PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 26, Fasc. 1 (2006), pp. 23–39

MONOTONICITY AND NON-MONOTONICITY OF DOMAINS OF STOCHASTIC INTEGRAL OPERATORS

BY

KEN-ITI SATO (NAGOYA)

To the memory of Kazimierz Urbanik

Abstract. A Lévy process on $\boldsymbol{R}^d$ with distribution μ at time 1 is denoted by $X^{(\mu)} = \{X_t^{(\mu)}\}$. If the improper stochastic integral $\int_0^{\infty-} f(s)\,dX_s^{(\mu)}$ of f with respect to $X^{(\mu)}$ is definable, its distribution is denoted by $\Phi_f(\mu)$. The class of all infinitely divisible distributions μ on $\boldsymbol{R}^d$ such that $\Phi_f(\mu)$ is definable is denoted by $\mathfrak{D}(\Phi_f)$. The class $\mathfrak{D}(\Phi_f)$, its two extensions $\mathfrak{D}_c(\Phi_f)$ and $\mathfrak{D}_{es}(\Phi_f)$ (compensated and essential), and its restriction $\mathfrak{D}^0(\Phi_f)$ (absolutely definable) are studied. It is shown that $\mathfrak{D}_{es}(\Phi_f)$ is monotonic with respect to f, which means that $|f_2| \leqslant |f_1|$ implies $\mathfrak{D}_{es}(\Phi_{f_1}) \subset \mathfrak{D}_{es}(\Phi_{f_2})$. Further, $\mathfrak{D}^0(\Phi_f)$ is monotonic with respect to f but neither $\mathfrak{D}(\Phi_f)$ nor $\mathfrak{D}_c(\Phi_f)$ is monotonic with respect to f. Furthermore, there exist μ, f_1, and f_2 such that $0 \leqslant f_2 \leqslant f_1$, $\mu \in \mathfrak{D}(\Phi_{f_1})$, and $\mu \notin \mathfrak{D}(\Phi_{f_2})$. An explicit example for this is related to some properties of a class of martingale Lévy processes.

2000 AMS Subject Classification: 60E07, 60G51, 60H05.

Key words and phrases: Improper stochastic integral, infinitely divisible distribution, Lévy process, martingale Lévy process, monotonic.

1. INTRODUCTION AND RESULTS

Let $ID(\boldsymbol{R}^d)$ be the class of infinitely divisible distributions on the d-dimensional Euclidean space $\boldsymbol{R}^d$. For each $\mu \in ID(\boldsymbol{R}^d)$ let $X^{(\mu)} = \{X_t^{(\mu)}, t \geqslant 0\}$ be the Lévy process on $\boldsymbol{R}^d$ satisfying $\mathscr{L}(X_1^{(\mu)}) = \mu$. Here $\mathscr{L}(Y)$ denotes the distribution of Y for any random element Y. Given $\mu \in ID(\boldsymbol{R}^d)$ and a real-valued measurable non-random function f on $[0, \infty)$, we say, as in [10], that f is *locally $X^{(\mu)}$-integrable* if the stochastic integral $\int_B f(s)\,dX_s^{(\mu)}$ of f with respect to $X^{(\mu)}$ is definable for each bounded Borel set B in $[0, \infty)$ in the sense of Urbanik and Woyczyński [14], Rajput and Rosinski [6], Kwapień and Woyczyński [5], and Sato [9]–[11]. We write

$$\int_0^t f(s)\,dX_s^{(\mu)} = \int_{[0,t]} f(s)\,dX_s^{(\mu)}.$$

Since this is an additive process in law, we use an additive process modification (see [8] for terminology). For μ fixed, let

$$L(X^{(\mu)}) = \{f: f \text{ is locally } X^{(\mu)}\text{-integrable}\}. \tag{1.1}$$

Characterization of $L(X^{(\mu)})$ in terms of the Lévy–Khintchine triplet of μ is given in [6], [5], [9], [10]. It is known that $L(X^{(\mu)})$ is a generalization of Orlicz spaces, one of whose properties is that $L(X^{(\mu)})$ is *monotonic*, which means that if

$$f_1 \text{ and } f_2 \text{ are measurable and } |f_2| \leqslant |f_1|, \tag{1.2}$$

then $f_2 \in L(X^{(\mu)})$ whenever $f_1 \in L(X^{(\mu)})$. Given f, let

$$D[f] = D[f; \mathbf{R}^d] = \{\mu \in ID(\mathbf{R}^d): f \text{ is locally } X^{(\mu)}\text{-integrable}\}. \tag{1.3}$$

Then (1.2) implies $D[f_1] \subset D[f_2]$. We express this property by saying that $D[f]$ is *monotonic with respect to* f.

Let $\mu \in ID(\mathbf{R}^d)$. We say that the *improper stochastic integral of* f *with respect to* $X^{(\mu)}$ *is definable* if $f \in L(X^{(\mu)})$ and if $\int_0^t f(s)\,dX_s^{(\mu)}$ is convergent in probability (equivalently, convergent almost surely) in $\mathbf{R}^d$ as $t \to \infty$. The limit is denoted by $\int_0^{\infty-} f(s)\,dX_s^{(\mu)}$. This notation will help to distinguish it from the stochastic integral (with random integrand in general) up to infinity of Cherny and Shiryaev [2]. We define

$$\Phi_f(\mu) = \mathscr{L}\Big(\int_0^{\infty-} f(s)\,dX_s^{(\mu)}\Big). \tag{1.4}$$

