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CAUCHY TRANSFORMS OF MEASURES VIEWED AS SOME FUNCTIONALS OF FOURIER TRANSFORMS*

BY

ZBIGNIEW J. JUREK (WROCŁAW)

In memory of Kazimierz Urbanik

Abstract. The Cauchy transform of a positive measure plays an important role in complex analysis and more recently in so-called free probability. We show here that the Cauchy transform restricted to the imaginary axis can be viewed as the Fourier transform of some corresponding measures. Thus this allows the full use of that classical tool. Furthermore, we relate restricted Cauchy transforms to classical compound Poisson measures, exponential mixtures, geometric infinite divisibility and free-infinite divisibility. Finally, we illustrate our approach with some examples.

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INTRODUCTION

Fourier transforms are very well established tools in analysis, differential equations or harmonic analysis. On the other hand, Cauchy transforms are used in complex analysis, in the approximation problem or in the moment problem and, relatively more recently, in so-called free probability. In Barndorff-Nielsen and Thorbjornsen (2002) and Jurek (2004) it was shown that Voiculescu transforms of free-infinitely divisible measures are closely related to Fourier transforms of some (classical) infinitely divisible measures expressed by random integrals (integration with respect to a Lévy process). That fact suggested that there might be an intrinsic relation between those two transforms, Fourier's

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and Cauchy's. This is what we present in this note. One may expect that these relations will shed more light on the fact that there are so many parallel results in classical and free probability theory.

1. THE CAUCHY TRANSFORM AS SOME FUNCTIONALS OF THE FOURIER TRANSFORMS

For a finite Borel measure m on the real line $\boldsymbol{R}$, let us recall that its *Cauchy transform* G_m is defined by

$$G_m(z) := \int_{\boldsymbol{R}} \frac{1}{z-x} m(dx) \quad \text{for } z \in \boldsymbol{C} \setminus \boldsymbol{R} = \{z \in \boldsymbol{C}\colon \Im z \neq 0\}. \tag{1}$$

Since $\overline{G_m(z)} = G_m(\bar{z})$, we may consider Cauchy transforms on half-planes either on $\boldsymbol{C}^+$ or on $\boldsymbol{C}^-$. This transform $G_m(z)$ is the key notion in so-called *free-probability* but in this note we restrict our investigations only to the Cauchy transforms and some functionals of them. From Akhiezer (1965), p. 125, or Lang (1975), p. 380, we have

$$m([a,b]) = -\lim_{y\to 0} \frac{1}{\pi} \int_a^b \Im G_m(x+iy)\,dx \quad \text{provided } m(\{a,b\}) = 0.$$

Thus G_m uniquely determines m but for that one needs to know Cauchy transform in strips $\{x+iy\colon x \in \boldsymbol{R}, 0 < y < \varepsilon\}$ for some $\varepsilon > 0$.

In some instances, as is the case here, we know (define) G_m *only* on the imaginary axis. Then it will be denoted by g_m and referred to as the *restricted Cauchy transform*, i.e., $g_m(it) := G_m(it)$, $t \in \boldsymbol{R}\setminus\{0\}$. Explicitly,

$$g_m(it) = -it \int_{\boldsymbol{R}} \frac{1}{t^2+x^2} m(dx) - \int_{\boldsymbol{R}} \frac{x}{t^2+x^2} m(dx) \quad \text{for } t \neq 0. \tag{2}$$

Of course, we also have $\overline{g_m(it)} = g_m(-it)$.

One of the main results here is

THEOREM 1. *The restricted Cauchy transform $g_m(it)$, $t \neq 0$, uniquely determines the measure m.*

Besides that we will relate restricted Cauchy transforms to some functionals of the Fourier transforms, to laws of product of independent random variables, to geometric infinite divisibility and some random integrals.

In the sequel, for a finite measure m its *Fourier transform* (in probability theory called its *characteristic function*, in short: char. f.), denoted by $\hat{m}$, is given as follows:

$$\hat{m}(t) := \int_{\boldsymbol{R}} e^{itx} m(dx) \quad \text{for } t \in \boldsymbol{R}. \tag{3}$$

1.1. Mixtures of measures and the restricted Cauchy transform. Let e denote an *exponential random variable* or an *exponential distribution*, i.e., it has probability density function $e^{-x}1_{(0,\infty)}$. Its Fourier transform is equal to $e^{\wedge}(t) = (1-it)^{-1}$. Let

$$m^{\langle e\rangle}(A) := \int_0^\infty m(s^{-1}A)e^{-s}ds \quad \text{for Borel subsets } A \subset \boldsymbol{R} \tag{4}$$

be the exponential mixture of a measure m. Note that if μ is the probability distribution of a random variable X and is independent of the exponential random variable e, then $\mu^{\langle e\rangle}$ is the probability distribution of $e \cdot X$. (In Jurek (1990), mixtures $m^{\langle \lambda\rangle}$ for σ-finite measures m on a Banach space and σ-finite measures λ on $(0, \infty)$ were studied.)

