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# COMPUTING THE PORTFOLIO CONDITIONAL VALUE-AT-RISK IN THE $\alpha$-STABLE CASE

BY

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*Abstract.* The class of $\alpha$-stable distributions is an attractive probabilistic model of asset returns distribution in the field of finance. When dealing with real issues, such as optimal portfolio selection, it is important that we can compute the Conditional Value-at-Risk (CVaR) accurately. The CVaR is also known as the expected tail loss (ETL) proposed in literature as a coherent risk measure. In our paper we propose an integral expression for the calculation of the CVaR of a stable law. We compare the current approach to some existing method and we demonstrate how to relate the derived result to some common multivariate distributional assumptions.

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## 1. INTRODUCTION

The theoretical properties of the class of $\alpha$-stable distributions allow the explanation of empirically observed phenomena such as: heavy tails and excess kurtosis. For this reason it has been proposed as a probabilistic model for asset returns distributions. Abandoning the classical assumption of normality, we need to rework the basic building blocks of financial theory and modeling. For various mathematical models and related discussions, see Rachev (2003), and Rachev and Mittnik (2000) and the references therein. A central problem that undergoes revision is the portfolio selection problem, see Ortobelli et al. (2003).

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The classic performance measure called *Sharpe Ratio*, founded on the Markowitz mean-variance framework, needs to be generalized to allow for comparison among portfolios with heavy-tailed, possibly skewed, returns distributions. The Sharpe Ratio is defined as:

$$SR(x'z) = \frac{E(x'z) - r_f}{\sigma_{x'z}},$$

where $z = (z_1, z_2, \ldots, z_n)'$ is the vector of assets returns, $x = (x_1, x_2, \ldots, x_n)'$ denotes a vector of portfolio weights, $r_f$ is the risk-free return and $\sigma_{x'z}$ is the standard deviation of portfolio returns. The assumption of heavy tails makes the standard deviation no longer a reasonable measure of risk.

In literature a set of axioms has been presented (see Artzner et al. (1997) and Artzner et al. (1998)) to define a "coherent" risk measure. The set of axioms is *complete*, i.e. if a measure does not satisfy some of them, it may lead to undesirable conclusions. If we consider a set $V$ of real-valued random variables, a function $\varrho: V \to \boldsymbol{R}$ is called a *coherent risk measure* if it is

1. monotonous: $X, Y \in V;\ Y \geqslant X \Rightarrow \varrho(Y) \leqslant \varrho(X)$,
2. sub-additive: $X, Y, X+Y \in V \Rightarrow \varrho(X+Y) \leqslant \varrho(Y) + \varrho(X)$,
3. positively homogeneous: $X \in V,\ h > 0,\ hX \in V \Rightarrow \varrho(hX) = h\varrho(X)$,
4. translation invariant: $X \in V,\ a \in \boldsymbol{R} \Rightarrow \varrho(X+a) = \varrho(X) - a$.

The *Conditional Value-at-Risk* (CVaR) of an absolutely continuous random variable $X$ at significance level $\varepsilon$ is defined as[1]

$$CVaR_\varepsilon(X) = -\boldsymbol{E}\big(X \mid X \leqslant -VaR_\varepsilon(X)\big), \tag{1}$$

where $VaR_\varepsilon(X)$ is implicitly defined by $\boldsymbol{P}\big(X \leqslant -VaR_\varepsilon(X)\big) = \varepsilon$ and it is the industry standard risk measure Value-at-Risk (VaR). It satisfies the axioms, hence it is a coherent risk measure; for a discussion see Bradley and Taqqu (2003). For a discussion on the CVaR and a comparison to VaR, see Yamai and Yoshiba (2002a) and Yamai and Yoshiba (2002b).

As an alternative to the Sharpe Ratio we can define other performance measures which assume $\alpha$-stable distributed portfolio returns. A new ratio, recently proposed in Martin et al. (2003) is the Stable Tail Adjusted Return Ratio (STARR) defined as

$$STARR_\varepsilon(x'z) = \frac{E(x'z) - r_f}{CVaR_\varepsilon(x'z)/D_\varepsilon},$$

where the normalization constant $D_\varepsilon = CVaR_\varepsilon(Y)$ and $Y$ follows the standard normal distribution. It is shown in Martin et al. (2003) that the new ratio generalizes the Sharpe Ratio in a reasonable way. Another example is the

[1] We tacitly assume that $X$ is interpreted as portfolio returns.

Rachev Ratio (RR):

$$RR_{\varepsilon_1,\varepsilon_2}(x'z) = \frac{CVaR_{\varepsilon_1}(r_f - x'z)}{CVaR_{\varepsilon_2}(x'z - r_f)}$$

capturing the asymmetry in portfolio returns distribution. It is reported in Biglova et al. (2004) that the Rachev Ratio seems to have the best ex-ante and ex-post performance when tested among a variety of utility functions, out-performing significantly the Sharpe Ratio.

For the efficient use of ratios involving the CVaR as a risk measure, it is very important to have an accurate method of computing the CVaR. The principal difficulty of working with the class of $\alpha$-stable laws is that their densities are not known in closed form and normally a researcher relies on approximations introducing a certain error. On the other hand, the power decay of the tail, an inherent property, makes Monte Carlo techniques unreliable.

The current paper develops an integral representation of $CVaR_\varepsilon(X)$, where $X$ follows $\alpha$-stable distribution with $1 < \alpha < 2$, which is numerically easy to handle. Throughout the paper we shall assume that if $X$ follows a stable law, $X \in S_\alpha(\sigma, \beta, \mu)$, then we have the following parameterization of the characteristic function $\varphi_X(t) = E\exp(itX)$:

$$\ln \varphi_X(t) = -\sigma^\alpha |t|^\alpha \left[ 1 - i\beta \operatorname{sign}(t) \tan\frac{\pi\alpha}{2} \right] + i\mu t, \tag{2}$$

were $\operatorname{sign}(t)$ denotes the sign of $t$ if $t \neq 0$ and $\operatorname{sign}(0) = 0$. We are assuming $1 < \alpha < 2$ to make sure that conditional expectation used in the definition of CVaR is well defined. For further details about the properties of $\alpha$-stable laws see Samorodnitsky and Taqqu (1994). In the next section we state the main results. The proofs are given in the Appendix. A comparison to Monte Carlo method and direct numerical integration follow. In Section 4 we demonstrate how to associate the main result with some common assumptions for the multivariate asset returns distribution. Finally, we provide tabulated values for $CVaR_{0.01}(X)$ and $CVaR_{0.05}(X)$.

## 2. INTEGRAL REPRESENTATION

The main result will be derived for the standardized case, i.e. $X \in S_\alpha(1, \beta, 0)$. As a matter of fact, using the properties of translation invariance and positive homogeneity of the CVaR, under the assumption that $\sigma > 0$ and $\mu \in \boldsymbol{R}$ are a scale and a location parameter, respectively, we obtain

$$CVaR_\varepsilon(\sigma X + \mu) = \sigma CVaR_\varepsilon(X) - \mu, \quad \text{where } \sigma X + \mu \in S_\alpha(\sigma, \beta, \mu).$$

PROPOSITION 1. *Let $X \in S_\alpha(1, \beta, 0)$ with $\alpha > 1$. If $VaR_\varepsilon(X) \neq 0$, then the Conditional VaR of X at significance level $\varepsilon$ admits the following integral* *representation*:

(3) $$CVaR_\varepsilon(X) = \frac{\alpha}{1-\alpha}\frac{|VaR_\varepsilon(X)|}{\pi\varepsilon}\int_{-\bar{\theta}_0}^{\pi/2} g(\theta)\exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)}v(\theta)\right)d\theta,$$

*where*

$$g(\theta) = \frac{\sin\left(\alpha(\bar{\theta}_0+\theta)-2\theta\right)}{\sin\alpha(\bar{\theta}_0+\theta)} - \frac{\alpha\cos^2\theta}{\sin^2\alpha(\bar{\theta}_0+\theta)},$$

$$v(\theta) = (\cos\alpha\bar{\theta}_0)^{1/(\alpha-1)}\left(\frac{\cos\theta}{\sin\alpha(\bar{\theta}_0+\theta)}\right)^{\alpha/(\alpha-1)}\frac{\cos\left(\alpha\bar{\theta}_0+(\alpha-1)\theta\right)}{\cos\theta},$$

$$\bar{\theta}_0 = \frac{1}{\alpha}\arctan\left(\bar{\beta}\tan\frac{\pi\alpha}{2}\right) \quad \textit{and} \quad \bar{\beta} = -\operatorname{sign}\left(VaR_\varepsilon(X)\right)\beta.$$

*Furthermore, if* $VaR_\varepsilon(X) = 0$, *then*

(4) $$CVaR_\varepsilon(X) = \frac{2\Gamma\left((\alpha-1)/\alpha\right)}{(\pi-2\theta_0)}\frac{\cos\theta_0}{(\cos\alpha\theta_0)^{1/\alpha}},$$

*where* $\theta_0 = \alpha^{-1}\arctan(\beta\tan(\pi\alpha/2))$.