Two extended notions and one restricted notion of definability of improper stochastic integrals are introduced in [10] and [11]. We say that the *compensated improper stochastic integral of* f *with respect to* $X^{(\mu)}$ *is definable* if $f \in L(X^{(\mu)})$ and if there is $q \in \mathbf{R}^d$ such that $\int_0^{\infty-} f(s)\,dX_s^{(\mu * \delta_{-q})}$ is definable. Here δ_{-q} is the distribution concentrated at $-q$. We say that the *essential improper integral of* f *with respect to* $X^{(\mu)}$ *is definable* if $f \in L(X^{(\mu)})$ and if there is a non-random $\mathbf{R}^d$-valued function q_t on $[0, \infty)$ such that $\int_0^t f(s)\,dX_s^{(\mu)} - q_t$ is convergent in probability in $\mathbf{R}^d$ as $t \to \infty$. We say that the *improper integral* $\int_0^{\infty-} f(s)\,dX_s^{(\mu)}$ *is absolutely definable* if $f \in L(X^{(\mu)})$ and if

$$\int_0^{\infty} \big|C_\mu(f(s)z)\big|\,ds < \infty \quad \text{for all } z \in \mathbf{R}^d. \tag{1.5}$$

Here $C_\mu(z)$ is the cumulant function of μ, that is, the complex-valued continuous function on $\mathbf{R}^d$ with $C_\mu(0) = 0$ such that the characteristic function $\hat{\mu}(z)$ of μ is expressed as $\hat{\mu}(z) = \exp(C_\mu(z))$. For any measurable function f on $[0, \infty)$, let

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}^0(\Phi_f; \mathbf{R}^d) = \Big\{\mu \in ID(\mathbf{R}^d): \int_0^{\infty-} f(s)\,dX_s^{(\mu)} \text{ is absolutely definable}\Big\},$$

$$\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_f; \boldsymbol{R}^d) = \{\mu \in ID(\boldsymbol{R}^d): \int_0^{\infty -} f(s)\, dX_s^{(\mu)} \text{ is definable}\},$$

$$\mathfrak{D}_c(\Phi_f) = \mathfrak{D}_c(\Phi_f; \boldsymbol{R}^d) = \{\mu \in ID(\boldsymbol{R}^d): \text{compensated improper integral of } f \text{ with respect to } X^{(\mu)} \text{ is definable}\},$$

$$\mathfrak{D}_{es}(\Phi_f) = \mathfrak{D}_{es}(\Phi_f; \boldsymbol{R}^d) = \{\mu \in ID(\boldsymbol{R}^d): \text{essential improper integral of } f \text{ with respect to } X^{(\mu)} \text{ is definable}\}.$$

Further let, for $\mu \in ID(\boldsymbol{R}^d)$,

$$L^{\infty -}(X^{(\mu)}) = \{f: f \text{ is measurable and } \mu \in \mathfrak{D}(\Phi_f; \boldsymbol{R}^d)\}.$$

It is known that

$$\mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_f) \subset \mathfrak{D}_c(\Phi_f) \subset \mathfrak{D}_{es}(\Phi_f). \tag{1.6}$$

We are interested in the problem whether $\mathfrak{D}(\Phi_f)$, $\mathfrak{D}^0(\Phi_f)$, $\mathfrak{D}_c(\Phi_f)$, and $\mathfrak{D}_{es}(\Phi_f)$ are monotonic with respect to f. Clearly, $\mathfrak{D}(\Phi_f)$ is monotonic with respect to f if and only if, for every $\mu \in ID(\boldsymbol{R}^d)$, $L^{\infty -}(X^{(\mu)})$ is monotonic.

Our results are the following

THEOREM 1.1. *The class $\mathfrak{D}_{es}(\Phi_f)$ is monotonic with respect to f.*

THEOREM 1.2. *The class $\mathfrak{D}^0(\Phi_f)$ is monotonic with respect to f.*

The class $\mathfrak{D}(\Phi_f)$ is not monotonic with respect to f. That is, for some $\mu \in ID(\boldsymbol{R}^d)$, $L^{\infty -}(X^{(\mu)})$ is not monotonic. In order to specify μ, we use the Lévy–Khintchine triplet (A, ν, γ) of $\mu \in ID(\boldsymbol{R}^d)$ in the sense that

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\boldsymbol{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right) \nu(dx) + i\langle \gamma, z\rangle,$$

where A is a $d \times d$ symmetric nonnegative-definite matrix, called the *Gaussian covariance matrix* of μ, ν is a measure on $\boldsymbol{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\boldsymbol{R}^d} (|x|^2 \wedge 1)\, \nu(dx) < \infty,$$

called the *Lévy measure* of μ, and γ is an element of $\boldsymbol{R}^d$, called the *location parameter* of μ. Sometimes we write $\mu = \mu_{(A,\nu,\gamma)}$. We say that a measure ϱ on $\boldsymbol{R}^d$ is *symmetric* if $\varrho(B) = \varrho(-B)$ for all Borel sets B.

THEOREM 1.3. *Let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\boldsymbol{R}^d)$ with A arbitrary and ν symmetric.*

(i) *If f_1 and f_2 satisfy* (1.2) *and if $\mu \in \mathfrak{D}_c(\Phi_{f_1})$, then $\mu \in \mathfrak{D}_c(\Phi_{f_2})$.*

(ii) *Assume that $\gamma = 0$. If f_1 and f_2 satisfy* (1.2) *and if $\mu \in \mathfrak{D}(\Phi_{f_1})$, then $\mu \in \mathfrak{D}(\Phi_{f_2})$. That is, $L^{\infty -}(X^{(\mu)})$ is monotonic.*

(iii) *Assume that $\gamma \neq 0$ and $\int_{|x|>1} |x|\, \nu(dx) < \infty$. Then $L^{\infty -}(X^{(\mu)})$ is not monotonic.*

A simple example for Theorem 1.3 (iii) is the case where $X^{(\mu)}$ is a Brownian motion with drift.

We ask a question whether there exist μ, f_1, and f_2 such that $0 \leqslant f_2 \leqslant f_1$, $f_1 \in L^{\infty-}(X^{(\mu)})$, and $f_2 \notin L^{\infty-}(X^{(\mu)})$. The next theorem gives more than the affirmative answer.

