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ON PC SOLUTIONS OF PARMA(p,q) MODELS

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Abstract. This note is concerned with the existence of periodically correlated solutions for the PARMA(p,q) system

$$x_n = \phi_n^1 x_{n-1} + \phi_n^2 x_{n-2} + \ldots + \phi_n^p x_{n-p} + \xi_n + \theta_n^1 \xi_{n-1} + \ldots + \theta_n^q \xi_{n-q}, \quad n \in \boldsymbol{Z},$$

where ξ_n is a white noise and the varying coefficients ϕ_n^i and θ_n^i are periodic in n with period T. Conditions which ensure the existence of periodically correlated solutions for such systems are obtained.

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1. INTRODUCTION

By a *stochastic process* we mean a two-sided sequence $x = \{x_n: n \in \boldsymbol{Z}\}$ of random variables with zero mean and finite variance, i.e. a sequence of functions in the Hilbert space $L_0^2(\Omega)$ of some probability space Ω. For our purpose in this note, we will take a stochastic process to mean a two-sided sequence $x = \{x_n: n \in \boldsymbol{Z}\}$ of vectors in a general Hilbert space $\boldsymbol{H}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Periodically correlated (PC) processes were first studied by Gladyshev in [5] and subsequently by several authors including [3], [6], [9], [10], and [12]–[14]. Like ARMA and AR systems for stationary processes, PARMA and PAR systems are important for modelling periodically correlated processes and have been studied by several authors including [1], [2], [3], [8], and [15]. A PC process $x = \{x_n: n \in \boldsymbol{Z}\}$ is said to be *PARMA*(p,q) if it satisfies an autoregressive periodic moving average system

$$(1) \qquad x_n = \phi_n^1 x_{n-1} + \phi_n^2 x_{n-2} + \ldots + \phi_n^p x_{n-p} + \xi_n + \theta_n^1 \xi_{n-1} + \ldots + \theta_n^q \xi_{n-q}, \quad n \in \boldsymbol{Z},$$

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where for each i the sequences $\{\phi_n^i\colon n\in \boldsymbol{Z}\}$ and $\{\theta_n^i\colon n\in \boldsymbol{Z}\}$ are periodic in n with period T and the process $\xi = \{\xi_n\colon n\in\boldsymbol{Z}\}$ is a *white noise*, namely a sequence of uncorrelated complex random variables with mean 0 and variance 1. It is essential to obtain conditions on the coefficients of the PARMA system (1) which ensure that it has a periodic solution. Some of such conditions were given in [15] and other ones can be found in [16] and [11]. Here we will obtain other conditions of such a kind. All these conditions generalize the following well-known theorem.

THEOREM 1. *If the polynomials*

$$\phi(z) = 1-\phi_1 z- \ldots -\phi_p z^p \quad \textit{and} \quad \theta(z) = 1+\theta_1 z+ \ldots +\theta_q z^q$$

have no common zeros, then the ARMA(p,q) system

$$x_n = \phi_1 x_{n-1}+\phi_2 x_{n-2}+ \ldots + \phi_p x_{n-p}+\xi_n+\theta_1 \xi_{n-1}+ \ldots +\theta_q \xi_{n-q}, \quad n\in \boldsymbol{Z},$$

has a unique stationary solution if $\phi(z)\neq 0$ *for all complex numbers z with* $|z|\leqslant 1$. *In this case the process* $x = \{x_n\colon n\in\boldsymbol{Z}\}$ *has a causal representation*

$$x_n = \sum_{j=0}^{\infty} \gamma_j \xi_{n-j}, \quad n\in\boldsymbol{Z},$$

where coefficients γ_j *are determined by the relation*

$$\gamma(z) = \sum_{j=0}^{\infty} \gamma_j z^j = \frac{\theta(z)}{\phi(z)}.$$

Next we state a multivariate extension of Theorem 1 which will be needed in Section 3. A vector process $\boldsymbol{X}_n = [x_n^1, x_n^2, \ldots, x_n^r]'$ consisting of random variables x_n^i is called a *multivariate* or an *r-variate* process. The r-variate process $\boldsymbol{X}_n$ is called *ARMA(p,q)* if it satisfies an ARMA(p,q) system

