_{n,h} \neq g^{(k)}_{n,h}$.

Now, if $\alpha_* \geqslant \frac{1}{2}$, then $D'(x) > 0$ in $[\alpha_*, \beta_*]$, and therefore $c_* = g^{(k)}_{n,h}(\alpha_*)$, which implies $\bar{B}^{(k)}_{n,h,\infty} = D(\alpha_*) = -g^{(k)}_{n,h}(\alpha_*)$. The bound is attained if

$$F^{-1}(u)-\mu = \begin{cases} -\sigma_\infty \cdot (1-\alpha_*)/\alpha_*, & 0 \leqslant x < \alpha_*, \\ \sigma_\infty, & \alpha_* < x \leqslant 1, \end{cases}$$

which easily implies the statement of the theorem for $\alpha_* \geqslant \frac{1}{2}$. Similar arguments lead to the conclusions in the case $\beta_* \leqslant \frac{1}{2}$. If $\alpha_* < \frac{1}{2} < \beta_*$, then $D'(x)$ has a unique zero at $x = \frac{1}{2}$ and D changes sign at $\frac{1}{2}$ from negative to positive. Therefore $\bar{B}^{(k)}_{n,h,\infty} = D(\frac{1}{2}) = -2G^{(k)}_{n,h}(\frac{1}{2})$ and the remaining conclusions of the theorem follow. ∎

To conclude our discussion we state without proof a general result on bounds for expectations of increments of record values. The proof of this result may be established similarly to the proofs of Theorems 1, 2 and 3 of [5]. The distributions for which the bounds are attained are also of the same form as in [5] with obvious modifications, so we do not specify them here.

THEOREM 4. *Let F be an arbitrary continuous distribution function with finite mean μ and pth central absolute moment σ_p for some $1 \leqslant p \leqslant \infty$ and let a_*, b_* be as in Lemma 8. For $k \geqslant 1$, $t \geqslant 1$ and $h > 0$*

$$\frac{ER^{(k)}_{t,h}}{\sigma_p} \leqslant C^{(k)}_{t,h,p},$$

where

(a) *if $1 < p < \infty$, then $C^{(k)}_{t,h,p} = \|\bar{\varphi}^{(k)}_{t,h} - c_*\|_q$ with $c_* = \varphi^{(k)}_{t,h}(y_*)$ and $y_* \in (a_*, b_*)$ being the unique solution to (14) in which $g^{(k)}_{n,h}$, α_* and β_* are replaced by $\varphi^{(k)}_{t,h}$, a_* and b_*, respectively;*

(b) *if $p = 1$, then $C^{(k)}_{t,h,1} = \frac{1}{2}(\varphi^{(k)}_{t,h}(b_*) - \varphi^{(k)}_{t,h}(a_*))$;*

(c) *if $p = \infty$, then*

$$C^{(k)}_{t,h,\infty} = \begin{cases} -\varphi^{(k)}_{t,h}(a_*) & \text{for } a_* \geqslant \frac{1}{2}, \\ -2\Phi^{(k)}_{t,h}(\frac{1}{2}) & \text{for } a_* < \frac{1}{2} < b_*, \\ \varphi^{(k)}_{t,h}(b_*) & \text{for } b_* \leqslant \frac{1}{2}. \end{cases}$$

Remark 3. For $t = m \in N$ and $h = n - m \in N$ we obtain results of [5], and if $t = n \in N$ and $h = 1$, we obtain results of [10].

4. NUMERICAL EXAMPLES

Our results admit direct numerical implementation. In Tables 1, 2 and 3 we give values of upper and lower bounds for $k = 3$, $n = 4$ and $h = 0.1 \ldots 0.9$ for three most popular scale units, i.e. $p = 1$, $p = 2$ and $p = \infty$, respectively.

TABLE 1. Numerical values of $-\underline{B}^{(3)}_{4,h,1}$, $\bar{B}^{(3)}_{4,h,1}$, $C^{(3)}_{4,h,1}$ and $C^{(3)}_{5-h,h,1}$

	$-\underline{B}^{(3)}_{4,h,1}$	$\bar{B}^{(3)}_{4,h,1}$	$C^{(3)}_{4,h,1}$	$C^{(3)}_{5-h,h,1}$
4.1	−0.0115	0.0079	0.0596	0.5739
4.2	−0.0206	0.0140	0.1200	0.5151
4.3	−0.0272	0.0182	0.1810	0.4551
4.4	−0.0313	0.0207	0.2428	0.3940
4.5	−0.0329	0.0214	0.3054	0.3316
4.6	−0.0318	0.0204	0.3688	0.2679
4.7	−0.0280	0.0177	0.4331	0.2029
4.8	−0.0215	0.0133	0.4983	0.1367
4.9	−0.0122	0.0075	0.5645	0.0690

TABLE 2. Numerical values of $-\underline{B}^{(3)}_{4,h,2}$, $\bar{B}^{(3)}_{4,h,2}$, $C^{(3)}_{4,h,2}$ and $C^{(3)}_{5-h,h,2}$

	$-\underline{B}^{(3)}_{4,h,2}$	$\bar{B}^{(3)}_{4,h,2}$	$C^{(3)}_{4,h,2}$	$C^{(3)}_{5-h,h,2}$
4.1	−0.0063	0.0076	0.0481	0.4285
4.2	−0.0113	0.0134	0.0960	0.3817
4.3	−0.0148	0.0174	0.1438	0.3346
4.4	−0.0169	0.0198	0.1915	0.2874
4.5	−0.0176	0.0204	0.2391	0.2399
4.6	−0.0169	0.0194	0.2865	0.1923
4.7	−0.0148	0.0169	0.3339	0.1445
4.8	−0.0113	0.0127	0.3811	0.0965
4.9	−0.0063	0.0071	0.4282	0.0483

TABLE 3. Numerical values of $-\underline{B}^{(3)}_{4,h,\infty}$, $\bar{B}^{(3)}_{4,h,\infty}$, $C^{(3)}_{4,h,\infty}$ and $C^{(3)}_{5-h,h,\infty}$

	$-\underline{B}^{(3)}_{4,h,\infty}$	$\bar{B}^{(3)}_{4,h,\infty}$	$C^{(3)}_{4,h,\infty}$	$C^{(3)}_{5-h,h,\infty}$
4.1	−0.0022	0.0072	0.0314	0.2439
4.2	−0.0038	0.0127	0.0618	0.2143
4.3	−0.0050	0.0164	0.0913	0.1854
4.4	−0.0056	0.0184	0.1199	0.1571
4.5	−0.0058	0.0189	0.1476	0.1294
4.6	−0.0055	0.0178	0.1745	0.1023
4.7	−0.0048	0.0153	0.2005	0.0759
4.8	−0.0036	0.0115	0.2258	0.0501
4.9	−0.0020	0.0064	0.2504	0.0247

These values are typical in the sense that for other values of n and k the expected error committed by replacing $Y_t^{(k)}$ with $(1-\{t\})\,Y_{[t]}^{(k)}+t\,Y_{[t]+1}^{(k)}$, measured in σ_p units, is at most a few percent. Therefore, the approximation of fractional record values by ordinary record values is very close. Moreover, for comparison the table contains the values of $C_{4,h,p}^{(3)}$ and $C_{5-h,h,p}^{(3)}$. These numbers are the bounds for $E(Y_{4+h}^{(k)}-Y_4^{(k)})$ and $E(Y_5^{(k)}-Y_{4+h}^{(k)})$, respectively.

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