Proof of Theorem 1. Step 1. Simple calculations give

$$(m^{\langle e\rangle})^{\wedge}(t) = \int_{\boldsymbol{R}} \frac{1}{1-itx} m(dx) = \int_0^\infty \hat{m}(st)e^{-s}ds \quad \text{for } t \in \boldsymbol{R}. \tag{5}$$

Since from the last equality we can get a Laplace transform of the function $\hat{m}(\cdot)$, we conclude that

$$m_1^{\langle e\rangle} = m_2^{\langle e\rangle} \quad \text{implies} \quad m_1 = m_2. \tag{6}$$

Step 2. Recall that $e(m) := e^{-m(\boldsymbol{R})}\sum_{k=0}^{\infty} m^{*k}/k!$ is said to be *compound Poisson distribution* (it corresponds to Poisson number of summands) and

$$\big(e(m)\big)^{\wedge}(t) = \exp\big(\hat{m}(t) - m(\boldsymbol{R})\big) = \exp \int_{\boldsymbol{R}} (e^{itx}-1)\, m(dx) \quad \text{for } t \in \boldsymbol{R}, \tag{7}$$

and, of course, $e(m)$ uniquely defines m. Furthermore, from (5) we get

$$\big(e(m^{\langle e\rangle})\big)^{\wedge}(t) = \exp \int_{\boldsymbol{R}} \left(\frac{1}{1-itx} - 1\right) m(dx). \tag{8}$$

Step 3. Finally, let us introduce new functionals of measures:

$$h_m(t) := \frac{1}{it} g_m\left(\frac{1}{it}\right), \; t \neq 0, \quad h_m(0) := \lim_{t\to 0} h_m(t),$$

i.e.,

$$g_m(is) = -\frac{i}{s} h_m\left(-\frac{1}{s}\right), \quad s \neq 0. \tag{9}$$

Thus, using (1) we have explicitly

$$h_m(t) = \int_{\boldsymbol{R}} \frac{1}{1-itx} m(dx), \quad h_m(0) = m(\boldsymbol{R}). \tag{10}$$

Combining (8) and (10) we obtain

$$\exp[h_m(t)-m(\boldsymbol{R})] = \left(e(m^{\langle e\rangle})\right)^{\wedge}(t). \tag{11}$$

Step 4. From the above and (6) we infer that h_m uniquely determines the measure m, which in turn by (9) means that $g_m(it)$, $t \neq 0$, uniquely identifies the measure m. This completes the proof of Theorem 1.

Here are some consequences of the above proof, rather than of the theorem itself, that relate the restricted Cauchy transform to some characteristic functions.

COROLLARY 1. (a) *The functions* $\left(m(\boldsymbol{R})\right)^{-1} h_m(t)$, $t \in \boldsymbol{R}$, *are Fourier transforms of random variables* $e \cdot X$, *where* e *and* X *are independent random variables with the exponential and* $m(\cdot)/m(\boldsymbol{R})$ *probability distributions, respectively.*

(b) *Let* e° *be the symmetrization of the standard exponential random variable* e *and independent of a random variable* X *whose probability distribution is* $m(\cdot)/m(\boldsymbol{R})$. *Then*

$$g_m(it) = -it^{-1} m(\boldsymbol{R})\left(\mathscr{L}(e^{\circ}\cdot X)\right)^{\wedge}(t^{-1}) - \int_{\boldsymbol{R}} \frac{x}{t^2+x^2}\, m(dx), \qquad t \neq 0.$$

Part (a) follows from (10) and the fact that for independent random variables we have

$$\left(\mathscr{L}(e\cdot X)\right)^{\wedge}(t) = E\left[(\mathscr{L}(e))^{\wedge}(tX)\right] = E\left[\frac{1}{1-itX}\right] = \int_{\boldsymbol{R}} \frac{1}{1-itx}\frac{m(dx)}{m(\boldsymbol{R})},$$

where $\mathscr{L}(Z)$ denotes the probability distribution for a random variable Z. Similarly we get part (b) using formula (2).

Finally we have the following algebraic relations between Cauchy and some Fourier transforms, which was suggested by a Boolean convolution introduced by Speicher and Woroudi (1997) and the mixtures given in (4).