$$\text{(2)} \qquad \boldsymbol{X}_n - \sum_{i=1}^{p} \boldsymbol{\Phi}_i \boldsymbol{X}_{n-i} = \boldsymbol{\Delta}_n + \sum_{j=1}^{q} \boldsymbol{\Theta}_j \boldsymbol{\Delta}_{n-j}, \quad n\in\boldsymbol{Z},$$

where $\boldsymbol{\Phi}_i$ and $\boldsymbol{\Theta}_j$ are some $r\times r$ matrices and $\boldsymbol{\Delta}_n$ is an r-variate white noise.

THEOREM 2 (cf. Theorem 11.3.1 in [4]). *If* $\det\left(\boldsymbol{I}-\sum_{i=1}^{p}\boldsymbol{\Phi}_i z^i\right)\neq 0$ *for all complex numbers z with* $|z|\leqslant 1$, *then the ARMA(p,q) system* (2) *has a unique r-variate solution. This solution is stationary and has a causal representation of the form*

$$\boldsymbol{X}_n = \sum_{j=0}^{\infty} \boldsymbol{\Gamma}_j \xi_{n-j}, \quad n\in\boldsymbol{Z}.$$

2. PRELIMINARIES

In this section we set up the notation and introduce the preliminaries we need in the next sections. A process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ is said to be *stationary* if its covariance function $R(m, n) = (x_m, x_n)$ depends only on $m-n$ or, equivalently, if for each integer k its covariance function $R(n+k, n)$ is constant in n. The process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ is said to be *periodically correlated* with period T (in short PC-T) if for every $k \in \boldsymbol{Z}$ its covariance function $R(n+k, n)$ is periodic in n with period T. Note that any PC-1 process is stationary and any stationary process is PC-T for all positive integers T. A multivariate process $X_n = [x_n^1, x_n^2, \ldots, x_n^r]'$ is said to be *stationary* if for each integer k its *Gramian covariance function* defined by

$$G(n+k, n) = (X_{n+k}, X_n) = [(x_{n+k}^i, x_n^j)]_{i,j=1}^r$$

is constant with respect to n. An r-variate process Δ_n is called a *multivariate white noise* if its Gramian covariance function satisfies

$$(\Delta_m, \Delta_n) = \begin{cases} 0 & \text{if } m \neq n, \\ I & \text{if } m = n. \end{cases}$$

There are several close ties between univariate PC processes and multivariate stationary processes which have been used successfully to study various properties of PC processes; cf. for example [12]–[15]. In the next lemma we give such a tie which will be needed later. We provide its easy proof here for the sake of completeness. For any process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ we define its *T-variate associate* Y_n by bundling its blocks of length T as follows:

$$Y_n = [x_{nT}, x_{nT-1}, \ldots, x_{nT-T+1}]'.$$

One can easily check that when the process ξ_n is a white noise, so is any one of its multivariate associates.

LEMMA 1. *A process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ is PC-T if and only if its T-variate associate $Y = \{Y_n\colon n \in \boldsymbol{Z}\}$ is stationary.*

Proof. ("only if part") Suppose that $x = \{x_n\colon n \in \boldsymbol{Z}\}$ is PC-T. Then for each $i, j = 0, 2, \ldots, T-1$ the (i, j)-th entry of the Gramian covariance function $G(n+k, n)$ of Y_n can be expressed as

$$G^{ij}(n+k, n) = (Y_{n+k}, Y_n)^{ij} = (x_{(n+k)T-i}, x_{nT-j}),$$

which shows that each entry $G^{ij}(n+k, n)$ of $G(n+k, n)$, and hence $G(n+k, n)$ itself, is constant in n.

("if part") Suppose that Y_n is stationary. Take two integers m and n. There are integers $p, q \in \boldsymbol{Z}$ and $i, j \in [0, T-1]$ such that $m = pT-i$ and $n = qT-j$.