The symmetric case ($\beta = 0$) yields the following

COROLLARY 1. *If* $X \in S_\alpha(1, 0, 0)$ *with* $\alpha > 1$ *and* $VaR_\varepsilon(X) \neq 0$, *then* $CVaR_\varepsilon(X)$ *admits the representation*:

(5) $$CVaR_\varepsilon(X) = \frac{\alpha}{1-\alpha}\frac{|VaR_\varepsilon(X)|}{\pi\varepsilon}\int_{0}^{\pi/2} g(\theta)\exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)}v(\theta)\right)d\theta,$$

*where*

$$g(\theta) = \frac{\sin(\alpha-2)\theta}{\sin\alpha\theta} - \frac{\alpha\cos^2\theta}{\sin^2\alpha\theta}, \qquad v(\theta) = \left(\frac{\cos\theta}{\sin\alpha\theta}\right)^{\alpha/(\alpha-1)}\frac{\cos(\alpha-1)\theta}{\cos\theta}.$$

*If* $VaR_\varepsilon(X) = 0$, *then*

$$CVaR_\varepsilon(X) = \frac{2\Gamma\left((\alpha-1)/\alpha\right)}{\pi}.$$

In addition, we have the following symmetry property of $\alpha$-stable $CVaR_\varepsilon(X)$ which they share with other zero mean absolutely continuous random variables.

COROLLARY 2. *Let* $X \in S_\alpha(1, \beta, 0)$ *with* $\alpha > 1$. *Then the following relation holds*:

$$CVaR_\varepsilon(X) = \frac{1-\varepsilon}{\varepsilon}CVaR_{1-\varepsilon}(-X).$$

Corollary 2 follows from the equalities

$$\begin{aligned} 0 = E(X) &= \varepsilon E\big(X \mid X \leqslant -VaR_\varepsilon(X)\big) + (1-\varepsilon) E\big(X \mid X > -VaR_\varepsilon(X)\big) \\ &= -\varepsilon CVaR_\varepsilon(X) - (1-\varepsilon) E\big(-X \mid -X < VaR_\varepsilon(X)\big) \\ &= -\varepsilon CVaR_\varepsilon(X) - (1-\varepsilon) E\big(-X \mid -X < -VaR_{1-\varepsilon}(-X)\big) \\ &= -\varepsilon CVaR_\varepsilon(X) + (1-\varepsilon) CVaR_{1-\varepsilon}(-X), \end{aligned}$$

where the fourth equality is a consequence of the VaR definition and

$$VaR_\varepsilon(X) = -VaR_{1-\varepsilon}(-X).$$

It is possible to show that in the Gaussian case ($\alpha = 2$) the integral expression is reduced to a closed form expression.

COROLLARY 3. *If* $X \in S_2(1, 0, 0) = N(0, 2)$, *then*

$$\begin{aligned} CVaR_\varepsilon(X) &= \frac{|VaR_\varepsilon(X)|}{\pi\varepsilon} \int_0^{\pi/2} \frac{1}{\sin^2\theta} \exp\left(-\frac{\big(VaR_\varepsilon(X)\big)^2}{4\sin^2\theta}\right) d\theta \\ &= \frac{1}{\varepsilon\sqrt{\pi}} \exp\left(-\frac{\big(VaR_\varepsilon(X)\big)^2}{4}\right). \end{aligned}$$

Proof. If we set $\alpha = 2$ in equation (3), for $CVaR_\varepsilon(X)$ we receive the expression

$$CVaR_\varepsilon(X) = \frac{2|VaR_\varepsilon(X)|}{\pi\varepsilon} \int_0^{\pi/2} \frac{1}{2\sin^2\theta} \exp\left(-\frac{\big(VaR_\varepsilon(X)\big)^2}{4\sin^2\theta}\right) d\theta.$$

Now we apply a change of variables

$$t = \frac{1}{\sin^2\theta} - 1 = \frac{\cos^2\theta}{\sin^2\theta}, \qquad dt = -\frac{2\cos\theta}{\sin^3\theta} d\theta,$$

and the result is

$$\begin{aligned} CVaR_\varepsilon(X) &= \frac{2|VaR_\varepsilon(X)|}{\pi\varepsilon} \int_0^\infty \frac{1}{4\sqrt{t}} \exp\left(-\frac{\big(VaR_\varepsilon(X)\big)^2 (t+1)}{4}\right) dt \\ &= \frac{|VaR_\varepsilon(X)|}{2\pi\varepsilon} \exp\left(-\frac{\big(VaR_\varepsilon(X)\big)^2}{4}\right) \int_0^\infty \frac{1}{\sqrt{t}} \exp\left(-\frac{\big(VaR_\varepsilon(X)\big)^2 t}{4}\right) dt. \end{aligned}$$

If we represent

$$|VaR_\varepsilon(X)| = \frac{|VaR_\varepsilon(X)|^2}{\sqrt{|VaR_\varepsilon(X)|^2}}$$

and combine it with the variable $t$, the integral is recognized as the gamma function

$$\int_0^{\infty} \frac{2}{\sqrt{|VaR_\varepsilon(X)|^2 t}} \exp\left(-\frac{(VaR_\varepsilon(X))^2 t}{4}\right) d\left(\frac{|VaR_\varepsilon(X)|^2 t}{4}\right) = \Gamma(1/2) = \sqrt{\pi},$$

which proves the statement. ∎

The integral representation given in Proposition 1 is well suited for numerical work. This is suggested by the properties of the integrand given in the next

PROPOSITION 2. *If* $VaR_\varepsilon(X) \neq 0$, *then the integrand in equation* (3)

$$z(\theta) = g(\theta)\exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)} v(\theta)\right)$$

*is a bounded function in* $[-\bar{\theta}_0, \pi/2]$. *Moreover,*

$$\lim_{\theta \to -\bar{\theta}_0} z(\theta) = 0 \quad \text{and} \quad \lim_{\theta \to \pi/2} z(\theta) = \begin{cases} -1 & \bar{\beta} > -1, \\ 1/\alpha - 1, & \bar{\beta} = -1. \end{cases}$$

We do not consider the Gaussian case because we have a nice closed form expression, see Corollary 3. The proofs of Propositions 1 and 2 will be given in the Appendix.

## 3. COMPARISON TO OTHER METHODS

In this section we explore the accuracy of some existing methods for the CVaR computation. It should be noted that the only source of error when using the expression in Proposition 1 is in the quadrature that numerically calculates the integral. Because of the nicely behaved integrand and the bounded integration range, this error is easily controllable.

**3.1. Monte Carlo.** From the definition of the Conditional Value-at-Risk it is clear that it could be computed using Monte Carlo scenarios according to the following algorithm:

1. Draw a large sample $X_1, X_2, \ldots, X_N$ from a stable law with parameters $\alpha = \alpha_0$, $\beta = \beta_0$, $\sigma = \sigma_0$ and $\mu = \mu_0$.

2. Sort the observations $X_{(1)} < X_{(2)} < \ldots < X_{(N)}$ and compute

$$(CVaR_\varepsilon(X))^\wedge = \frac{1}{[\varepsilon N]} \sum_{i=1}^{[\varepsilon N]} X_{(i)}, \tag{6}$$

where by $[a]$ we mean the largest integer smaller than or equal to $a$.

By the law of large numbers, the statistic (6) converges almost surely to the true value $CVaR_\varepsilon(X)$. For comparison, Figure 1 (see Section 7) depicts the surface of CVaR with $\varepsilon = 0.1$ for a large part of the entire $(\alpha, \beta)$ space computed by using the integral representation we have developed, and Figure 2 (Section 7) depicts the same surface computed by using the Monte Carlo method with $N = 1\,000\,000$. Because of the heavy tail, we observe larger fluctuations in Monte Carlo estimates as $\alpha$ decreases. Very few extreme outliers have been ignored because they destroyed the scale. The fluctuations are significant even when $\varepsilon$ is as high as 0.1, which means that in the computation of the sample average we have effectively used 100 000 simulations. Another study of the $CVaR_\varepsilon(X)$ calculation done with the Monte Carlo method can be found in Yamai and Yoshiba (2002b). They consider only the symmetric case ($\beta = 0$) and provide 95% confidence bounds for the estimates.