THEOREM 2. *For probability Borel measures* μ *and* ν *and their restricted Cauchy transforms* g_μ *and* g_ν *there exists a unique probability measure* ϱ *such that its restricted Cauchy transform is given by*

$$g_\varrho(it) = \frac{g_\mu(it)\cdot g_\nu(it)}{g_\mu(it)+g_\nu(it)-itg_\mu(it)\cdot g_\nu(it)} \quad \textit{for } t \neq 0. \tag{12}$$

If e *denotes the standard exponential probability measure, then the above for probability Borel measures* μ_1 *and* μ_2 *means that*

$$\frac{(\mu_1^{\langle e\rangle})^{\wedge}(t)\cdot(\mu_2^{\langle e\rangle})^{\wedge}(t)}{(\mu_1^{\langle e\rangle})^{\wedge}(t)+(\mu_2^{\langle e\rangle})^{\wedge}(t)-(\mu_1^{\langle e\rangle})^{\wedge}(t)\cdot(\mu_2^{\langle e\rangle})^{\wedge}(t)} = (\varrho^{\langle e\rangle})^{\wedge}(t) \quad \textit{for } t \in \boldsymbol{R}.$$

Equivalently, we have

$$\left[(\mu_1^{\langle e\rangle})^{\wedge}+(\mu_2^{\langle e\rangle})^{\wedge}\right]\cdot(\varrho^{\langle e\rangle})^{\wedge} = (\mu_1^{\langle e\rangle})^{\wedge}\cdot(\mu_2^{\langle e\rangle})^{\wedge}\left(1+(\varrho^{\langle e\rangle})^{\wedge}\right)$$

or

$$\frac{(\mu_1^{\langle e\rangle})^\wedge+(\mu_2^{\langle e\rangle})^\wedge}{(\mu_1^{\langle e\rangle})^\wedge\cdot(\mu_2^{\langle e\rangle})^\wedge}=1+\frac{1}{(\varrho^{\langle e\rangle})^\wedge}.$$

Proof. Step 1. For a measure μ and its Cauchy transform G_μ let us define the transform

$$E_\mu(z):=z-\frac{1}{G_\mu(z)},\quad \text{i.e. } G_\mu(z)=\frac{1}{z-E_\mu(z)}, \tag{13}$$

which is an analytic function that maps $\boldsymbol{C}^+$ to $\boldsymbol{C}^-\cup\boldsymbol{R}$ and $E_\mu(z)/z\to 0$ as $z\to\infty$ non-tangentially (i.e. such that the ratio $\Re z/\Im z$ is bounded). Conversely, if $E\colon \boldsymbol{C}^+\to\boldsymbol{C}^-\cup\boldsymbol{R}$ is an analytic function so that $E(z)/z\to 0$ as $z\to\infty$ non-tangentially, then there exists a measure μ such that $E=E_\mu$. That fact led Speicher and Woroudi (1997) to the following notion of so-called *Boolean convolution* $\oplus$: for measures μ and ν there exists a unique measure $\varrho\equiv\mu\oplus\nu$ such that

$$E_\varrho(z)=E_{\mu\oplus\nu}(z)=E_\mu(z)+E_\nu(z)\quad \text{for } z\in\boldsymbol{C}^+. \tag{14}$$

Step 2. Combining (13) and (14) we get

$$G_\varrho(z)=\frac{G_\mu(z)\cdot G_\nu(z)}{G_\mu(z)+G_\nu(z)-zG_\mu(z)\cdot G_\nu(z)}$$

from which we get equality (12). Using the characteristic functions h_μ from Corollary 1 (a), we arrive at

$$h_\varrho(t)=\frac{1}{it}G_\varrho\left(\frac{1}{it}\right)=\frac{h_\mu(t)\cdot h_\nu(t)}{h_\mu(t)+h_\nu(t)-h_\mu(t)\cdot h_\nu(t)},$$

which concludes the proof of the second part because $h_\mu(t)=(\mu^{\langle e\rangle})^\wedge(t)$ by Corollary 1.

1.1.1. Remark. The above proof is based on structural characterizations of some analytic functions with a specific behavior at infinity. An open question is to find a more direct, more probabilistic argument for the above factorizations.

1.1.2. Remark. From (12) we note that for Dirac measures δ_a and δ_b ($a, b\in\boldsymbol{R}$) we have $\delta_a\oplus\delta_b=\delta_{a+b}$ $(=\delta_a*\delta_b)$. Similarly, if γ_a denotes the Cauchy distribution with a parameter $a>0$ (cf. Section 3.2 below), then using (29) we get $\gamma_a\oplus\gamma_b=\gamma_{a+b}$ $(=\gamma_a*\gamma_b)$.