Hence we can write

$$(x_m, x_n) = (x_{pT-i}, x_{qT-j}) = (Y_p^i, Y_q^j) = (Y_p, Y_q)^{ij} = G^{ij}(p, q) = G^{ij}(p+1, q+1)$$
$$= (Y_{p+1}, Y_{q+1})^{ij} = (x_{(p+1)T-i}, x_{(q+1)T-j}) = (x_{T+m}, x_{T+n}).$$

Therefore the covariance function $R(m, n) = (x_m, x_n)$ of $x = \{x_n \colon n \in \boldsymbol{Z}\}$ is periodic in n with period T, which means that $x = \{x_n \colon n \in \boldsymbol{Z}\}$ is PC-T. ∎

3. MAIN RESULT

In this section we generalize Theorem 1 and prove our main result Theorem 4, which establishes sufficient conditions for the existence of PC solutions for the PARMA(p,q) system (1). First we prove Proposition 1, which gives some conditions for the existence of PC solutions for the PAR(p) system

$$x_n = \phi_n^1 x_{n-1} + \phi_n^2 x_{n-2} + \ldots + \phi_n^p x_{n-p} + \xi_n, \tag{3}$$

and we show Proposition 2, which gives a converse of Proposition 1 that is crucial for our proofs.

In the rest of the paper we work with the notation set in the following definition:

Definition 1. Starting with the PAR(p) system (3) and its autoregressive coefficients ϕ_n^i we define $A_n^{k,r}$ for integers $n \in \boldsymbol{Z}$ and $k \in [1, p]$ to be

$$A_n^{k,r} = \begin{cases} \phi_n^k & \text{if } r = 1, \\ \sum \phi_n^{i_1} \phi_{n-i_1}^{i_2} \ldots \phi_{n-i_1-\ldots-i_{r-1}}^{i_r} & \text{if } r = 2, 3, \ldots, T. \end{cases}$$

The sum here is over all possible strings $(i_1, i_2, \ldots, i_r)$ of integers in $[1, p]$ which add up to k. We denote $A_n^{k,T}$ simply by A_n^k and put

$$S_n^k = \sum_{r=1}^{T-1} A_n^{k,r}.$$

We now consider $\boldsymbol{\Phi}_k$ and $\boldsymbol{\Theta}_k$ for $k = 1, 2, \ldots, p$ to be those $T \times T$ matrices which are (uniquely) defined by the following properties:

(a) $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Theta}_1$ are upper triangular,
(b) $\boldsymbol{\Phi}_p$ and $\boldsymbol{\Theta}_p$ are lower triangular,
(c) diagonal entries of $\boldsymbol{\Theta}_1$ are all 1,
(d) the rest of entries of these $2p$ matrices are given by

$$\boldsymbol{\Phi}_k^{ij} = A_{nT+1-i}^{kT+j-i} \quad \text{and} \quad \boldsymbol{\Theta}_k^{ij} = S_{nT+1-i}^{kT-T+j-i}.$$

Proposition 1. *If the PAR(p) system* (3) *has a PC-T solution* $x = \{x_n \colon n \in \boldsymbol{Z}\}$, *then the multivariate ARMA(p,q) system*

$$Y_n - \sum_{i=1}^{p} \boldsymbol{\Phi}_i Y_{n-i} = \sum_{j=0}^{p} \boldsymbol{\Theta}_j \Delta_{n-j} \tag{4}$$

must have a T-variate stationary solution.

Proof. Iterating the system (3) once, we get

$$x_n=\sum_{i=1}^{p}\phi_n^i\Big(\sum_{j=1}^{p}\phi_{n-i}^j x_{n-i-j}+\xi_{n-i}\Big)+\xi_n=\sum_{i=1}^{p}\sum_{j=1}^{p}\phi_n^i\phi_{n-i}^j x_{n-i-j}+\sum_{i=1}^{p}\phi_n^i\xi_{n-i}+\xi_n.$$