THEOREM 1.4. *Let $f_1(s)$ be a real-valued function which vanishes on $[0, a)$ and is continuous on $[a, \infty)$ with some $a \geqslant 0$. Let $\mu \in \mathfrak{D}(\Phi_{f_1}) \backslash \mathfrak{D}^0(\Phi_{f_1})$. Then there is a nonrandom open set D in $[a, \infty)$ such that $\mu \notin \mathfrak{D}(\Phi_{f_2})$ for $f_2(s) = f_1(s) 1_D(s)$.*

Notice that this theorem and Theorem 1.2 give a characterization of the property that $\mathfrak{D}(\Phi_{f_1}) \backslash \mathfrak{D}^0(\Phi_{f_1}) \neq \emptyset$.

We say that $f(s) \asymp g(s)$ as $s \to \infty$ if there are positive constants c_1 and c_2 such that $0 < c_1 f(s) \leqslant g(s) \leqslant c_2 f(s)$ for all large s.

EXAMPLE 1.5. Let $f_1(s)$ be a locally square-integrable function on $[0, \infty)$. Suppose that $f_1(s) \asymp s^{-1}$ as $s \to \infty$ and that there are positive constants c and s_0 such that

$$\int_{s_0}^{\infty} |f_1(s) - cs^{-1}| \, ds < \infty.$$

Then Theorem 2.8 of [11] states that the class $\mathfrak{D}(\Phi_{f_1}) \backslash \mathfrak{D}^0(\Phi_{f_1})$ is nonempty and that

$$\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f_1}) \backslash \mathfrak{D}^0(\Phi_{f_1})$$

if and only if

$$\int_{|x|>1} |x| \, \nu(dx) < \infty, \qquad \int_{\boldsymbol{R}^d} x \mu(dx) = 0,$$

$$\lim_{t \to \infty} \int_{s_0}^{t} s^{-1} \, ds \int_{|x|>s} x \nu(dx) \text{ exists in } \boldsymbol{R}^d, \quad \text{and} \quad \int_{s_0}^{\infty} s^{-1} \Big| \int_{|x|>s} x \nu(dx) \Big| \, ds = \infty.$$

Distributions satisfying these conditions will be given in Example 1.7.

We show that the class $\mathfrak{D}_c(\Phi_f)$ is not monotonic with respect to f.

THEOREM 1.6. *Let $f_1(s) = s^{-1} 1_{[1,\infty)}(s)$. Suppose that $\mu \in \mathfrak{D}(\Phi_{f_1})$ and that the Lévy measure ν of μ satisfies*

$$\Big| \int_{|x|>s} x_j \nu(dx) \Big| \sim \frac{c}{\log s} \quad \text{as } s \to \infty \tag{1.7}$$

for some $j \in \{1, \ldots, d\}$ and $c > 0$. Then there is a nonrandom open set D in $[1, \infty)$ such that $\mu \notin \mathfrak{D}_c(\Phi_{f_2})$ for $f_2(s) = f_1(s) 1_D(s)$.

Here x_j is the jth coordinate of $x \in \boldsymbol{R}^d$. In Theorem 1.6 we recall that $\mu \in \mathfrak{D}(\Phi_{f_1})$ implies $\int_{|x|>1} |x| \, \nu(dx) < \infty$ by virtue of Theorem 2.8 of [11].

EXAMPLE 1.7. In Example 2.9 of [11] we have introduced the measure ν concentrated on $\{x \in \boldsymbol{R}^d\colon |x| = 2, 3, \ldots\}$ given by

$$\nu(B) = \int_{S_0} \lambda(d\xi) \sum_{n \in \boldsymbol{Z}} 1_B(n\xi)\, a_n \quad \text{for Borel sets } B,$$

where S_0 is a nonempty Borel set on the unit sphere $\{|\xi| = 1\}$ satisfying $S_0 \cap (-S_0) = \emptyset$, λ is a finite measure on S_0 satisfying $\int_{S_0} \xi \lambda(d\xi) \neq 0$, $\boldsymbol{Z}$ is the class of all integers, and a_n, $n \in \boldsymbol{Z}$, are such that $a_0 = a_1 = a_{-1} = 0$ and, for positive integers n, m,

$$a_n = \frac{1}{n}\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right), \quad a_{-n} = 0 \quad \text{if } 2^{m^2} < n < 2^{(m+1)^2},\ m \text{ odd},$$

$$a_n = 0, \quad a_{-n} = \frac{1}{n}\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right) \quad \text{if } 2^{m^2} < n < 2^{(m+1)^2},\ m \text{ even},$$

$$a_n = \frac{1}{n}\left(\frac{1}{\log n} + \frac{1}{\log(n+1)}\right), \quad a_{-n} = 0 \quad \text{if } n = 2^{m^2},\ m \text{ even},$$

$$a_n = 0, \quad a_{-n} = \frac{1}{n}\left(\frac{1}{\log n} + \frac{1}{\log(n+1)}\right) \quad \text{if } n = 2^{m^2},\ m \text{ odd}.$$

It is shown that $\sum_{|n| \geq 2} |n|\, a_n < \infty$ and that, for $k = 3, 4, \ldots$,

$$\sum_{|n| \geq k} n a_n = \begin{cases} (\log k)^{-1} & \text{if } 2^{m^2} < k \leq 2^{(m+1)^2},\ m \text{ odd}, \\ -(\log k)^{-1} & \text{if } 2^{m^2} < k \leq 2^{(m+1)^2},\ m \text{ even}. \end{cases}$$

Thus

$$\int_{|x|>1} |x|\, \nu(dx) = \int_{S_0} \lambda(d\xi) \sum_{|n| \geq 2} |n|\, a_n < \infty,$$

$$\int_1^\infty s^{-1}\, ds \Big| \int_{|x|>s} x\nu(dx) \Big| = \Big| \int_{S_0} \xi\lambda(d\xi) \Big| \int_1^\infty s^{-1}\, ds \Big| \sum_{|n|>s} n a_n \Big| = \infty,$$

$$\int_{|x|>s} x_j\, \nu(dx) = \int_{S_0} \xi_j \lambda(d\xi) \sum_{|n|>s} n a_n.$$