1.2. Random integrals and the restricted Cauchy transform. In the past it was shown that many classes of probability distributions can be identified as classes of distributions of some random integrals of the form

$$\int_A f(t)\,dY_\nu(r(t)),\qquad A\subset[0,\infty),\ Y_\nu \text{ is a Lévy process and } \mathscr{L}(Y_\nu(1))=\nu,$$

where f and r are deterministic functions; see for instance Jurek (1985), (1988), (2004) or Jurek and Vervaat (1983) or Iksanov et al. (2004) [see www.math.uni.wroc.pl/~zjjurek/conjecture.]

For purposes of this note let us introduce, after Jurek (2004), a *random integral* and its corresponding *random integral mapping* $\mathcal{K}$ as follows:

$$(15) \qquad \mathcal{K}(\nu) := \mathcal{L}\big(\int_0^\infty t\,dY_\nu(1-e^{-t})\big) \in ID,$$

where Y_ν is a Lévy process with càdlàg paths, the integral is defined (as simple as possible) by formal integration by parts, and $\mathcal{L}(X)$ denotes the probability distribution of a random variable X. In terms of Fourier transforms, (15) means that

$$(16) \qquad (\mathcal{K}(\nu))^\wedge(t) := \exp\Big[\int_0^\infty \log\big(\mathcal{L}(Y_\nu(1))^\wedge(st)\,e^{-s}ds\big)\Big] \quad \text{for } t \in \mathbb{R}.$$

From (16), using the Laplace transform argument (the same way as for (6)), we infer that $\mathcal{K}$ is a one-to-one mapping; for details see Jurek (2004), Proposition 3.

COROLLARY 2. (a) *For a finite measure m and its restricted Cauchy transform g_m we have*

$$g_m(it) = -it^{-1}\big(m(\mathbb{R}) + \log\big(\mathcal{K}(e(m))\big)^\wedge(-t^{-1})\big), \qquad t \neq 0.$$

(b) *For a finite measure m we have*

$$\mathcal{L}\big(\int_0^\infty t\,dY_{e(m)}(1-e^{-t})\big) \equiv \mathcal{K}\big(e(m)\big) = e\big(m^{\langle e\rangle}\big).$$

This means that the random integration with respect to a compound Poisson process $Y_{e(m)}(t)$, $t \geqslant 0$, is the same as the exponential mixing of an exponent measure m in a compound Poisson measure $e(m)$.

Proof. Putting $e(m)$ for ν into (16) and using (7) we get

$$\log\big(\mathcal{K}(e(m))\big)^\wedge(t) = \int_0^\infty\int_{\mathbb{R}} (e^{itsx}-1)\,m(dx)\,e^{-s}ds = \int_{\mathbb{R}}\Big(\frac{1}{1-itx}-1\Big)m(dx),$$

that is, $\log\big(\mathcal{K}(e(m))\big)^\wedge(t) = h_m(t) - m(\mathbb{R})$ and (11) give part (b). Finally, (9) implies equality in (a).

1.2.1. Remark. From part (b) we also infer the property (6) because $\mathcal{K}$ is a one-to-one mapping.

1.3. Geometric infinite divisibility and the restricted Cauchy transform. After Klebanov et al. (1984) (cf. also Ramachandran (1997)), we say that a random variable X has a *geometric infinitely divisible distribution* if

$$(17) \qquad \forall (0 < p < 1)\ \exists (\text{rv's } G_p, X_1^{(p)}, X_2^{(p)}, X_3^{(p)}, \ldots)\ X \overset{d}{=} \sum_{j=1}^{G_p} X_j^{(p)},$$

where $X_j^{(p)}$, $j = 1, 2, \ldots$, are independent and identically distributed and G_p is independent of them and has the geometric distribution with parameter p — the moment of the first success in the Bernoulli trials, i.e., $P(G_p = j) = (1-p)^{j-1}p$, $j = 1, 2, 3, \ldots$ By *GID* we denote the class of all geometric infinitely divisible distributions (random variables or characteristic functions).

From (17) one infers that for any $c > 0$ functions

$$\text{(18)} \qquad \boldsymbol{R} \ni t \to \frac{1}{1+c\big(1-\phi(t)\big)} \in GID \quad \text{provided } \phi \text{ is a char. f.}$$

Moreover, characteristic functions of the form (18) play the role of the compound Poisson measures $e(m)$ for the class *GID*.

COROLLARY 3. *For $c > 0$ and a finite measure m functions*

$$k_{c,m}(t) := \big(1 + c\big(m(\boldsymbol{R}) - h_m(t)\big)\big)^{-1}, \qquad t \in \boldsymbol{R},$$

are Fourier transforms of geometric infinitely divisible distributions.