Iterating it $T-1$ more times we get

$$\begin{aligned}(5)\qquad x_n=&\sum_{i_1=1}^{p}\sum_{i_2=1}^{p}\dots\sum_{i_T=1}^{p}\phi_n^{i_1}\phi_{n-i_1}^{i_2}\dots\phi_{n-i_1-i_2\dots i_{T-1}}^{i_T}x_{n-i_1-i_2-\dots-i_T}\\&+\sum_{i_1=1}^{p}\sum_{i_2=1}^{p}\dots\sum_{i_{T-1}=1}^{p}\phi_n^{i_1}\phi_{n-i_1}^{i_2}\dots\phi_{n-i_1-i_2\dots i_{T-2}}^{i_{T-1}}\xi_{n-i_1-i_2-\dots-i_{T-1}}\\&+\dots+\sum_{i=1}^{p}\phi_n^i\xi_{n-i}+\xi_n.\end{aligned}$$

Using matrices $A_n^{k,r}$ and A_n^k as in Definition 1 we can write (5) in the compact form

$$(6)\qquad x_n=\sum_{k=T}^{Tp}A_n^k x_{n-k}+\xi_n+\sum_{k=1}^{(T-1)p}\Big(\sum_{r=1}^{T-1}A_n^{k,r}\Big)\xi_{n-k}.$$

If in (6) we replace n with nT, $nT-1$, …, $nT-T+1$, respectively, we get the following system of T equations:

(7)
$$\begin{cases}x_{nT}=\sum_{k=T}^{pT}A_{nT}^k x_{nT-k} & +\sum_{k=1}^{(T-1)p}\Big(\sum_{r=1}^{T-1}A_{nT}^{k,r}\Big)\xi_{nT-k}+\xi_{nT},\\ x_{nT-1}=\sum_{k=T}^{pT}A_{nT-1}^k x_{nT-k-1} & +\sum_{k=1}^{(T-1)p}\Big(\sum_{r=1}^{T-1}A_{nT-1}^{k,r}\Big)\xi_{nT-k-1}\,\xi_{nT-1},\\ \dots\dots\dots & \dots\dots\dots\\ x_{nT-T+1}=\sum_{k=T}^{pT}A_{nT-T+1}^k x_{nT-T+1-k}+ & \sum_{k=1}^{(T-1)p}\Big(\sum_{r=1}^{T-1}A_{nT-T+1}^{k,r}\Big)\xi_{nT-T+1-k}+\xi_{nT-T+1}.\end{cases}$$

The system of T equations in (7) can be written as the single matricial equation

$$(8)\qquad \begin{bmatrix}x_{nT}\\x_{nT-1}\\ \dots\\x_{nT-T+1}\end{bmatrix}=\begin{bmatrix}A_{nT}^T & A_{nT}^{T+1} & \dots & A_{nT}^{2T-1}\\ 0 & A_{nT-1}^T & \dots & A_{nT-1}^{2T-2}\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & A_{nT-T+1}^T\end{bmatrix}\begin{bmatrix}x_{nT-T}\\x_{nT-T-1}\\ \dots\\x_{nT-2T+1}\end{bmatrix}$$
$$+\begin{bmatrix}A_{nT}^{2T} & A_{nT}^{2T+1} & \dots & A_{nT}^{3T-1}\\ A_{nT-1}^{2T-1} & A_{nT-1}^{2T} & \dots & A_{nT-1}^{3T-2}\\ \dots & \dots & \dots & \dots\\ A_{nT-T+1}^{T+1} & A_{nT-T+1}^{T+2} & \dots & A_{nT-T+1}^{2T}\end{bmatrix}\begin{bmatrix}x_{nT-2T}\\x_{nT-2T-1}\\ \dots\\x_{nT-3T+1}\end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\begin{bmatrix} A_{nT}^{pT} & 0 & \dots & 0 \\ A_{nT-1}^{pT} & A_{nT-1}^{pT-2} & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{nT-T+1}^{pT-T+1} & A_{nT-T+1}^{pT-T+2} & \dots & A_{nT-T+1}^{pT} \end{bmatrix} \begin{bmatrix} x_{nT-pT} \\ x_{nT-pT-1} \\ \cdots \\ x_{nT-pT-T+1} \end{bmatrix}$$