**3.2. Direct numerical integration.** We can also calculate the CVaR of a stable law using equation (11) and computing the integrals numerically. We can verify if the derived formula (3) is advantageous from the practical viewpoint by, checking that the error from the numerical integration is small. Since stable densities are not known in closed form, it is possible to use either the Zolotarev's integral representation given in Theorem 1 or the FFT-based approximation developed in Rachev and Mittnik (2000). We replace the infinite upper limit of the first integral with a large constant $K$:

$$(CVaR_\varepsilon(X))^{\sim} = \frac{1}{\varepsilon}\int_0^K u f_X(u;\, \alpha,\, -\beta)\, du - \frac{1}{\varepsilon}\int_0^{|VaR_\varepsilon(X)|} u f_X\big(u;\, \alpha,\, -\operatorname{sign}(VaR_\varepsilon(X))\,\beta\big)\, du.$$

The absolute error is given by

$$\delta_{\varepsilon,K}(X) = \left|CVaR_\varepsilon(X) - (CVaR_\varepsilon(X))^{\sim}\right| = \frac{1}{\varepsilon}\int_K^\infty u f_X(u;\, \alpha,\, -\beta)\, du. \tag{7}$$

Note that $\delta_{\varepsilon,K}(X)$ does not incorporate the error arising from the use of a pdf approximation instead of the pdf itself. The following result provides a way to compute $\delta_{\varepsilon,K}(X)$.

Proposition 3. *Let* $X \sim S_\alpha(1, \beta, 0)$ *with* $\alpha > 1$. *The absolute error* $\delta_{\varepsilon,K}(X)$ *defined in equation* (7) *admits the following representation*:

$$\delta_{\varepsilon,K}(X) = \frac{\alpha}{1-\alpha}\frac{K}{\pi\varepsilon}\int_{\theta_0}^{\pi/2} g(\theta) \exp\big(-K^{\alpha/(\alpha-1)} v(\theta)\big)\, d\theta, \tag{8}$$

*where*

$$g(\theta) = \frac{\sin(\alpha(\theta-\theta_0) - 2\theta)}{\sin\alpha(\theta-\theta_0)} - \frac{\alpha\cos^2\theta}{\sin^2\alpha(\theta-\theta_0)},$$

$$v(\theta) = (\cos\alpha\theta_0)^{1/(\alpha-1)}\left(\frac{\cos\theta}{\sin\alpha(\theta-\theta_0)}\right)^{\alpha/(\alpha-1)}\frac{\cos((\alpha-1)\theta - \alpha\theta_0)}{\cos\theta},$$

$$\theta_0 = \frac{1}{\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right).$$

Proof. Equation (7) is the same as equation (8), the only difference is that the lower bound is the large positive constant $K$ instead of $VaR_\varepsilon(X)$. Therefore $\delta_{\varepsilon,K}(X)$ admits the same integral representation as $CVaR_\varepsilon(X)$ if $VaR_\varepsilon(X) > 0$; see Proposition 1. In this case $\bar{\beta} = -\beta$, $\bar{\theta}_0 = -\theta_0$ and we obtain equation (8) from equation (3). ■

Tables 1 and 2 (see Section 7) show the absolute and relative errors in the case $\varepsilon = 0.01$ and $K = 100$ for a large part of the entire $(\alpha, \beta)$ space. This choice of $\varepsilon$ would be typical when measuring the risk of extreme losses. Clearly, the error $\delta_{\varepsilon,K}(X)$ is not negligible even when $\alpha \approx 1.7$, which would normally be the case when considering stock returns.

## 4. APPLICATION TO PORTFOLIO THEORY

In this section we briefly show how to relate the derived expression to some common assumptions on the multivariate distribution of assets returns.

### 4.1. Multivariate $\alpha$-stable distributions

**4.1.1.** *General case.* Let us assume that the vector of assets returns $z = (z_1, z_2, \ldots, z_n)'$ follows a multivariate $\alpha$-stable distribution with $\alpha > 1$, a vector of expected returns $\mu = (\mu_1, \mu_2, \ldots, \mu_n)'$, and a spectral measure $\Gamma$. The notation is $z \in S_\alpha(\Gamma, \mu)$. We can claim (see Samorodnitsky and Taqqu (1994), p. 67) that portfolio returns $x'z \in S_\alpha(\sigma_P, \beta_P, \mu_P)$ with

$$\sigma_P = \Big(\int_{S^n} |x's|^\alpha \Gamma(ds)\Big)^{1/\alpha}, \qquad \beta_P = \frac{\int_{S^n} |x's|^\alpha \operatorname{sign}(x's)\, \Gamma(ds)}{\int_{S^n} |x's|^\alpha \Gamma(ds)}, \qquad \mu_P = x'\mu,$$

where $S^n$ denotes the $n$-dimensional unit sphere, i.e. $S^n = \{u \in R^n\colon \|u\| = 1\}$. Then for the Conditional Value-at-Risk of portfolio returns we have

$$CVaR_\varepsilon(x'z) = \sigma_P\, CVaR_\varepsilon\left(\frac{x'z - \mu_P}{\sigma_P}\right) - \mu_P, \tag{9}$$

where

$$\frac{x'z - \mu_P}{\sigma_P} \in S_\alpha(1, \beta_P, 0).$$

The entire class of multivariate $\alpha$-stable laws is quite general and the spectral measure is hard to estimate. The sub-Gaussian family is often considered in the applications.

**4.1.2.** *Sub-Gaussian stable laws.* The sub-Gaussian stable laws are shifted symmetric multivariate $\alpha$-stable laws with characteristic function

$$\varphi_z(t) = E \exp(it'z) = \exp\big(-(t'Qt)^{\alpha/2} + it'\mu\big),$$

where $Q$ is a positive-definite matrix called the *dispersion* matrix and $\mu = Ez$ is the vector of expected returns (recall that we assume $\alpha > 1$). For further details about these distributions and their parameters estimation see Ortobelli et al. (2003) and Samorodnitsky and Taqqu (1994). For this multivariate distributional assumption, portfolio returns follow the symmetric stable law $x'z \in S_\alpha(\sigma_P, 0, \mu_P)$ with

$$\sigma_P = \sqrt{x'Qx}, \qquad \mu_P = x'\mu.$$

Therefore equation (9) can be rewritten as

$$CVaR_\varepsilon(x'z) = (\sqrt{x'Qx})\, CVaR_\varepsilon\left(\frac{x'z - \mu_P}{\sigma_P}\right) - x'\mu, \tag{10}$$

where

$$\frac{x'z - \mu_P}{\sigma_P} \in S_\alpha(1, 0, 0).$$

Notice that $CVaR_\varepsilon(X)\big((x'z - \mu_P)/\sigma_P\big)$ is constant with respect to the vector of portfolio weights $x$.

The sub-Gaussian stable laws form a special case of the more general class of the elliptical distributions. We mention this class of models in the subsection below for comparison, even though it is somewhat outside of the framework of this paper.

**4.2. Elliptical distributions.** Let the vector of portfolio returns follow an elliptical distribution with a vector of expected returns $\mu$ (we consider the case with finite expectation), a non-negative definite symmetric matrix $\Sigma$ and characteristic generator $\phi(\cdot)$, $z \in E_n(\mu, \Sigma, \phi)$; for further details see Embrechts et al. (2003). In this case, the CVaR of portfolio returns equals

$$CVaR_\varepsilon(x'z) = (\sqrt{x'\Sigma x})\, CVaR_\varepsilon(Y) - x'\mu,$$

where $Y$ is distributed according to a standardized univariate elliptical law, i.e. $Y \in E_1(0, 1)$. The multivariate $t$-distribution is an elliptical distribution that exhibits heavy tails for which $CVaR_\varepsilon(Y)$ can be explicitly given:

$$CVaR_\varepsilon(Y) = \frac{\Gamma\big((\nu+1)/2\big)}{\Gamma(\nu/2)} \frac{\sqrt{\nu}}{(\nu-1)\,\varepsilon\sqrt{\pi}} \left(1 + \frac{VaR_\varepsilon(Y)^2}{\nu}\right)^{(1-\nu)/2},$$

where $Y \in t(\nu)$ with $\nu > 1$ degrees of freedom.

## 5. CONCLUSION

Accurate computation of the Conditional Value-at-Risk of univariate stable laws is a non-trivial task. The current paper develops an easily computable integral representation and compares it to some existing methods – Monte

Carlo and direct numerical integration. The first approach is hampered by the fact that $\alpha$-stable distributions exhibit heavy tails. This produces large fluctuations of the estimated values. The second approach is hampered by the lack of closed form expressions of the stable densities. As a result of the comparison, the proposed integral representation appears superior to the two traditional approaches. Tables 3 and 4 (see Section 7) provide tabulated values for $CVaR_{0.01}(X)$ and $CVaR_{0.05}(X)$. These values 0.01 or 0.05 would represent a common choice for $\varepsilon$ in portfolio selection problems.