In other words, for $c > 0$ and the restricted Cauchy transform g_m there exists a geometric infinitely divisible characteristic function $k_{c,m}$ such that

$$g_m(is) = i\left[\frac{1}{csk_{c,m}(-s^{-1})} - \frac{m(\boldsymbol{R}) + c^{-1}}{s}\right], \qquad s \neq 0.$$

Proof. Since $k_{c,m}$ is of the form (18) and, by Corollary 1 (a), $\big(m(\boldsymbol{R})\big)^{-1} h_m(\cdot)$ is a characteristic function, it follows that $k_{c,m} \in GID$. Moreover, from (9) we get the second equality, i.e., the restricted Cauchy transform g_m in terms of *GID* Fourier transform.

1.4. Free-infinite divisibility and geometric infinite divisibility. For a probability measure μ one defines $F_\mu(z) := 1/G_\mu(z)$, where G_μ is the Cauchy transform from (1). Furthermore, the Voiculescu transform V_μ is defined as $V_\mu(z) := F_\mu^{-1}(z) - z$, where one proves that the inverse function F_μ^{-1} exists in some Stolz angles; for more details cf. Bercovici and Voiculescu (1993), Corollary 5.5. A measure μ is said to be *free-infinitely divisible* if for each $n \geqslant 2$ there exists a probability measure μ_n such that $V_\mu(z) = V_{\mu_n}(z) + \ldots + V_{\mu_n}(z)$ (n times). From Barndorff-Nielsen and Thorbjornsen (2002), Proposition 5.2, we have the following free-probability analog of the Lévy–Khintchine formula:

μ is free-infinitely divisible iff its Voiculescu transform V_μ is such that

$$\text{(19)} \qquad zV_\mu\left(\frac{1}{z}\right) = iaz - \sigma^2 z^2 + \int_{\boldsymbol{R}} \left[\frac{1}{1-zx} - 1 - zx 1_{(|x| \leqslant 1)}(x)\right] M(dx), \qquad z \in \boldsymbol{C}^-,$$

with the three parameters a, σ^2 and a measure M the same as in the classical Lévy–Khintchine formula.

COROLLARY 4. *Suppose that* $c > 0$ *and* V_μ *is the Voiculescu transform of a free-infinitely divisible probability measure* μ. *Then functions*

$$w_{c,\mu}(t) := \big(1 - c\,(it)\,V_\mu\big((it)^{-1}\big)\big)^{-1} \quad \text{are GID char. f.} \tag{20}$$

More explicitly,

$$w_{c,\mu}(t) = \frac{1}{1 - c\left[iat - \sigma^2 t^2 + \int_{\mathbf{R}\setminus\{0\}} \big((1-itx)^{-1} - 1 - itx\,1_{\{x:\,\|x\|\leqslant 1\}}(x)\big)\,M(dx)\right]}. \tag{21}$$

Here $a \in \mathbf{R}$, $\sigma^2 \geqslant 0$ *and* M *is a* σ*-finite measure that integrates* $\min(1, x^2)$ *over the real line, and this triplet is uniquely associated with the measure* μ.

Proof. This is so because $\lim_{t\to 0} w_{c,\mu}(t) = 1$ by (19) (note the integrability condition for M) and

$$\exp\left[1 - \frac{1}{w_{c,\mu}(t)}\right] = \exp\left[it\,V_\mu\big((it)^{-1}\big)\right] \in ID \text{ (infinite divisible char. f.)}$$

by Jurek (2004), Corollaries 5 and 6. (More precisely, these are characteristics of integral (15); class $\mathscr{E} \subset ID$.) Consequently, by Ramachandran (1997) we conclude that $w_{c,\mu} \in GID$. The remaining part follows from Jurek (2004), Corollary 6, or Barndorff-Nielsen and Thorbjornsen (2002), Proposition 5.2.

2. REMARKS ON THE FUNCTIONS F_m

As we have seen in Section 1.4, in the free-probability theory besides the Cauchy transform G_m an important role is played by a companion function $F_m(z) := 1/G_m(z)$. If one would like to consider the Voiculescu transform V_μ *only* on the imaginary axis, then the invertibility of $F_m(it)$ must be settled. Here are preliminary results in that direction.

Let $f_m(it) := F_m(it)$, $t \neq 0$, be the companion function of the restricted Cauchy transform.

PROPOSITION 1. (a) *For each finite and non-zero Borel measure* m *on* $\mathbf{R}$ *its restricted Cauchy transform* $g_m(i\,\cdot)$ *never vanishes on* $\mathbf{R}\setminus\{0\}$, *the function* $t \to t^{-1}\Im g_m(it)$ *is one-to-one on the half-line* $(0, \infty)$ *and* $\lim_{t\to+\infty}(it)\,g_m(it) = m(\mathbf{R})$. *The analogous result holds for the negative half-line.*

(b) *The imaginary part of the function* $\mathbf{R}^+ \ni t \to f_m(it) := 1/g_m(it) \in \mathbf{C}^+$ *satisfies the inequality* $0 < m(\mathbf{R})\,t \leqslant \Im f_m(it)$ *and* $\lim_{t\to+\infty}(it)^{-1} f_m(it) = \big(m(\mathbf{R})\big)^{-1}$. *Furthermore, if* m *is a measure satisfying these conditions such that* $\Re g_m(it) = 0$, *then there exists a constant* $0 \leqslant d_m < \infty$ *such that* $i(d_m, \infty) \subseteq f_m(i\mathbf{R}^+) \subseteq \mathbf{C}^+$.