Further it is shown that $\int_1^t s^{-1}\, ds \int_{|x|>s} x\nu(dx)$ is convergent as $t \to \infty$. Let $\mu = \mu_{(A,\nu,\gamma)}$ with $\gamma = -\int_{\boldsymbol{R}^d} x|x|^2(1+|x|^2)^{-1}\, \nu(dx)$ and A arbitrary. Then

$$\int_{\boldsymbol{R}^d} x\mu(dx) = 0 \quad \text{and} \quad \mu \in \mathfrak{D}(\Phi_{f_1}) \backslash \mathfrak{D}^0(\Phi_{f_1}) \text{ for } f_1(s) = s^{-1} 1_{[1,\infty)}(s),$$

since the conditions stated in Example 1.5 are satisfied. Choosing $j \in \{1, \ldots, d\}$ such that $\int_{S_0} \xi_j \lambda(d\xi) \neq 0$, we can apply Theorem 1.6 to this distribution μ. We can also apply Theorem 1.4 to this f_1 and this μ. If $A = 0$, then the process $X^{(\mu)}$ is a compensated compound Poisson process.

Using the measure ν above, consider

$$\tilde{\nu}(B) = \nu(B) + \frac{1}{2\log 2} \int_{S_0} \lambda(d\xi)\, 1_B(2\xi) \quad \text{for Borel sets } B$$

and define $\tilde{\mu} \in ID(\boldsymbol{R}^d)$ by

$$C_{\tilde{\mu}}(z) = \int_{\boldsymbol{R}^d} (e^{i\langle z,x\rangle} - 1)\, \tilde{\nu}(dx).$$

Then

$$\int_{\boldsymbol{R}^d} x\tilde{\mu}(dx) = \int_{\boldsymbol{R}^d} x\tilde{\nu}(dx) = \int_{\boldsymbol{R}^d} x\nu(dx) + \frac{1}{\log 2} \int_{S_0} \xi\lambda(d\xi) = \bar{0}.$$

The distribution $\tilde{\mu}$ also belongs to $\mathfrak{D}(\Phi_{f_1}) \backslash \mathfrak{D}^0(\Phi_{f_1})$ for $f_1(s) = s^{-1} 1_{[1,\infty)}(s)$ and Theorem 1.6 applies to $\tilde{\mu}$ by the same reason as for μ. Theorem 1.4 also applies to $f_1(s) = s^{-1} 1_{[1,\infty)}(s)$ and $\tilde{\mu}$. The Lévy process $X^{(\tilde{\mu})}$ associated with $\tilde{\mu}$ is a compound Poisson process with mean 0.

In Section 2 we will give proofs of all theorems stated above. The process $X^{(\mu)}$ associated with μ in Theorem 1.6 is a martingale Lévy process and the processes $\int_0^t f_1(s)\, dX_s^{(\mu)}$ and $\int_0^t f_2(s)\, dX_s^{(\mu)}$ have intriguing properties, which we will discuss in Section 3. Applications of Theorems 1.1 and 1.2 to some types of f will be given in Section 4. Determination of $\mathfrak{D}(\Phi_f)$ for some f is made.

2. PROOFS

In the following three propositions let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\boldsymbol{R}^d)$ and $f \in \boldsymbol{L}(X^{(\mu)})$. We present necessary and sufficient conditions for μ to belong to $\mathfrak{D}(\Phi_f)$, $\mathfrak{D}^0(\Phi_f)$, or $\mathfrak{D}_{\mathrm{es}}(\Phi_f)$.

PROPOSITION 2.1. *The following three statements are equivalent*:

(a) $\mu \in \mathfrak{D}(\Phi_f)$.

(b) $\int_0^t C_\mu(f(s)z)\, ds$ *is convergent in* $\boldsymbol{C}$ *as* $t \to \infty$ *for each* $z \in \boldsymbol{R}^d$.

(c) μ *satisfies the following*:

$$\int_0^\infty f(s)^2 (\operatorname{tr} A)\, ds < \infty, \tag{2.1}$$

$$\int_0^\infty ds \int_{\boldsymbol{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx) < \infty, \tag{2.2}$$

$$\int_0^t f(s) \left(\gamma + \int_{\boldsymbol{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) ds \tag{2.3}$$

is convergent in $\boldsymbol{R}^d$ *as* $t \to \infty$.

Proof. See Proposition 5.5 of [10] and Propositions 2.2 and 2.6 of [11]. It follows from $f \in \boldsymbol{L}(X^{(\mu)})$ that $\int_0^t |C_\mu(f(s)z)|\,ds < \infty$ and that

$$\int_0^t \left| f(s)\left(\gamma + \int_{\boldsymbol{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\right| ds < \infty$$

for $t \in (0, \infty)$, as is shown in Proposition 2.17 and Corollary 2.19 of [10]. ∎

PROPOSITION 2.2. *A distribution μ is in $\mathfrak{D}_{\mathrm{es}}(\Phi_f)$ if and only if* (2.1) *and* (2.2) *are satisfied.*

Proof. See Proposition 5.6 of [10] or Proposition 2.6 of [11]. ∎

PROPOSITION 2.3. *A distribution μ is in $\mathfrak{D}^0(\Phi_f)$ if and only if* (2.1), (2.2), *and*

$$\int_0^\infty \left| f(s)\left(\gamma + \int_{\boldsymbol{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\right| ds < \infty. \tag{2.4}$$

Proof. For fixed $u \in \boldsymbol{R}$ denote by μ^u a probability measure such that $\mu^u(B) = \int 1_B(ux)\,\mu(dx)$ for all Borel sets B. Let (A^u, ν^u, γ^u) be the triplet of μ^u. Then $A^u = u^2 A$, $\nu^u(B) = \int 1_B(ux)\,\nu(dx)$, and

$$\gamma^u = u\gamma + \int_{\boldsymbol{R}^d} ux\left(\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2}\right)\nu(dx).$$