## 6. APPENDICES

### APPENDIX 1 – PROOF OF PROPOSITION 1

For the proof of Proposition 1, we need some preliminary results given in the sequel.

### 6.1. Preliminary results

LEMMA 1. *If $X$ is distributed according to a standard stable law, $X \in S_\alpha(1, \beta, 0)$, with density $f_X(u; \alpha, \beta)$, then*

$$CVaR_\varepsilon(X) = \frac{1}{\varepsilon}\int_0^\infty u f_X(u; \alpha, -\beta)\,du - \frac{1}{\varepsilon}\int_0^{|VaR_\varepsilon(X)|} u f_X\big(u; \alpha, -\operatorname{sign}(VaR_\varepsilon(X))\beta\big)\,du. \tag{11}$$

Proof. From the definition of the conditional expectation we have the following representation of $CVaR_\varepsilon(X)$:

$$CVaR_\varepsilon(X) = \frac{1}{\varepsilon}\int_{VaR_\varepsilon(X)}^\infty u f_X(-u; \alpha, \beta)\,du \tag{12}$$

that can easily be transformed into

$$CVaR_\varepsilon(X) = \frac{1}{\varepsilon}\int_0^\infty u f_X(-u; \alpha, \beta)\,du - \frac{1}{\varepsilon}\int_0^{VaR_\varepsilon(X)} u f_X(-u; \alpha, \beta)\,du \quad \text{if } VaR_\varepsilon(X) > 0$$

and

$$CVaR_\varepsilon(X) = \frac{1}{\varepsilon}\int_0^\infty u f_X(-u; \alpha, \beta)\,du - \frac{1}{\varepsilon}\int_0^{-VaR_\varepsilon(X)} u f_X(u; \alpha, \beta)\,du \quad \text{if } VaR_\varepsilon(X) \leqslant 0.$$

The statement follows immediately from the symmetry property of stable densities

$$f_X(-u; \alpha, \beta) = f_X(u; \alpha, -\beta)$$

and the $CVaR_\varepsilon(X)$ expressions from above. ∎

We will take advantage of the integral expressions of the stable densities with $1 < \alpha < 2$ given in Nolan (1997). For this purpose we restate them in the current parameterization (2). The results are summarized in

THEOREM 1. *Let $X \in S_\alpha(1, \beta, 0)$. The density function of $X$ when $x > 0$ is given by*

$$f_X(x;\alpha,\beta) = \frac{\alpha x^{1/(\alpha-1)}}{\pi|\alpha-1|} \int_{-\theta_0}^{\pi/2} v(\theta;\alpha,\beta)\exp\left(-x^{\alpha/(\alpha-1)} v(\theta;\alpha,\beta)\right) d\theta,$$

*where*

$$\theta_0 = \frac{1}{\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right),$$

$$v(\theta;\alpha,\beta) = (\cos\alpha\theta_0)^{1/(\alpha-1)}\left(\frac{\cos\theta}{\sin\alpha(\theta_0+\theta)}\right)^{\alpha/(\alpha-1)} \frac{\cos(\alpha\theta_0+(\alpha-1)\theta)}{\cos\theta}.$$

*Moreover, $v(\theta;\alpha,\beta)$ is continuous, positive, strictly monotone on $(-\theta_0, \pi/2)$, and*

(13) $$\lim_{\theta\to-\theta_0} v(\theta;\alpha,\beta) = \infty, \qquad \lim_{\theta\to\pi/2} v(\theta;\alpha,\beta) = 0,$$

*with the second limit under the assumption $\beta > -1$. The limit is finite and positive for $\beta = -1$.*

Proofs of these results can be found in Nolan (1997), Zolotarev (1983), Buckle (1995) and the references therein. We also need the following

LEMMA 2. *The function*

(14) $$L(\theta;\alpha,\beta) = \frac{\alpha}{1-\alpha}\cos\theta\left(\frac{\cos(\alpha\theta_0+(\alpha-1)\theta)}{\cos\alpha\theta_0\cos\theta}\right)^{1/\alpha}$$

*is a primitive function of*

$$\left(v(\theta;\alpha,\beta)\right)^{(1-\alpha)/\alpha} = (\cos\alpha\theta_0)^{-1/\alpha}\frac{\sin\alpha(\theta_0+\theta)}{\cos\theta}\left(\frac{\cos(\alpha\theta_0+(\alpha-1)\theta)}{\cos\theta}\right)^{(1-\alpha)/\alpha}$$

*and has the following properties:*

$$\lim_{\theta\to-\theta_0^+} L(\theta;\alpha,\beta) = \frac{\alpha\cos\theta_0}{(1-\alpha)(\cos\alpha\theta_0)^{1/\alpha}}, \qquad \lim_{\theta\to\pi/2^-} L(\theta;\alpha,\beta) = 0,$$

*where $v(\theta;\alpha,\beta)$ and $\theta_0$ are as above.*

Proof. The most straightforward way to show that the statement holds is to differentiate $L(\theta)$. First note that since

$$\sin\left(\alpha(\theta_0+\theta)-\theta+\theta\right) = \sin\left(\alpha\theta_0+(\alpha-1)\theta\right)\cos\theta + \cos\left(\alpha\theta_0+(\alpha-1)\theta\right)\sin\theta,$$

we have

$$\big(v(\theta;\alpha,\beta)\big)^{(1-\alpha)/\alpha} = C_1 \sin\big(\alpha\theta_0+(\alpha-1)\theta\big)\left(\frac{\cos\big(\alpha\theta_0+(\alpha-1)\theta\big)}{\cos\theta}\right)^{(1-\alpha)/\alpha}$$

$$+C_1\sin\theta_0\left(\frac{\cos\big(\alpha\theta_0+(\alpha-1)\theta\big)}{\cos\theta}\right)^{1/\alpha},$$

where $C_1 = (\cos\alpha\theta_0)^{-1/\alpha}$. Writing

(15) $$L(\theta;\alpha,\beta) = \frac{\alpha}{1-\alpha} C_1 \big(\cos\big(\alpha\theta_0+(\alpha-1)\theta\big)\big)^{1/\alpha} (\cos\theta)^{(\alpha-1)/\alpha}$$

and differentiating we obtain the equation for $\big(v(\theta;\alpha,\beta)\big)^{(1-\alpha)/\alpha}$ in Lemma 2. The limit properties are easy to compute from (15). ■

Some elementary facts that we shall take advantage of are gathered in the following

LEMMA 3. *Let $\theta_0$ be defined as in Theorem* 1. *Then $\theta_0 = \theta_0(\beta)$ is a decreasing function of $\beta$. Furthermore,*

(16) $$-\frac{\pi}{2} < -\frac{\pi(2-\alpha)}{2\alpha} \leqslant \theta_0(\beta) \leqslant \frac{\pi(2-\alpha)}{2\alpha} < \frac{\pi}{2},$$

*where $\beta\in[-1, 1]$. Moreover,*

(17) $$0 \leqslant \sin\alpha(\theta_0+\theta) \leqslant 1 \quad \textit{for } \theta\in[-\theta_0, \pi/2]$$

*and* $\sin\alpha(\theta_0+\theta) = 0$ *if $\theta = -\theta_0$ for any choice of $\beta$ or if $\theta = \pi/2$ and $\beta = -1$.*

Proof. Since

$$\tan\frac{\pi\alpha}{2} = \tan\left(\frac{\pi\alpha}{2}-\pi\right) = \tan\left(\frac{\pi(\alpha-2)}{2}\right) = -\tan\left(\frac{\pi(2-\alpha)}{2}\right),$$

we have

$$\theta_0 = \theta_0(\beta) = -\frac{1}{\alpha}\arctan\left(\beta\tan\left(\frac{\pi(2-\alpha)}{2}\right)\right),$$

and therefore $\theta_0(\beta)' < 0$. The chain of inequalities (16) follows directly, observing that by assumption $1<\alpha<2$. If $\theta\in[-\theta_0, \pi/2]$, then

(18) $$0 \leqslant \alpha(\theta_0+\theta) \leqslant \alpha(\theta_0+\pi/2) \leqslant \pi.$$

The last inequality turns into equality if $\theta_0 = \theta_0(-1) = \pi(2-\alpha)/(2\alpha)$ and we obtain equation (17) together with the limit cases. ■

**6.2. Proof of the main result.** The proof follows directly from the integral representation of the density function $f_X(u;\alpha,\beta)$ given in Theorem 1. We substitute the expression of the density into equation (11):