Proof. Since, by (1) and (2),

$$\Im g_m(it) = -t\int_{\mathbf{R}} \frac{1}{t^2+x^2}\, m(dx), \qquad \Re g_m(it) = -\int_{\mathbf{R}} \frac{x}{t^2+x^2}\, m(dx), \qquad t \neq 0,$$

for $s \geqslant t > 0$ the equality $t^{-1} g_m(it) = s^{-1} g_m(is)$ implies that

$$t^{-1}\Im g_m(it) = s^{-1}\Im g_m(is),$$

and thus

$$\int_{\mathbf{R}} \frac{s^2-t^2}{(t^2+x^2)(s^2+x^2)}\, m(dx) = 0.$$

Hence, for $m \neq 0$, the above implies that $s = t$. Consequently, the function $0 < t \to t^{-1}\Im g_m(it)$ is one-to-one, which completes the proof of part (a).

For part (b) let us introduce the notation:

$$a_t := \int_{\mathbf{R}} \frac{1}{t^2+x^2}\, m(dx) > 0, \qquad b_t := \int_{\mathbf{R}} \frac{x}{t^2+x^2}\, m(dx) \in \mathbf{R} \qquad \text{for } t > 0.$$

Thus $g_m(it) = -ita_t - b_t$ and, consequently,

$$f_m(it) = \frac{1}{g_m(it)} = it\frac{a_t}{t^2 a_t^2 + b_t^2} - \frac{b_t}{t^2 a_t^2 + b_t^2} \in \mathbf{C}^+ \qquad \text{whenever } t > 0.$$

Assuming that m is a probability measure and using the Schwarz inequality we obtain $(ta_t)^2 + b_t^2 \leqslant a_t$, which with the above definition of f_m gives the inequality for the imaginary part of $f_m(it)$. A similar argument holds for an arbitrary finite measure m.

Finally, for $s > 0$, in order to have $is = f_m(it)$ for some $t > 0$ one needs

$$b_t = 0 \quad \text{and} \quad s = 1/(ta_t), \text{ i.e., } s^{-1} = t\int_{\mathbf{R}} \frac{1}{t^2+x^2}\, m(dx).$$

But the function $t \to ta_t$ is continuous and $\lim_{t\to\infty} ta_t = 0$. Putting

$$1/d_m := \sup\{ta_t\colon\ 0 < t < \infty\}$$

we see that the equation above holds for $s > d_m$.

3. EXAMPLES

We will illustrate our results and technics by some examples. For the computations below, from the definition (1) and formula (2), one needs to keep in mind that

$$\text{if } \Im z > 0 \text{ then } \Im(G_m(z)) < 0 \quad \text{and} \quad \text{if } \Im z < 0 \text{ then } \Im(G_m(z)) > 0.$$

Consequently, for the restricted Cauchy transform $g_m(it)$ we get

$$\Im\big(g_m(it)\big) < 0 \text{ for } t > 0 \quad \text{and} \quad \Im\big(g_m(it)\big) > 0 \text{ for } t < 0.$$

3.1. Semicircle law. From Voiculescu (1999), p. 299, let us consider a probability measure μ_α, $\alpha > 0$, such that its Cauchy transform is equal to

$$G_{\mu_\alpha}(z) = \frac{z + \sqrt{z^2 - \alpha^2}}{\alpha^2/2}, \tag{22}$$

and assume we do not know the measure μ_α. Hence the restricted Cauchy transform is equal to

$$g_{\mu_\alpha}(it) = G_{\mu_\alpha}(it) = i2\alpha^{-2}\big(t - \operatorname{sign}(t)\sqrt{t^2 + \alpha^2}\big), \qquad t \neq 0.$$

Thus, by (9) and Corollary 1 (a),

$$h_{\mu_\alpha}(t) = \frac{1}{it} g_{\mu_\alpha}\left(\frac{1}{it}\right) = \frac{2}{1 + \sqrt{1 + \alpha^2 t^2}}, \qquad t \in \boldsymbol{R}, \tag{23}$$

is a Fourier transform of the random variable $e \cdot X_\alpha$, where these two are independent variables and μ_α is the probability distribution of X_α. Hence

$$E[\exp(ite \cdot X_\alpha)] = \int_0^\infty \hat{\mu}_\alpha(ts) e^{-s}\, ds = \frac{2}{1 + \sqrt{1 + \alpha^2 t^2}}, \qquad t \in \boldsymbol{R}. \tag{24}$$