Notice that

$$\int_{\boldsymbol{R}^d} |ux|\left|\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2}\right|\nu(dx) \leqslant \int_{\boldsymbol{R}^d} \frac{|ux|\,(|x|^2+|ux|^2)}{(1+|ux|^2)(1+|x|^2)}\nu(dx) < \infty. \tag{2.5}$$

Let

$$\varphi(u) = \operatorname{tr} A^u + \int_{\boldsymbol{R}^d} (|x|^2 \wedge 1)\,\nu^u(dx) + |\gamma^u|. \tag{2.6}$$

When $u = f(s)$, μ^u and (A^u, ν^u, γ^u) are written as $\mu^{f(s)}$ and $(A^{f(s)}, \nu^{f(s)}, \gamma^{f(s)})$. The properties (2.1), (2.2), and (2.4) combined are expressed by

$$\int_0^\infty \varphi(f(s))\,ds < \infty. \tag{2.7}$$

We note that

$$\begin{aligned} |C_\mu(f(s)z)| &= |C_{\mu^{f(s)}}(z)| \\ &\leqslant \frac{|z|^2}{2}\operatorname{tr} A^{f(s)} + 3(1+|z|^2)\int_{\boldsymbol{R}^d} (|x|^2 \wedge 1)\,\nu^{f(s)}(dx) + |z|\,|\gamma^{f(s)}| \end{aligned}$$

(see, in [10], (2.5)–(2.7) and line 3 of the proof of Theorem 3.14). Hence, if (2.1), (2.2), and (2.4) are satisfied, then (1.5) is satisfied, that is, $\mu \in \mathfrak{D}^0(\Phi_f)$.

Conversely, assume that $\mu \in \mathfrak{D}^0(\Phi_f)$. Then $\mu \in \mathfrak{D}(\Phi_f)$ and (2.1) and (2.2) follow from Proposition 2.1. We have

$$\operatorname{Im} C_\mu(f(s)z) = \operatorname{Im} C_{\mu^{f(s)}}(z) = \int_{\mathbf{R}^d} \left(\sin\langle z, x\rangle - \frac{\langle z, x\rangle}{1+|x|^2} \right) \nu^{f(s)}(dx) + \langle \gamma^{f(s)}, z\rangle.$$

For fixed z,

$$\sin\langle z, x\rangle - \frac{\langle z, x\rangle}{1+|x|^2} = \begin{cases} O(|x|^3) & \text{as } |x| \to 0, \\ O(1) & \text{as } |x| \to \infty. \end{cases}$$

Hence it follows from (2.2) that

$$\int_{\mathbf{R}^d} \left| \sin\langle z, x\rangle - \frac{\langle z, x\rangle}{1+|x|^2} \right| \nu^{f(s)}(dx) \leqslant c_z \int_{\mathbf{R}^d} (|x|^2 \wedge 1)\, \nu^{f(s)}(dx)$$
$$= c_z \int_{\mathbf{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx),$$

where c_z is a constant depending on z. Thus we obtain

$$\int_0^\infty |\langle \gamma^{f(s)}, z\rangle|\, ds < \infty$$

using the relation $\int_0^\infty |\operatorname{Im} C_\mu(f(s)z)|\, ds < \infty$. Choosing $z = (\delta_{jk})_{1 \leqslant k \leqslant d}$, $1 \leqslant j \leqslant d$, we obtain (2.4). ∎

Proof of Theorem 1.1. Use Proposition 2.2. Let f_1 and f_2 satisfy (1.2). Suppose that $\mu \in \mathfrak{D}_{\mathrm{es}}(\Phi_{f_1})$. Then (2.1) and (2.2) hold with f_1 in place of f. Since $|f_2| \leqslant |f_1|$, it follows that (2.1) and (2.2) hold with f_2 in place of f. This means that $\mu \in \mathfrak{D}_{\mathrm{es}}(\Phi_{f_2})$. ∎

Proof of Theorem 1.2. Use Proposition 2.3. Let f_1 and f_2 satisfy (1.2) and suppose that $\mu \in \mathfrak{D}^0(\Phi_{f_1})$. Using the function $\varphi(u)$ in (2.6) induced by $\mu = \mu_{(A,\nu,\gamma)}$, we have

$$\int_0^\infty \varphi(f_1(s))\, ds < \infty. \tag{2.8}$$

Let us use

$$\tilde{\varphi}(u) = \operatorname{tr} A^u + \int_{\mathbf{R}^d} (|x|^2 \wedge 1)\, \nu^u(dx) + \sup_{v \in \mathbf{R}, |v| \leqslant |u|} |\gamma^v|. \tag{2.9}$$

We have $\varphi(u) \leqslant \tilde{\varphi}(u) \leqslant (3/2)\varphi(u)$ as in Proposition 3.10 of [10]. Thus (2.8) is equivalent to

$$\int_0^\infty \tilde{\varphi}(f_1(s))\, ds < \infty. \tag{2.10}$$

The function $\tilde{\varphi}$ enjoys the property that $\tilde{\varphi}(f_2(s)) \leqslant \tilde{\varphi}(f_1(s))$ whenever $|f_2(s)| \leqslant |f_1(s)|$. Hence $\int_0^\infty \tilde{\varphi}(f_2(s))\, ds < \infty$. This means $\mu \in \mathfrak{D}^0(\Phi_{f_2})$. ∎

Proof of Theorem 1.3. (i) Let f_1 and f_2 satisfy (1.2). Assume that $\mu \in \mathfrak{D}_c(\Phi_{f_1})$. Then there is $q \in \boldsymbol{R}^d$ such that $\mu * \delta_{-q} \in \mathfrak{D}(\Phi_{f_1})$. Thus

$$\int_0^\infty f_1(s)^2 (\operatorname{tr} A)\, ds < \infty, \qquad \int_0^\infty ds \int_{\boldsymbol{R}^d} (|f_1(s)x|^2 \wedge 1)\, \nu(dx) < \infty,$$

and $\int_0^t f_1(s)(\gamma - q)\, ds$ is convergent, since ν is symmetric. We may and do choose $q = \gamma$. Then we see that $\mu * \delta_{-q} \in \mathfrak{D}(\Phi_{f_2})$. It follows that $\mu \in \mathfrak{D}_c(\Phi_{f_2})$.