(19) $$CVaR_\varepsilon(X) = \frac{1}{\varepsilon} I_1 - \frac{1}{\varepsilon} I_2$$

with

$$I_1 = \int_0^\infty u \int_{\theta_0}^{\pi/2} \frac{\alpha u^{1/(\alpha-1)}}{\pi(\alpha-1)} v(\theta;\, \alpha,\, -\beta) \exp\left(-u^{\alpha/(\alpha-1)} v(\theta;\, \alpha,\, -\beta)\right) d\theta du,$$

$$I_2 = \int_0^{|VaR_\varepsilon(X)|} u \int_{-\bar{\theta}_0}^{\pi/2} \frac{\alpha u^{1/(\alpha-1)}}{\pi(\alpha-1)} v(\theta;\, \alpha,\, \bar{\beta}) \exp\left(-u^{\alpha/(\alpha-1)} v(\theta;\, \alpha,\, \bar{\beta})\right) d\theta du,$$

where

$$\bar{\theta}_0 = \frac{1}{\alpha} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right) \quad \text{and} \quad \bar{\beta} = -\operatorname{sign}\left(VaR_\varepsilon(X)\right)\beta.$$

The lower bound of the integral in the density representation in $I_1$ is $\theta_0$ since $\arctan(-x) = -\arctan(x)$. We shall consider first $I_2$ assuming that $VaR_\varepsilon(X) \neq 0$. The case $VaR_\varepsilon(X) = 0$ is trivial since $I_2 = 0$. Switching the integration order yields

$$I_2 = \frac{\alpha}{\pi(\alpha-1)} \int_{-\bar{\theta}_0}^{\pi/2} \left( \int_0^{|VaR_\varepsilon(X)|} u^{\alpha/(\alpha-1)} v(\theta;\, \alpha,\, \bar{\beta}) \exp\left(-u^{\alpha/(\alpha-1)} v(\theta;\, \alpha,\, \bar{\beta})\right) du \right) d\theta$$

and after a change of variables $t = u^{\alpha/(\alpha-1)} v(\theta;\, \alpha,\, \bar{\beta})$ we obtain the expression

$$I_2 = \frac{1}{\pi} \int_{-\bar{\theta}_0}^{\pi/2} v(\theta;\, \alpha,\, \bar{\beta})^{(1-\alpha)/\alpha} \left( \int_0^{a(\theta)} t^{(\alpha-1)/\alpha} e^{-t} dt \right) d\theta,$$

where $a(\theta) = |VaR_\varepsilon(X)|^{\alpha/(\alpha-1)} v(\theta;\, \alpha,\, \bar{\beta})$. The equation above can be rewritten as

$$I_2 = \frac{1}{\pi} \int_{-\bar{\theta}_0}^{\pi/2} \left( \int_0^{a(\theta)} t^{(\alpha-1)/\alpha} e^{-t} dt \right) dL(\theta;\, \alpha,\, \bar{\beta}),$$

where $L(\theta;\, \alpha,\, \bar{\beta}) = v(\theta;\, \alpha,\, \bar{\beta})^{(1-\alpha)/\alpha}$. Lemma 2 gives the particular expression for $L(\theta;\, \alpha,\, \bar{\beta})$ and some useful function properties. Integration by parts leads to

$$\begin{aligned} I_2 &= \frac{1}{\pi} L(\theta;\, \alpha,\, \bar{\beta}) \left( \int_0^{a(\theta)} t^{(\alpha-1)/\alpha} e^{-t} dt \right)\Big|_{-\bar{\theta}_0}^{\pi/2} - \frac{1}{\pi} \int_{-\bar{\theta}_0}^{\pi/2} L(\theta;\, \alpha,\, \bar{\beta})\, d\left( \int_0^{a(\theta)} t^{(\alpha-1)/\alpha} e^{-t} dt \right) \\ &= \frac{\Gamma\left((\alpha-1)/\alpha\right)}{\pi} \frac{\cos \bar{\theta}_0}{(\cos \alpha\bar{\theta}_0)^{1/\alpha}} - \frac{1}{\pi} \int_{-\bar{\theta}_0}^{\pi/2} L(\theta;\, \alpha,\, \bar{\beta}) \left(a(\theta)\right)^{(\alpha-1)/\alpha} e^{-a(\theta)} da(\theta) \\ &:= \frac{\Gamma\left((\alpha-1)/\alpha\right)}{\pi} \frac{\cos \bar{\theta}_0}{(\cos \alpha\bar{\theta}_0)^{1/\alpha}} + I_3. \end{aligned}$$

The second equality holds because:

- At the left endpoint the value of $L(\theta;\alpha,\bar{\beta})$ is given in Lemma 2 and from the properties of the function $v(\theta;\alpha,\beta)$ given in Theorem 1 we obtain

$$\lim_{\theta\to-\bar{\theta}_0}\Big(\int_0^{a(\theta)} t^{(\alpha-1)/\alpha}e^{-t}\,dt\Big)=\Gamma\left(\frac{2\alpha-1}{\alpha}\right)=\frac{\alpha-1}{\alpha}\Gamma\left(\frac{\alpha-1}{\alpha}\right)$$

when computing the value at the left endpoint.

- At the right endpoint the value of $L(\theta;\alpha,\bar{\beta})$ is zero, and so is the value of the incomplete Gamma function since the upper integral bound approaches the lower one. Hence, at the right endpoint the first term is zero.

- Finally,

$$\frac{d}{dt}\Big(\int_0^{a(t)} f(u)\,du\Big)=f(a(t))\,a'(t).$$

Integration by parts is used once again:

$$I_3=-\frac{1}{\pi}\int_{-\bar{\theta}_0}^{\pi/2} L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}e^{-a(\theta)}\,da(\theta)=\frac{1}{\pi}\int_{-\bar{\theta}_0}^{\pi/2} L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}\,de^{-a(\theta)}$$

$$=\frac{1}{\pi}L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}e^{-a(\theta)}\Big|_{-\bar{\theta}_0}^{\pi/2}-\frac{1}{\pi}\int_{-\bar{\theta}_0}^{\pi/2} e^{-a(\theta)}\,d\Big(L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}\Big).$$

Since $\lim_{\theta\to\pi/2^-}a(\theta)=0=\lim_{\theta\to\pi/2^-}L(\theta;\alpha,\bar{\beta})$, the first summand is zero at the right endpoint. At the left endpoint we have

$$\lim_{\theta\to-\bar{\theta}_0}L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}e^{-a(\theta)}=\lim_{\theta\to-\bar{\theta}_0}L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{-1/\alpha}a(\theta)e^{-a(\theta)}$$

$$=\lim_{\theta\to-\bar{\theta}_0}L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{-1/\alpha}\lim_{\theta\to-\bar{\theta}_0}a(\theta)e^{-a(\theta)}.$$

We can easily see that $\lim_{\theta\to-\bar{\theta}_0}a(\theta)e^{-a(\theta)}=0$ from the properties of the function $v(\theta;\alpha,\beta)$ given in Theorem 1. It remains to compute

$$\lim_{\theta\to-\bar{\theta}_0}L(\theta;\alpha,\bar{\beta})\big(a(\theta)\big)^{-1/\alpha}=|VaR_\varepsilon(X)|^{1/(1-\alpha)}\lim_{\theta\to-\bar{\theta}_0}L(\theta;\alpha,\bar{\beta})\big(v(\theta;\alpha,\bar{\beta})\big)^{-1/\alpha}$$

$$=\frac{\alpha}{1-\alpha}|VaR_\varepsilon(X)|^{1/(1-\alpha)}(\cos\alpha\bar{\theta}_0)^{1/(1-\alpha)}\lim_{\theta\to-\bar{\theta}_0}\cos\theta\left(\frac{\sin\alpha(\bar{\theta}_0+\theta)}{\cos\theta}\right)^{1/(\alpha-1)}=0.$$

The last equality holds because it follows from equation (16) that

$$\lim_{\theta\to-\bar{\theta}_0}\cos\theta\neq0\quad\text{for any } 1<\alpha<2 \text{ and } -1\leqslant\beta\leqslant1.$$

Therefore the limit at the left endpoint is zero. The term behind the differential in the expression for $I_3$ equals

$$L(\theta;\,\alpha,\,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha} = |VaR_\varepsilon(X)|\frac{\alpha}{1-\alpha}\frac{\cos\big(\alpha\bar{\theta}_0+(\alpha-1)\,\theta\big)}{\sin\alpha\,(\bar{\theta}_0+\theta)}\cos\theta.$$