$$+\begin{bmatrix} 1 & S_{nT}^{1} & S_{nT}^{2} & S_{nT}^{3} & \dots & S_{nT}^{T-1} \\ 0 & 1 & S_{nT-1}^{1} & & \dots & S_{nT-1}^{T-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & S_{nT-T+2}^{1} \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{nT} \\ \xi_{nT-1} \\ \cdots \\ \xi_{nT-T+1} \end{bmatrix}$$

$$+\begin{bmatrix} S_{nT}^{T} & S_{nT}^{T+1} & \dots & S_{nT}^{2T-1} \\ S_{nT-1}^{T-1} & S_{nT-1}^{T-2} & \dots & S_{nT-1}^{2T-2} \\ \cdots & \cdots & \cdots & \cdots \\ S_{nT-T+1}^{1} & S_{nT-T+1}^{2} & \dots & S_{nT-T+1}^{T} \end{bmatrix} \begin{bmatrix} \xi_{nT-T} \\ \xi_{nT-T-1} \\ \cdots \\ \xi_{nT-2T+1} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\begin{bmatrix} S_{nT}^{pT-T} & 0 & \dots & 0 \\ S_{nT-1}^{pT-T-1} & S_{nT-1}^{pT-T} & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ S_{nT-T+1}^{pT-2T+1} & S_{nT-T+1}^{pT-2T+2} & \dots & S_{nT-T+1}^{pT-T} \end{bmatrix} \begin{bmatrix} \xi_{nT-pT+p} \\ \xi_{nT-pT+p-1} \\ \cdots \\ \xi_{nT-(p+1)(T-1)} \end{bmatrix}.$$

In terms of the associated T-variate processes

$$Y_n = [x_{nT}, x_{nT-1}, \dots, x_{nT-T+1}]' \quad \text{and} \quad \Delta_n = [\xi_{nT}, \xi_{nT-1}, \dots, \xi_{nT-T+1}]'$$

of x_n and ξ_n the equation (8) becomes (4) and completes the proof. ∎

The following converse of Proposition 1 is crucial in our proof of the next theorem.

PROPOSITION 2. *If the multivariate ARMA(p,p) system* (4) *has a unique stationary solution, then the PAR(p) system* (3) *has a PC-T solution.*

Proof. Suppose that $Y = \{Y_n\colon n \in \boldsymbol{Z}\}$ is the unique T-variate stationary solution of the ARMA(p,p) system (4). We unbundle this process to get the univariate PC-T process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ to which $Y = \{Y_n\colon n \in \boldsymbol{Z}\}$ serves as the associate (cf. Lemma 1). We want to show that $x = \{x_n\colon n \in \boldsymbol{Z}\}$ is a solution for the PAR(p) system (3). To show this it is natural to try to follow steps in the proof of Proposition 1 backward, and that is how we are going to proceed. Since $Y = \{Y_n\colon n \in \boldsymbol{Z}\}$ satisfies the system (4), its unbundled process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ must satisfy the matricial equation (8), which is the same as the system of T equations in (7). Since these T equations are clearly equivalent to the single equation (6), the process $x = \{x_n\colon n \in \boldsymbol{Z}\}$ satisfies (6). Finally, since (6) is the compact form of (5), $x = \{x_n\colon n \in \boldsymbol{Z}\}$ must satisfy (5). It remains to show the last step of going from (5) to (3). But, in general, a solution for (5) is not necessarily a solution for (3). However, in our case this is true. To verify this claim we need to introduce some more notation. For any process $x = \{x_n\colon n \in \mathbf{Z}\}$ we define Lx to be the process given by

$$(Lx)_n = \sum_{i=1}^{p} a_n^i x_{n-i} + \xi_n.$$

With this notation, $x = \{x_n\colon n \in \mathbf{Z}\}$ being a solution of the PAR(1) system (3) means $x = \{x_n\colon n \in \mathbf{Z}\}$ being a fixed point for L, and $y = \{y_n\colon n \in \mathbf{Z}\}$ being a solution of (5) means $y = \{y_n\colon n \in \mathbf{Z}\}$ being a fixed point for L^T. We showed that our PC-T process x satisfies (5), which means that $x = \{x_n\colon n \in \mathbf{Z}\}$ satisfies the equation