(ii) Look back to the argument above with $\gamma = 0$. Then the proof is evident.

(iii) Let

$$f_1(s) = \begin{cases} 0 & \text{if } 0 \leqslant s < 1, \\ s^{-1} & \text{if } n \leqslant s < n+1 \text{ with } n \text{ odd}, \\ -s^{-1} & \text{if } n \leqslant s < n+1 \text{ with } n \text{ even} \end{cases}$$

and let $f_2(s) = s^{-1} 1_{[1,\infty)}(s)$. Then $|f_2| = |f_1|$. Applying Theorem 2.8 of [11], we see that $f_2 \notin L^{\infty-}(X^{(\mu)})$ since $\gamma \neq 0 = \int_{\boldsymbol{R}^d} x|x|^2(1+|x|^2)^{-1}\, \nu(dx)$. On the other hand, $f_1 \in L^{\infty-}(X^{(\mu)})$ by virtue of Proposition 2.1. Indeed, $\int_0^\infty ds \int_{\boldsymbol{R}^d} (|f_1(s)x|^2 \wedge 1)\, \nu(dx) < \infty$ by the same reasoning as in the proof of Lemma 2.7 of [11], and

$$\int_0^t f_1(s) \left(\gamma + \int_{\boldsymbol{R}^d} x \left(\frac{1}{1+|f_1(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) ds = \int_0^t f_1(s)\, ds\, \gamma,$$

which is convergent in $\boldsymbol{R}^d$ as $t \to \infty$. Hence $L^{\infty-}(X^{(\mu)})$ is not monotonic. ∎

Proof of Theorem 1.4. Let (A, ν, γ) be the triplet of μ. We use an $\boldsymbol{R}^d$-valued function

$$h(s) = f_1(s)\gamma + \int_{\boldsymbol{R}^d} f_1(s)x \left(\frac{1}{1+|f_1(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx). \tag{2.11}$$

Using (2.5), we see that $h(s)$ is continuous on $[a, \infty)$. Since $\mu \in \mathfrak{D}(\Phi_{f_1})$, we have

$$\int_a^\infty f_1(s)^2 (\operatorname{tr} A)\, ds < \infty, \qquad \int_a^\infty ds \int_{\boldsymbol{R}^d} (|f_1(s)x|^2 \wedge 1)\, \nu(dx) < \infty,$$

and $\int_a^t h(s)\, ds$ is convergent in $\boldsymbol{R}^d$ as $t \to \infty$ (Proposition 2.1). Since $\mu \notin \mathfrak{D}^0(\Phi_{f_1})$, we have $\int_a^\infty |h(s)|\, ds = \infty$ (Proposition 2.3). Choose and fix $j \in \{1, \ldots, d\}$ such that $\int_a^\infty |h_j(s)|\, ds = \infty$, where $h_j(s)$ is the jth coordinate of $h(s)$. Define $D^+ = \{s \geqslant a\colon h_j(s) > 0\}$, $D^- = \{s \geqslant a\colon h_j(s) < 0\}$, and $D^0 = \{s \geqslant a\colon h_j(s) = 0\}$. Then D^+ and D^- are open in $[a, \infty)$. Let $h_j^+(s) = h_j(s) \vee 0$ and $h_j^-(s) = h_j^+(s) - h_j(s)$. We see that $\int_a^\infty h_j^+(s)\, ds = \infty$ and $\int_a^\infty h_j^-(s)\, ds = \infty$. Let $D = D^+$ or D^-

(either will do). Let $f_2(s) = f_1(s)\,1_D(s)$. Then

$$\int_0^t \left(f_2(s)\gamma + \int_{\mathbf{R}^d} f_2(s)\,x \left(\frac{1}{1+|f_2(s)\,x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) ds$$

$$= \int_a^t f_1(s)\,1_D(s) \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+|f_1(s)\,x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) ds$$

$$= \int_a^t 1_D(s)\,h(s)\,ds.$$

If $D = D^+$, then

$$\int_a^t 1_D(s)\,h_j(s)\,ds = \int_a^t h_j^+(s)\,ds \to \infty \quad \text{as } t \to \infty.$$

If $D = D^-$, then

$$\int_a^t 1_D(s)\,h_j(s)\,ds = -\int_a^t h_j^-(s)\,ds \to -\infty \quad \text{as } t \to \infty.$$

Hence $\int_a^t 1_D(s)\,h(s)\,ds$ is not convergent in $\mathbf{R}^d$. Consequently, $\mu \notin \mathfrak{D}(\Phi_{f_2})$ by virtue of Proposition 2.1. ∎

Proof of Theorem 1.6. Let the triplet of μ be (A, ν, γ). In order to prove that $\mu \notin \mathfrak{D}_c(\Phi_{f_2})$ it is enough to show that, for every $q \in \mathbf{R}^d$,

$$\int_1^t s^{-1}\,1_D(s) \left(\gamma - q + \int_{\mathbf{R}^d} x \left(\frac{1}{1+|s^{-1}\,x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) ds$$

is not convergent in $\mathbf{R}^d$ as $t \to \infty$. Let $h(s)$ be as in (2.11). This is an $\mathbf{R}^d$-valued function, continuous on $[1, \infty)$. Since $\gamma = -\int_{\mathbf{R}^d} x\,|x|^2\,(1+|x|^2)^{-1}\,\nu(dx)$ (see Theorem 2.8 of [11]), we have

$$h(s) = \int_{\mathbf{R}^d} s^{-1}\,x \left(\frac{1}{1+|s^{-1}\,x|^2} - 1 \right) \nu(dx) \quad \text{for } s \geqslant 1.$$