Decomposing $\cos\big(\alpha\bar{\theta}_0+(\alpha-1)\,\theta\big) = \cos\alpha\,(\bar{\theta}_0+\theta)\cos\theta+\sin\alpha\,(\bar{\theta}_0+\theta)\sin\theta$ and computing the derivative of $L(\theta;\,\alpha,\,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}$ we arrive at

$$\frac{d}{d\theta}\Big(L(\theta;\,\alpha,\,\bar{\beta})\big(a(\theta)\big)^{(\alpha-1)/\alpha}\Big) := h(\theta)$$
$$= |VaR_\varepsilon(X)|\frac{\alpha}{1-\alpha}\left(\frac{\sin\big(\alpha(\bar{\theta}_0+\theta)-2\theta\big)}{\sin\alpha\,(\bar{\theta}_0+\theta)}-\frac{\alpha\cos^2\theta}{\sin^2\alpha\,(\bar{\theta}_0+\theta)}\right).$$

Finally, for $I_2$ we get

$$I_2 = \frac{\Gamma\big((\alpha-1)/\alpha\big)}{\pi}\frac{\cos\bar{\theta}_0}{(\cos\alpha\bar{\theta}_0)^{1/\alpha}}-\frac{1}{\pi}\int_{-\bar{\theta}_0}^{\pi/2} h(\theta)\,e^{-a(\theta)}\,d\theta.$$

The first integral $I_1$ is handled similarly:

$$I_1 = \frac{\alpha}{\pi\,(\alpha-1)}\int_{\theta_0}^{\pi/2}\Big(\int_0^\infty u^{\alpha/(\alpha-1)}\,v(\theta;\,\alpha,\,-\beta)\exp\big(-u^{\alpha/(\alpha-1)}\,v(\theta;\,\alpha,\,-\beta)\big)\,du\Big)\,d\theta.$$

Since the upper bound is infinity, we get the complete Gamma function after the corresponding change of variables:

$$I_1 = \frac{1}{\pi}\int_{\theta_0}^{\pi/2} v(\theta;\,\alpha,\,-\beta)^{(1-\alpha)/\alpha}\Big(\int_0^\infty t^{(\alpha-1)/\alpha}\,e^{-t}\,dt\Big)\,d\theta$$
$$= \frac{\Gamma\big((2\alpha-1)/\alpha\big)}{\pi}\int_{\theta_0}^{\pi/2}\,[v(\theta;\,\alpha,\,-\beta)]^{(1-\alpha)/\alpha}\,d\theta$$
$$= \frac{\Gamma\big((2\alpha-1)/\alpha\big)}{\pi}L(\theta;\,\alpha,\,-\beta)\Big|_{\theta_0}^{\pi/2} = \frac{\Gamma\big((\alpha-1)/\alpha\big)}{\pi}\frac{\cos\theta_0}{(\cos\alpha\theta_0)^{1/\alpha}}.$$

Substituting the relevant expressions for $I_1$ and $I_2$ in (19) and bearing in mind that the cosine is an even function, we prove the main statement. If $VaR_\varepsilon(X) = 0$, then $I_2 = 0$. We can easily see it from equation (11). Hence if $VaR_\varepsilon(X) = 0$, then

$$CVaR_\varepsilon(X) = \frac{1}{\varepsilon}I_1$$

where, according to Nolan (1997) and after the corresponding change of parameterizations,

$$\varepsilon = P(X < 0) = \frac{1}{2} - \frac{\theta_0}{\pi}$$

and we obtain the second statement in the proposition. ∎

## APPENDIX 2 – PROOF OF PROPOSITION 2

From the properties of the function $v(\theta; \alpha, \bar{\beta})$ given in Theorem 1, and noticing that $v(\theta; \alpha, \bar{\beta}) = v(\theta)$, we obtain

$$\lim_{\theta \to \pi/2} \exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)} v(\theta)\right) = 1.$$

Therefore

$$\lim_{\theta \to \pi/2} z(\theta) = \lim_{\theta \to \pi/2} g(\theta) = \lim_{\theta \to \pi/2} \frac{\sin\left(\alpha(\bar{\theta}_0 + \theta) - 2\theta\right)}{\sin \alpha(\bar{\theta}_0 + \theta)} - \lim_{\theta \to \pi/2} \frac{\alpha \cos^2 \theta}{\sin^2 \alpha(\bar{\theta}_0 + \theta)}$$

$$= \frac{\sin\left(\alpha(\bar{\theta}_0 + \pi/2) - \pi\right)}{\sin \alpha(\bar{\theta}_0 + \pi/2)} = -1 \quad \text{if } \bar{\beta} > -1.$$

The case $\bar{\beta} = -1$ is more complicated because the denominator turns into zero (see Lemma 3). Applying the intermediate result in formula (18) and l'Hôpital's rule, we obtain

$$\lim_{\theta \to \pi/2} \frac{\sin\left(\alpha(\bar{\theta}_0 + \theta) - 2\theta\right)}{\sin \alpha(\bar{\theta}_0 + \theta)} = \lim_{\theta \to \pi/2} \frac{\sin\left(\pi + \alpha(\theta - \pi/2) - 2\theta\right)}{\sin\left(\pi + \alpha(\theta - \pi/2)\right)} = \frac{2-\alpha}{\alpha}$$

and

$$\lim_{\theta \to \pi/2} \frac{\alpha \cos^2 \theta}{\sin^2 \alpha(\bar{\theta}_0 + \theta)} = \lim_{\theta \to \pi/2} \frac{\alpha \cos^2 \theta}{\sin^2\left(\pi + \alpha(\theta - \pi/2)\right)} = \frac{1}{\alpha}.$$

Therefore, if $\bar{\beta} = -1$, $\lim_{\theta \to \pi/2} z(\theta) = (1-\alpha)/\alpha$. The other limit can be seen as a product of limits. That is,

$$\lim_{\theta \to -\bar{\theta}_0} z(\theta) = \lim_{\theta \to -\bar{\theta}_0} \frac{g(\theta)}{v(\theta; \alpha, \bar{\beta})} v(\theta; \alpha, \bar{\beta}) \exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)} v(\theta; \alpha, \bar{\beta})\right)$$

$$= \lim_{\theta \to -\bar{\theta}_0} \frac{g(\theta)}{v(\theta; \alpha, \bar{\beta})} \lim_{\theta \to -\bar{\theta}_0} v(\theta; \alpha, \bar{\beta}) \exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)} v(\theta; \alpha, \bar{\beta})\right).$$

Using the properties of the function $v(\theta; \alpha, \bar{\beta})$ given in Theorem 1, it is easy to see that

$$\lim_{\theta \to -\bar{\theta}_0} v(\theta; \alpha, \bar{\beta}) \exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)} v(\theta; \alpha, \bar{\beta})\right) = 0.$$

It remains to calculate the first limit:

$$\lim_{\theta\to-\bar\theta_0}\frac{g(\theta)}{v(\theta;\alpha,\bar\beta)}=(\cos\alpha\bar\theta_0)^{1/(1-\alpha)}\lim_{\theta\to-\bar\theta_0}\left(\sin\left(\alpha(\bar\theta_0+\theta)-2\theta\right)\sin\alpha(\bar\theta_0+\theta)-\alpha\cos^2\theta\right)$$

$$\times\frac{\left(\sin\alpha(\bar\theta_0+\theta)\right)^{(2-\alpha)/(\alpha-1)}}{(\cos\theta)^{1/(\alpha-1)}\cos\left(\alpha\bar\theta_0+(\alpha-1)\theta\right)}=0.$$

The last equality holds because we consider the case $1<\alpha<2$, and hence $(2-\alpha)/(\alpha-1)>0$.

We can easily see that the integrand is a bounded function because of the properties of $v(\theta;\alpha,\beta)$:

$$0\leqslant\exp\left(-|VaR_\varepsilon(X)|^{\alpha/(\alpha-1)}v(\theta;\alpha,\bar\beta)\right)\leqslant 1.$$

The function

$$g(\theta)=\frac{\sin\left(\alpha(\bar\theta_0+\theta)-2\theta\right)\sin\alpha(\bar\theta_0+\theta)-\alpha\cos^2\theta}{\sin^2\alpha(\bar\theta_0+\theta)}$$

has a bounded numerator. The denominator turns into zero only when $\theta=-\bar\theta_0$ for any $\alpha$ and $\bar\beta$ or when $\theta=\pi/2$ and $\bar\beta=-1$. This result is contained in Lemma 3. Those limit cases have already been considered and we know that at these points the entire integrand has finite limits. Clearly, since the denominator does not turn into zero for any middle point, it follows that $g(\theta)$ is bounded for any $(\alpha,\beta)$ pair, and so is the entire integrand. ■