$$x = L^T x. \tag{9}$$

Putting $w = Lx$ we can write (9) as

$$x = L^{T-1} w. \tag{10}$$

Applying L to both sides of (10) we get $Lx = L(L^{T-1} w)$, which clearly gives

$$w = L^T w. \tag{11}$$

Therefore (9) and (11) show that both x and w are solutions of (5), and hence they must be the same:

$$x = w; \tag{12}$$

otherwise they generate two different solutions to the ARMA(p,p) system (4), contradicting the uniqueness assumption. Finally, substituting $w = Lx$ in (12), we get $x = Lx$, which means that the unbundled process $x = \{x_n\colon n \in \mathbf{Z}\}$ is a solution for the PAR(p) system (3). ∎

An application of Theorem 2 followed by an application of Proposition 2 proves the following result:

THEOREM 3. *With the notation of Definition* 1, *if* $\det\left(\mathbf{I} - \sum_{i=1}^{p} \boldsymbol{\Phi}_i z^i\right) \neq 0$ *for all complex numbers z with $|z| \leqslant 1$, then the PAR(p) system* (3) *has a unique solution which is PC-T.*

In the rest of this section we establish our main result Theorem 4 which extends Theorem 3 to the general case of the PARMA(p,q) system (1). We start with the following extension of Proposition 1.

PROPOSITION 3. *Suppose that the PARMA(p,q) system* (1) *has a PC-T solution $x = \{x_n\colon n \in \mathbf{Z}\}$. Then there exists a positive integer s and some $T \times T$ matrices $\boldsymbol{\Psi}_k$, $k = 0, 1, 2, \ldots, s$, such that*

$$Y_n - \sum_{i=1}^{p} \boldsymbol{\Phi}_i Y_{n-i} = \sum_{k=0}^{s} \boldsymbol{\Psi}_k \Delta_{n-k}, \tag{13}$$

where Y_n, Δ_n and $\boldsymbol{\Phi}_i$ are as before.

Proof. The proof of this proposition is similar to that of Proposition 1, modified by replacing each ξ_m by $\sum_{j=0}^{q} \theta_m^j \xi_{m-j}$. To be more precise, starting with a solution $x = \{x_n: n \in \boldsymbol{Z}\}$ of the PARMA(p,q) system (1) and following the first four steps in the proof of Proposition 1, modifying each step as mentioned earlier, we arrive at the modified version of the system (7). Now, grouping the coefficients here in matrices ϕ_i as before and some new matrices $\boldsymbol{\Psi}_k$ we arrive at the multivariate ARMA(p,s) system (13), which completes the proof. ∎

One can similarly modify the proof of Proposition 2 to get the following converse of Proposition 3:

PROPOSITION 4. *With the notation of Proposition* 3 *and Definition* 1, *if the multivariate ARMA*(p,s) *system* (13) *has a unique solution, then the PARMA*(p,q) *system* (1) *must have a PC-T solution.*

Now we state our main result, which extends Theorem 3 to the general case of PARMA(p,q) system (1). Its proof is a consequence of an application of Theorem 2 followed by an application of Proposition 3.

THEOREM 4. *If* $\det\left(\boldsymbol{I}-\sum_{i=1}^{p} \boldsymbol{\Phi}_i z^i\right) \neq 0$ *for all complex numbers* z *with* $|z| \leqslant 1$, *then the PARMA*(p,q) *system* (1) *has a unique PC-T solution.*

4. EXAMPLES

In this section we examine some specific examples. The first example establishes Theorem 3 in [7].

EXAMPLE 1 (*PAR*(1) *system*). *If* $\{a_n\}$ *is periodic with period* T *and* $|a_T a_{T-1} \ldots a_1| > 1$, *then the PAR*(1) *system*

$$x_n = a_n x_{n-1} + \xi_n$$

has a unique solution which is PC-T.