Substituting $1/u$ for t $(u > 0)$ and changing variable one gets

$$\int_0^\infty \hat{\mu}_\alpha(x) e^{-sx}\, dx = \frac{2}{\alpha} \frac{\alpha}{s + \sqrt{s^2 + \alpha^2}} = \frac{2}{\alpha} \int_0^\infty \frac{\mathcal{J}_1(\alpha x)}{x} e^{-sx}\, dx, \qquad s > 0, \tag{25}$$

where $\mathcal{J}_1$ is the Bessel function of the first kind of order one. The last equality is from Gradshteyn and Ryzhik (1994), Section 17.13, formula No 103 on p. 1182. This, with Theorem 1 (iv) in Jurek (2003), gives

$$\hat{\mu}_\alpha(t) = \frac{2}{\alpha} \frac{\mathcal{J}_1(\alpha t)}{t} = \frac{1}{B_1(i\alpha t)} = \int_{\boldsymbol{R}} e^{itx} \frac{2\sqrt{\alpha^2 - x^2}}{\pi\alpha^2} 1_{[-\alpha,\alpha]}(x)\, dx, \tag{26}$$

where $B_1(t)$ is a Fourier transform of a selfdecomposable distribution (given by series of independent Laplace random variables multiplied by zeros of a Bessel function) and $1/B_1(it)$ is again a Fourier transform. This is an example of a pair of Fourier transforms from the van Dantzig class $\mathcal{D}$ (van Dantzig property); cf. Jurek (2003), Theorem 1 (i), (iv), and Section 4 on p. 218. More importantly, in (26) we recognize that μ_α has the semicircle law. And this is what we should get because, indeed, (22) is the Cauchy transform of the semicircle law; cf. Voiculescu (1999), p. 299.

Similarly, from Corollary 1 (a) and (23) we infer that

$$(27)\qquad \big(\mathscr{K}(e(\mu_\alpha))\big)^{\wedge}(t) = \exp\left[\frac{1-\sqrt{1+\alpha^2 t^2}}{1+\sqrt{1+\alpha^2 t^2}}\right] = \big(e((\mu_\alpha)^{\langle e\rangle})\big)^{\wedge}(t), \qquad t \in \boldsymbol{R},$$

is a Fourier transform of a compound Poisson measure.

Furthermore, from Corollary 3 with $c = 1$ we get

$$(28)\qquad \frac{1}{2-h_{\mu_\alpha}(t)} = \frac{1+\sqrt{1+\alpha^2 t^2}}{2\sqrt{1+\alpha^2 t^2}} \in GID,$$

i.e., it is a Fourier transform and it corresponds to a symmetric geometric infinitely divisible distribution.

3.2. Cauchy distribution. This time we know that γ_a is the Cauchy random variable with the probability density $a/(\pi(a^2+x^2))$, $x \in \boldsymbol{R}$ ($a > 0$ is a parameter) and with the Fourier transform $\exp(-a|t|)$, $t \in \boldsymbol{R}$. By (9) and Corollary 1 (a), we conclude that

$$(29)\qquad h_{\gamma_a}(t) = \frac{1}{it} g_{\gamma_a}\left(\frac{1}{it}\right) = E[\exp(ite \cdot \gamma_a)] = \int_0^\infty e^{-a|t|s} e^{-s}\, ds = \frac{1}{1+a|t|},$$

where again we got a selfdecomposable distribution. This with Corollary 2 (a) implies that

$$\big(\mathscr{K}(e(\gamma_a))\big)^{\wedge}(t) = \exp\left[-\frac{a|t|}{1+a|t|}\right] = \big(e((\gamma_a)^{\langle e\rangle})\big)^{\wedge}(t), \qquad t \in \boldsymbol{R},$$

is a Fourier transform of a compound Poisson measure. And Corollary 3 allows us to conclude that

$$(30)\qquad \frac{1}{2-h_{\gamma_a}(t)} = \frac{1+a|t|}{1+2a|t|} \in GID,$$

i.e., it is a Fourier transform and it corresponds to a symmetric geometric infinitely divisible distribution. Finally, from (29) and (9) we retrieve the restricted Cauchy transform for the Cauchy distribution γ_a:

$$G_{\gamma_a}(is) = \frac{\operatorname{sign}(s)}{i(|s|+a)} \qquad \text{for } s \neq 0$$

(note that the formula on p. 302 in Voiculescu (1999) is valid only in a half-plane).