## APPENDIX 3 – RELATION TO THE CLOSED FORM EXPRESSION OF $E|X|$, $\alpha>1$

The closed form expression (4) can be related to the closed form expression of the first absolute moment of $X\in S_\alpha(1,\beta,0)$:

$$\text{(20)}\quad E|X|=\frac{2\Gamma(1-1/\alpha)}{\pi}\left(1+\beta^2\tan^2\frac{\pi\alpha}{2}\right)^{1/2\alpha}\cos\left(\frac{1}{\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right)\right)$$

$$=\frac{2\Gamma(1-1/\alpha)}{\pi}\left(1+\beta^2\tan^2\frac{\pi\alpha}{2}\right)^{1/2\alpha}\cos\theta_0,$$

where $\theta_0$ is defined in Theorem 1 (for details see Samorodnitsky and Taqqu, p. 18). Since the integral in the expectation can be equivalently rewritten as

$$E|X|=\int_0^\infty uf(u)\,du+\int_0^\infty uf(-u)\,du,$$

in the case of stable laws we get

$$E|X| = \int_0^\infty u f(u;\alpha,\beta)\,du + \int_0^\infty u f(u;\alpha,-\beta)\,du$$

$$= \varepsilon_0^* CVaR_{\varepsilon_0^*}(-X) + \varepsilon_0 CVaR_{\varepsilon_0}(X),$$

where $-X \in S_\alpha(1,-\beta,0)$, $\varepsilon_0 = P(X<0)$ and $\varepsilon_0^* = P(X \geqslant 0)$. Therefore

(21) $$E|X| = \frac{2\Gamma((\alpha-1)/\alpha)}{\pi}\frac{\cos\theta_0}{(\cos\alpha\theta_0)^{1/\alpha}}.$$

It is possible to show, using some algebra, that equation (21) equals the closed form expression (20).

## 7. PLOTS AND TABLES

$X \in S_\alpha(1,\beta,0)$, the integral representation

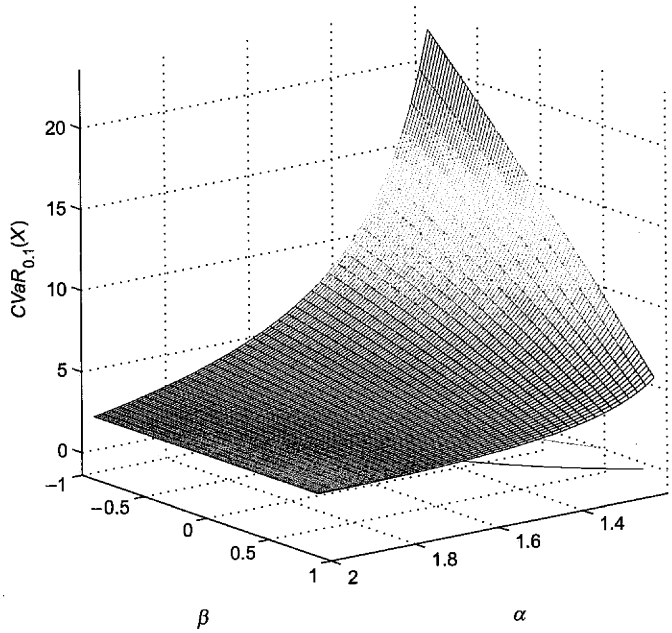


FIGURE 1. $CVaR_{0.1}(X)$ computed with the integral representation for different $(\alpha, \beta)$ pairs

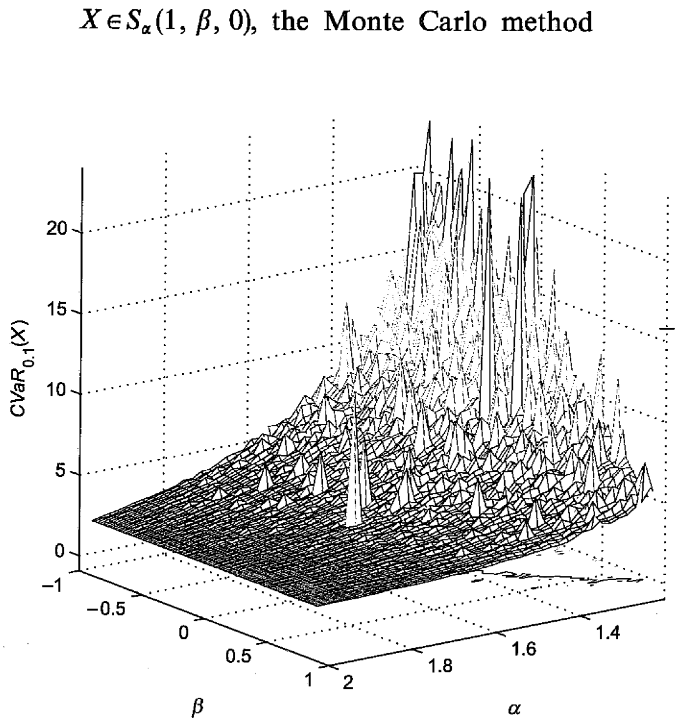


FIGURE 2. $CVaR_{0.1}(X)$ computed with the Monte Carlo method using 1 million simulations for different $(\alpha, \beta)$ pairs

TABLE 1. The absolute error $\delta_{\varepsilon,K}(X)$, where $\varepsilon = 0.01$ and $K = 500$ for different $(\alpha, \beta)$ pairs

| | | $\beta$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | −0.5 | −0.3 | −0.1 | 0 | 0.1 | 0.3 | 0.5 |
| | 1.20 | 72.163 | 62.550 | 52.933 | 48.124 | 43.315 | 33.694 | 24.070 |
| | 1.30 | 25.642 | 22.224 | 18.806 | 17.097 | 15.388 | 11.969 | 8.550 |
| | 1.40 | 9.987 | 8.656 | 7.324 | 6.658 | 6.004 | 4.661 | 3.330 |
| $\alpha$ | 1.50 | 4.034 | 3.479 | 2.944 | 2.676 | 2.409 | 1.873 | 1.338 |
| | 1.60 | 1.606 | 1.392 | 1.178 | 1.071 | 0.963 | 0.750 | 0.536 |
| | 1.70 | 0.616 | 0.534 | 0.450 | 0.408 | 0.364 | 0.272 | 0.206 |
| | 1.80 | 0.205 | 0.174 | 0.157 | 0.143 | 0.129 | 0.100 | 0.071 |
| | 1.90 | 0.056 | 0.049 | 0.041 | 0.037 | 0.034 | 0.026 | 0.018 |

TABLE 2. The relative error $\delta_{\varepsilon,K}(X)/CVaR_{\varepsilon}(X)$ in percent, where $\varepsilon = 0.01$ and $K = 500$ for different $(\alpha, \beta)$ pairs

| | | β | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | −0.5 | −0.3 | −0.1 | 0 | 0.1 | 0.3 | 0.5 |
| | 1.20 | 53.886 | 52.550 | 51.008 | 50.137 | 49.179 | 46.920 | 43.932 |
| | 1.30 | 35.983 | 34.757 | 33.365 | 32.588 | 31.743 | 29.782 | 27.267 |
| | 1.40 | 22.843 | 21.883 | 20.804 | 20.208 | 19.601 | 18.085 | 16.229 |
| α | 1.50 | 13.894 | 13.148 | 12.392 | 11.976 | 11.530 | 10.517 | 9.258 |
| | 1.60 | 7.959 | 7.517 | 7.027 | 6.759 | 6.473 | 5.833 | 5.048 |
| | 1.70 | 4.304 | 4.035 | 3.732 | 3.560 | 3.365 | 2.865 | 2.567 |
| | 1.80 | 2.032 | 1.859 | 1.817 | 1.728 | 1.633 | 1.421 | −1.163 |
| | 1.90 | 0.842 | 0.771 | 0.693 | 0.650 | 0.605 | 0.504 | 0.388 |