Choose j as in (1.7) and let $h_j(s)$ be the jth coordinate of $h(s)$. Since $\int_1^t h(s)\,ds$ is convergent in $\mathbf{R}^d$ as $t \to \infty$ (see Theorem 2.8 of [11]), $\int_1^t h_j(s)\,ds$ is convergent in $\mathbf{R}$. We claim that

$$\int_1^t |h_j(s)|\,ds \sim c \log\log t \quad \text{as } t \to \infty. \tag{2.12}$$

Indeed,

$$\int_1^t |h_j(s)|\,ds \leqslant \int_1^t s^{-1} \left| \int_{|x|>s} x_j\,\nu(dx) \right| ds + I_1 + I_2,$$

where

$$I_1 = \int_1^t s^{-1} \left| \int_{|x|>s} \frac{x_j \nu(dx)}{1+|s^{-1}x|^2} \right| ds, \quad I_2 = \int_1^t s^{-1} \left| \int_{1<|x|\leqslant s} \frac{x_j |s^{-1}x|^2 \nu(dx)}{1+|s^{-1}x|^2} \right| ds.$$

We will denote positive constants by $c_1, c_2, \ldots$ The quantities I_1 and I_2 are bounded in t, since

$$I_1 \leqslant \int_1^\infty s^{-1} ds \int_{|x|>s} \frac{|x| \nu(dx)}{1+|s^{-1}x|^2} = \int_{|x|>1} |x| \nu(dx) \int_1^{|x|} \frac{s^{-1} ds}{1+|s^{-1}x|^2}$$

$$= \int_{|x|>1} |x| \left(\log|x| - \frac{1}{2}\log(1+|x|^2) + \frac{1}{2}\log 2 \right) \nu(dx) \leqslant c_1 \int_{|x|>1} |x| \nu(dx)$$

and since

$$I_2 \leqslant \int_1^\infty s^{-1} ds \int_{1<|x|\leqslant s} \frac{|x| |s^{-1}x|^2 \nu(dx)}{1+|s^{-1}x|^2} = \int_{|x|>1} |x| \nu(dx) \int_{|x|}^\infty \frac{s^{-1}|s^{-1}x|^2 ds}{1+|s^{-1}x|^2}$$

$$= \frac{\log 2}{2} \int_{|x|>1} |x| \nu(dx).$$

Thus we obtain, for some $s_0 > 1$,

$$\int_1^t |h_j(s)| ds \leqslant c_2 \int_{s_0}^t \frac{ds}{s \log s} + c_3 \leqslant c_2 \log\log t + c_4$$

from condition (1.7). Similarly,

$$\int_1^t |h_j(s)| ds \geqslant c_5 \int_{s_0}^t \frac{ds}{s \log s} - c_6 \geqslant c_5 \log\log t - c_7.$$

Looking back more carefully, we see that (2.12) holds. We have, *a fortiori*,

$$\int_1^\infty |h_j(s)| ds = \infty.$$

Now define D^+, D^-, D^0, $h_j^+(s)$, and $h_j^-(s)$ as in the proof of Theorem 1.4. Then

$$\int_1^\infty h_j^+(s) ds = \infty \quad \text{and} \quad \int_1^\infty h_j^-(s) ds = \infty.$$

We have

$$\int_2^\infty 1_{D^0}(s) \frac{ds}{s \log s} < \infty, \tag{2.13}$$

because it follows from

$$|h_j(s)| \geqslant s^{-1} \left| \int_{|x|>s} x_j \nu(dx) \right| - \int_{|x|>s} \frac{s^{-1}|x| \nu(dx)}{1+|s^{-1}x|^2} - \int_{|x|>s} \frac{|s^{-1}x|^3 \nu(dx)}{1+|s^{-1}x|^2}$$

that

$$0 = \int_0^\infty 1_{D^0}(s)\,|h_j(s)|\,ds \geqslant \int_1^\infty 1_{D^0}(s)\Big|\int_{|x|>s} x_j\,\nu(dx)\Big|\frac{ds}{s} - c_8$$

$$\geqslant c_9 \int_{s_0}^\infty 1_{D^0}(s)\frac{ds}{s\log s} - c_{10}.$$

Using (2.13), we see that

$$\limsup_{t\to\infty}\frac{1}{\log\log t}\int_2^t 1_{D^+}(s)\frac{ds}{s\log s} + \limsup_{t\to\infty}\frac{1}{\log\log t}\int_2^t 1_{D^-}(s)\frac{ds}{s\log s} \geqslant 1.$$

Choose $D = D^+$ or D^- in such a way that

$$\limsup_{t\to\infty}\frac{1}{\log\log t}\int_2^t 1_D(s)\frac{ds}{s\log s} > 0.$$

Choose $t_n \to \infty$ such that $(\log\log t_n)^{-1}\int_2^{t_n} 1_D(s)(s\log s)^{-1}\,ds$ tends to some $b > 0$. Then

$$\frac{1}{\log\log t_n}\int_2^{t_n} 1_D(s)\frac{ds}{s} \to \infty, \tag{2.14}$$

since, for any $k > 0$,

$$\frac{1}{\log\log t_n}\int_2^{t_n} 1_D(s)\frac{ds}{s} \geqslant \frac{\log k}{\log\log t_n}\int_k^{t_n} 1_D(s)\frac{ds}{s\log s} \to b\log k.$$

We claim that, for any choice of $q \in \boldsymbol{R}^d$, $\int_2^{t_n} 1_D(s)\big(h_j(s) - s^{-1}q_j\big)\,ds$ is divergent as $n \to \infty$. If $q_j = 0$, then this is divergent since

$$\int_2^{t_n} 1_D(s)\,h_j(s)\,ds = \int_2^{t_n} h_j^+(s)\,ds \text{ or } \int_2^{t_n} h_j^-(s)\,ds,$$

which diverges to ∞ or to $-\infty$. If $q_j \neq 0$, then

$$\int_2^{t_n} 1_D(s)\,h_j(s)\,ds - q_j\int_2^{t_n} 1_D(s)\,s^{-1}\,ds$$

is divergent, because

$$\int_2^{t_n} 1_D(s)\,|h_j(s)|\,ds \leqslant \int_2^{t_n} |h_j(s)|\,ds \sim c\log\log t_n$$

from (2.12) and in virtue of (2.14). The proof is complete. ∎

3. REMARKS ON MARTINGALE LÉVY PROCESSES

We have the following general result. Recall that $f \in L(X^{(\mu)})$ for all $\mu \in ID(\boldsymbol{R}^d)$ if and only if f is locally square-integrable on $[0, \infty)$ (see [10]).