We note that in this case there is only one autoregressive coefficient matrix $\boldsymbol{\Phi}$ and it is given by

$$\boldsymbol{\Phi} = [A_{T+1-i}^{T+j-i}] = \begin{bmatrix} A_{nT}^{T} & A_{nT}^{T+1} & \ldots & A_{nT}^{2T-1} \\ 0 & A_{nT-1}^{T} & \ldots & A_{nT-1}^{2T-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & A_{nT-T+1}^{T} \end{bmatrix}$$

$$= \begin{bmatrix} P & A_{nT}^{T+1} & \ldots & A_{nT}^{2T-1} \\ 0 & P & \ldots & A_{nT-1}^{2T-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & P \end{bmatrix},$$

where P is used to denote $a_T a_{T-1} \dots a_1$. Hence we get

$$\det(\boldsymbol{I}-\boldsymbol{\Phi}z) = \det\begin{bmatrix} 1-Pz & -A_{nT}^{T+1} & \dots & -A_{nT}^{2T-1} \\ 0 & 1-Pz & \dots & -A_{nT-1}^{2T-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1-Pz \end{bmatrix} = (1-Pz)^T.$$

Now all we need to do is to apply Theorem 3.

EXAMPLE 2 (*PAR*(1,*q*) *system*). *Suppose that* $\{a_n\}$ *is periodic with period* T *and* $|a_T a_{T-1} \dots a_1| > 1$. *Then the PARMA*(1,*q*) *system*

$$x_n = a_n x_{n-1} + \xi_n + \sum_{j=1}^{q} \theta_j \xi_{n-j}$$

has a unique solution which is PC-T.

This can be verified exactly as in Example 1 except here we need to invoke Theorem 4 instead of Theorem 3.

EXAMPLE 3 (*PARMA*(2,*q*) *system*). *Consider the system*

$$x_n = a_n x_{n-1} + b_n x_{n-2} + \xi_n + \sum_{j=1}^{q} \theta_j \xi_{n-j}, \tag{14}$$

where the sequences $\{a_n\}$ *and* $\{b_n\}$ *are periodic with period 2. In this case there are two autoregressive coefficients* Φ_1 *and* Φ_2 *given by*

$$\Phi_1 = [A_{T+1-i}^{T+j-i}] = \begin{bmatrix} A_2^2 & A_2^3 \\ 0 & A_1^2 \end{bmatrix} = \begin{bmatrix} a_2 a_1 & a_2 b_1 + a_2 b_2 \\ 0 & a_2 a_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ 0 & a_1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & a_2 \end{bmatrix},$$

$$\Phi_2 = [A_{T+1-i}^{2T+j-i}] = \begin{bmatrix} A_2^4 & 0 \\ A_1^3 & A_1^4 \end{bmatrix} = \begin{bmatrix} b_2 b_2 & 0 \\ a_1 b_2 & b_1 b_1 \end{bmatrix} = \begin{bmatrix} b_2 & 0 \\ a_1 & b_1 \end{bmatrix}\begin{bmatrix} b_2 & 0 \\ a_1 & b_1 \end{bmatrix}.$$

The PARMA(2,*q*) *system* (14) *has a unique PC*-2 *solution if*

$$\det\left(\boldsymbol{I} - \begin{bmatrix} a_2 & b_2 \\ 0 & a_1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & a_2 \end{bmatrix} z - \begin{bmatrix} b_2 & 0 \\ a_1+b_1 & b_1 \end{bmatrix}\begin{bmatrix} b_2 & 0 \\ a_1 & b_1 \end{bmatrix} z^2\right) \neq 0 \tag{15}$$

for all complex numbers z *with* $|z| \leqslant 1$.

This can be seen by using Theorem 4.

EXAMPLE 4. *If* $|b_1| \geqslant 1$ *and* $|b_2| \geqslant 1$, *then the PARMA system*

$$x_n = b_n x_{n-2} + \sum_{j=0}^{q} \theta^j \xi_{n-j}$$

has a unique PC-2 *solution.*

Note that this example is a special case of Example 3 and now the equation (15) reduces to

$$\det\left(I-\begin{bmatrix} b_2 b_2 & 0 \\ b_1 b_2 & b_1 b_1 \end{bmatrix} z^2\right) = (1-b_2 b_2 z)(1-b_1 b_1 z) \neq 0.$$

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