3.3. Gaussian distribution. Let $\mathscr{N}$ denote the standard normal distribution (variable) with the probability density function $(2\pi)^{-1/2}\exp(-x^2/2)$, $x \in \boldsymbol{R}$.

From (9) and Corollary 1 (a) we infer that

$$
\begin{aligned}
(31)\qquad h_{\mathcal{N}}(t) &= \frac{1}{it}\, g_{\mathcal{N}}\left(\frac{1}{it}\right) = E\left[\exp(it\boldsymbol{e}\cdot\mathcal{N})\right] = \int_{0}^{\infty} \exp\left(-(ts)^2/2\right) e^{-s}\, ds \\
&= \exp\left(1/(2t^2)\right) \int_{0}^{\infty} \exp\left(-2^{-1}(st+t^{-1})^2\right) ds \\
&= \exp\left(1/(2t^2)\right) t^{-1} \int_{t^{-1}}^{\operatorname{sign}(t)\cdot\infty} \exp(-w^2/2)\, dw \\
&= (2\pi)^{1/2}\, t^{-1} \exp\left(1/(2t^2)\right) \left[\Phi(\operatorname{sign}(t)\cdot\infty) - \Phi(t^{-1})\right] \\
&= (2\pi)^{1/2}\, |t|^{-1} \exp\left(1/(2t^2)\right) \Phi(-|t|^{-1}),
\end{aligned}
$$

where Φ denotes the cumulative distribution function of the standard normal distribution $\mathcal{N}$, is a Fourier transform of $\boldsymbol{e}\cdot\mathcal{N}$. Furthermore,

$$
(32)\qquad g_{\mathcal{N}}(iw) = -i\sqrt{2\pi}\, \exp(w^2/2)\, \operatorname{sign}(w)\, \Phi(-|w|), \qquad w \neq 0,
$$

is the restricted Cauchy transform of $\mathcal{N}$.

4. COMMENTS AND REMARKS

Remark 1. For the class *GID*, a Cauchy probability distribution (with the probability density $2^{-1}\exp[-|x[]$ and the Fourier transform $(1+t^2)^{-1}$) corresponds to a standard normal distribution (Gaussian) in *ID*; cf. Klebanov et al. (1984), at the bottom of page 758. In this context, let us mention that in free-infinite divisibility, the semicircle distribution (the example in Section 3.1) plays the role of a standard normal distribution.

Remark 2. For a finite measure m, let us define a function $\mathfrak{u}_m(t)$ by the following equality:

$$
(33)\qquad \mathfrak{u}_m(t) := h_m(t) - m(\boldsymbol{R}) = \int_{\boldsymbol{R}} \left(\frac{1}{1-itx} - 1\right) m(dx), \qquad t \in \boldsymbol{R};
$$

then on the right-hand side we recognize a functional of the Voiculescu transforms (via (19)) for free-infinitely divisible measures. But, as in the case of the classical *ID*, not all infinitely divisible characteristic functions are of the form (7), so not all functionals of free-infinitely divisible distributions have transforms of the form (33). In fact, (7) "encourages us to abandon the assumption that m is finite" writes Stroock (1994), p. 136. In a similar spirit, if we assume that a measure M integrates $\min(1, x^2)$, then (33) naturally extends to

$$
\mathfrak{u}_M(t) := \int_{\boldsymbol{R}} \left(\frac{1}{1-itx} - 1 - itx\, 1_{\{u:|u|\leqslant 1\}}(x)\right) M(dx),
$$

which coincides with 'Poissonian' analog of free-infinitely divisible distribution, and Lévy exponents of class $\mathscr{E}$ probability measures; see Barndorff-Nielsen and Thorbjornsen (2002), Proposition 5.2, and Jurek (2004), Corollary 6. Note that the integrand above is bounded by $\mathrm{const}\cdot\min(1, x^2)$.

Remark 3. Let m be a finite measure on $\boldsymbol{R}^d$ and let us begin with the definition

$$h_m(t) := \int_{\boldsymbol{R}^d} \frac{1}{1-i\langle t, x\rangle} m(dx), \qquad t \in \boldsymbol{R}^d, \tag{34}$$

where $\langle\cdot, \cdot\rangle$ denotes the scalar product in $\boldsymbol{R}^d$. Then many of presented here results will hold true with some obvious modifications; note that in Jurek (2004) or in Iksanov et al. (2004) random integrals are given for Banach space valued Lévy processes. See Araujo and Giné (1980), Chapter 3, for the classical infinite divisibility on Banach spaces. Hence one may use (34) as the stepping stone for free-probability in finite (infinite) linear spaces.

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Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: zjjurek@math.uni.wroc.pl
www.math.uni.wroc.pl/~zjjurek

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