TABLE 3. Tabulated values of $CVaR_{0.01}(X)$ for different $(\alpha, \beta)$ pairs

| | | β | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | −0.2 | −0.1 | −0.05 | 0 | 0.05 | 0.1 | 0.2 |
| | 1.04 | 841.464 | 774.929 | 741.581 | 708.177 | 674.712 | 641.185 | 573.924 |
| | 1.07 | 441.074 | 407.107 | 390.055 | 372.954 | 355.802 | 338.595 | 304.004 |
| | 1.10 | 284.534 | 263.190 | 252.457 | 241.686 | 230.862 | 219.993 | 198.100 |
| | 1.13 | 202.553 | 187.749 | 180.293 | 172.799 | 165.266 | 157.689 | 142.397 |
| | 1.16 | 152.885 | 141.993 | 136.500 | 130.973 | 125.411 | 119.810 | 108.484 |
| | 1.19 | 120.012 | 111.676 | 107.466 | 103.225 | 98.952 | 94.646 | 85.922 |
| | 1.22 | 96.9163 | 90.3500 | 87.0290 | 83.6808 | 80.3040 | 76.8966 | 69.9821 |
| | 1.25 | 79.9703 | 74.6833 | 72.0058 | 69.3040 | 66.5763 | 63.8211 | 58.2210 |
| | 1.28 | 67.1182 | 62.7867 | 60.5903 | 58.3720 | 56.1304 | 53.8641 | 49.2503 |
| | 1.31 | 57.1118 | 53.5126 | 51.6854 | 49.8384 | 47.9704 | 46.0800 | 42.2259 |
| | 1.34 | 49.1521 | 46.1263 | 44.5884 | 43.0327 | 41.4579 | 39.8629 | 36.6063 |
| | 1.37 | 42.7058 | 40.1369 | 38.8299 | 37.5067 | 36.1662 | 34.8074 | 32.0294 |
| | 1.40 | 37.4043 | 35.2052 | 34.0853 | 32.9506 | 31.8002 | 30.6332 | 28.2444 |
| | 1.43 | 32.9859 | 31.0900 | 30.1236 | 29.1438 | 28.1497 | 27.1406 | 25.0724 |
| | 1.46 | 29.2596 | 27.6154 | 26.7765 | 25.9254 | 25.0614 | 24.1837 | 22.3828 |
| | 1.49 | 26.0836 | 24.6505 | 23.9186 | 23.1757 | 22.4211 | 21.6539 | 20.0784 |
| α | 1.52 | 23.3504 | 22.0960 | 21.4549 | 20.8038 | 20.1420 | 19.4688 | 18.0850 |
| | 1.55 | 20.9773 | 19.8756 | 19.3122 | 18.7396 | 18.1574 | 17.5648 | 16.3457 |
| | 1.58 | 18.8996 | 17.9296 | 17.4331 | 16.9284 | 16.4149 | 15.8921 | 14.8156 |
| | 1.61 | 17.0661 | 16.2104 | 15.7723 | 15.3266 | 14.8731 | 14.4111 | 13.4592 |
| | 1.64 | 15.4356 | 14.6802 | 14.2932 | 13.8994 | 13.4986 | 13.0901 | 12.2481 |
| | 1.67 | 13.9748 | 13.3081 | 12.9663 | 12.6186 | 12.2644 | 11.9035 | 11.1593 |
| | 1.70 | 12.6564 | 12.0686 | 11.7673 | 11.4607 | 11.1484 | 10.8301 | 10.1739 |
| | 1.73 | 11.4573 | 10.9409 | 10.6761 | 10.4067 | 10.1323 | 9.8527 | 9.2765 |
| | 1.76 | 10.3586 | 9.9072 | 9.6758 | 9.4405 | 9.2009 | 8.9569 | 8.4548 |
| | 1.79 | 9.3440 | 8.9529 | 8.7526 | 8.5490 | 8.3420 | 8.1313 | 7.6985 |
| | 1.82 | 8.3999 | 8.0658 | 7.8949 | 7.7215 | 7.5454 | 7.3665 | 7.0000 |
| | 1.85 | 7.5152 | 7.2361 | 7.0938 | 6.9495 | 6.8034 | 6.6552 | 6.3528 |
| | 1.88 | 6.6809 | 6.4562 | 6.3420 | 6.2266 | 6.1098 | 5.9919 | 5.7518 |
| | 1.91 | 5.8904 | 5.7204 | 5.6344 | 5.5477 | 5.4602 | 5.3719 | 5.1931 |
| | 1.94 | 5.1388 | 5.0244 | 4.9667 | 4.9087 | 4.8503 | 4.7916 | 4.6732 |
| | 1.97 | 4.4223 | 4.3646 | 4.3356 | 4.3065 | 4.2772 | 4.2479 | 4.1891 |
| | 2.00 | 3.7692 | 3.7692 | 3.7692 | 3.7692 | 3.7692 | 3.7692 | 3.7692 |

TABLE 4. Tabulated values of $CVaR_{0.05}(X)$ for different $(\alpha, \beta)$ pairs

| | | $\beta$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | −0.2 | −0.1 | −0.05 | 0 | 0.05 | 0.1 | 0.2 |
| | 1.04 | 176.684 | 163.708 | 157.204 | 150.689 | 144.162 | 137.624 | 124.508 |
| | 1.07 | 96.7741 | 89.8624 | 86.3927 | 82.9133 | 79.4238 | 75.9235 | 68.8887 |
| | 1.10 | 65.0979 | 60.5758 | 58.3024 | 56.0203 | 53.7289 | 51.4279 | 46.7952 |
| | 1.13 | 48.2309 | 44.9718 | 43.3310 | 41.6823 | 40.0253 | 38.3595 | 34.9999 |
| | 1.16 | 37.8213 | 35.3350 | 34.0816 | 32.8210 | 31.5528 | 30.2766 | 27.6987 |
| | 1.19 | 30.7951 | 28.8254 | 27.8313 | 26.8306 | 25.8229 | 24.8079 | 22.7547 |
| | 1.22 | 25.7570 | 24.1541 | 23.3442 | 22.5282 | 21.7059 | 20.8769 | 19.1976 |
| | 1.25 | 21.9829 | 20.6520 | 19.9787 | 19.3000 | 18.6154 | 17.9247 | 16.5238 |
| | 1.28 | 19.0602 | 17.9377 | 17.3693 | 16.7958 | 16.2170 | 15.6327 | 14.4461 |
| | 1.31 | 16.7367 | 15.7781 | 15.2923 | 14.8018 | 14.3065 | 13.8060 | 12.7888 |
| | 1.34 | 14.8500 | 14.0230 | 13.6036 | 13.1799 | 12.7519 | 12.3191 | 11.4387 |
| | 1.37 | 13.2905 | 12.5714 | 12.2063 | 11.8375 | 11.4645 | 11.0873 | 10.3194 |
| | 1.40 | 11.9822 | 11.3526 | 11.0328 | 10.7095 | 10.3826 | 10.0518 | 9.3779 |
| | 1.43 | 10.8704 | 10.3162 | 10.0346 | 9.7498 | 9.4617 | 9.1701 | 8.5759 |
| | 1.46 | 9.9149 | 9.4251 | 9.1761 | 8.9241 | 8.6692 | 8.4112 | 7.8853 |
| | 1.49 | 9.0856 | 8.6512 | 8.4304 | 8.2069 | 7.9807 | 7.7518 | 7.2851 |
| $\alpha$ | 1.52 | 8.3595 | 7.9735 | 7.7771 | 7.5785 | 7.3775 | 7.1740 | 6.7592 |
| | 1.55 | 7.7187 | 7.3752 | 7.2005 | 7.0237 | 6.8449 | 6.6638 | 6.2950 |
| | 1.58 | 7.1492 | 6.8434 | 6.6879 | 6.5306 | 6.3715 | 6.2106 | 5.8827 |
| | 1.61 | 6.6397 | 6.3677 | 6.2295 | 6.0897 | 5.9484 | 5.8054 | 5.5144 |
| | 1.64 | 6.1812 | 5.9398 | 5.8172 | 5.6932 | 5.5679 | 5.4413 | 5.1837 |
| | 1.67 | 5.7664 | 5.5528 | 5.4445 | 5.3350 | 5.2243 | 5.1125 | 4.8852 |
| | 1.70 | 5.3893 | 5.2013 | 5.1060 | 5.0097 | 4.9126 | 4.8142 | 4.6155 |
| | 1.73 | 5.0448 | 4.8802 | 4.7969 | 4.7133 | 4.6286 | 4.5432 | 4.3700 |
| | 1.76 | 4.7288 | 4.5867 | 4.5147 | 4.4422 | 4.3691 | 4.2954 | 4.1462 |
| | 1.79 | 4.4379 | 4.3165 | 4.2551 | 4.1933 | 4.1310 | 4.0683 | 3.9416 |
| | 1.82 | 4.1691 | 4.0672 | 4.0158 | 3.9641 | 3.9120 | 3.8597 | 3.7540 |
| | 1.85 | 3.9198 | 3.8365 | 3.7946 | 3.7516 | 3.7101 | 3.6675 | 3.5816 |
| | 1.88 | 3.6878 | 3.6224 | 3.5895 | 3.5565 | 3.5233 | 3.4899 | 3.4229 |
| | 1.91 | 3.4715 | 3.4232 | 3.3989 | 3.3746 | 3.3502 | 3.3257 | 3.2764 |
| | 1.94 | 3.2690 | 3.2372 | 3.2213 | 3.2053 | 3.1893 | 3.1733 | 3.1411 |
| | 1.97 | 3.0789 | 3.0633 | 3.0554 | 3.0475 | 3.0397 | 3.0318 | 3.0160 |
| | 2.00 | 2.9171 | 2.9171 | 2.9171 | 2.9171 | 2.9171 | 2.9171 | 2.9171 |

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