PROPOSITION 3.1. *Let $X^{(\mu)}$ be a martingale Lévy process on $\mathbf{R}^d$. Let f be locally square-integrable on $[0, \infty)$. Then $Y_t = \int_0^t f(s)\,dX_s^{(\mu)}$ is a martingale additive process.*

Proof. Let $\mu = \mu_{(A,\nu,\gamma)}$. We have $E|X_t^{(\mu)}| < \infty$ and

$$0 = EX_t^{(\mu)} = t\left(\gamma + \int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\,\nu(dx)\right).$$

Let (A_t, ν_t, γ_t) be the triplet of Y_t. Proposition 2.6 of [11] states that

$$A_t = \int_0^t f(s)^2 A\,ds,$$

$$\nu_t(B) = \int_0^t ds \int_{\mathbf{R}^d} 1_B(f(s)x)\,\nu(dx) \quad \text{for } B \text{ Borel with } B \not\ni 0,$$

$$\gamma_t = \int_0^t f(s)\left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds.$$

Hence, recalling that $\int_{|x|>1} |x|\,\nu(dx) < \infty$, we obtain

$$\begin{aligned}
\int_{|x|>1} |x|\,\nu_t(dx) &= \int_0^t ds \int_{|f(s)x|>1} |f(s)x|\,\nu(dx) \\
&\leqslant \int_0^t ds \int_{|x|>1} |f(s)x|\,\nu(dx) + \int_0^t ds \int_{|x|\leqslant 1} |f(s)x|^2\,\nu(dx) \\
&= \int_0^t |f(s)|\,ds \int_{|x|>1} |x|\,\nu(dx) + \int_0^t f(s)^2\,ds \int_{|x|\leqslant 1} |x|^2\,\nu(dx) < \infty.
\end{aligned}$$

Thus $E|Y_t| < \infty$. Now we have

$$\begin{aligned}
EY_t &= \gamma_t + \int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\,\nu_t(dx) \\
&= \int_0^t f(s)\left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds \\
&\quad + \int_0^t ds \int_{\mathbf{R}^d} \frac{f(s)x|f(s)x|^2\,\nu(dx)}{1+|f(s)x|^2} \\
&= \int_0^t f(s)\left(\gamma + \int_{\mathbf{R}^d} \frac{x|x|^2\,\nu(dx)}{1+|x|^2}\right)ds = 0,
\end{aligned}$$

that is, $\{Y_t\}$ is a martingale additive process. ∎

Remark on Proposition 3.1. If $X^{(\mu)}$ is a martingale Lévy process and if $f \in \boldsymbol{L}(X^{(\mu)})$, it is not necessarily true that $Y_t = \int_0^t f(s)\,dX_s^{(\mu)}$ is a martingale additive process. In fact, $E|Y_t|$ may be infinite. For example, let $X^{(\mu)}$ be a com-

pound Poisson process on $\boldsymbol{R}^d$ with mean zero. Then any measurable function f belongs to $L(X^{(\mu)})$ as Example 4.4 of [10] states. If $\int_0^{t_0} |f(s)|\,ds = \infty$, then $E|Y_{t_0}| = \infty$, because, choosing $a > 0$ such that $0 < \int_{|x|>a} |x|\,\nu(dx) < \infty$ for the Lévy measure ν of $X^{(\mu)}$, we have, for the Lévy measure ν_{t_0} of Y_{t_0},

$$\int_{|x|>1} |x|\,\nu_{t_0}(dx) = \int_0^{t_0} ds \int_{|f(s)x|>1} |f(s)x|\,\nu(dx)$$

$$\geqslant \int_{|x|>a} |x|\,\nu(dx) \int_{[0,t_0]\cap\{|f(s)|>1/a\}} |f(s)|\,ds = \infty,$$

which implies that $E|Y_{t_0}| = \infty$.

Remark on martingale additive processes related to Theorem 1.6. The Lévy process $X^{(\mu)}$ associated with μ in Theorem 1.6 is a martingale, that is, it satisfies $E|X_t^{(\mu)}| < \infty$ and $EX_t^{(\mu)} = 0$. Consider the case $d = 1$. Let $h(s)$, D^+, D^-, and D^0 be as in the proof of Theorem 1.6. Thus

$$h(s) = s^{-1}\gamma + \int_{|x|\geqslant 2} s^{-1}x\left(\frac{1}{1+(s^{-1}x)^2} - \frac{1}{1+x^2}\right)\nu(dx) \quad \text{for } s \geqslant 1$$

and D^+, D^-, or D^0 is the set of $s \geqslant 1$ at which $h(s)$ is positive, negative, or zero, respectively. Let

$$Y_t = \int_0^t s^{-1}\,1_{[1,\infty)}(s)\,dX_s^{(\mu)}, \qquad Y_t^p = \int_0^t s^{-1}\,1_{D^p}(s)\,dX_s^{(\mu)} \text{ for } p = +, -, 0.$$

Then $\{Y_t\}$, $\{Y_t^+\}$, $\{Y_t^-\}$, $\{Y_t^0\}$, and $\{Y_t^+ + Y_t^-\}$ are martingale additive processes, as is shown in Proposition 3.1. We can show that

$$Y_t^+ \to \infty \quad \text{and} \quad Y_t^- \to -\infty \quad \text{a.s. as } t \to \infty, \tag{3.1}$$

$$Y_t,\ Y_t^+ + Y_t^-, \text{ and } Y_t^0 \text{ are convergent in } \boldsymbol{R} \text{ a.s. as } t \to \infty, \tag{3.2}$$

$$E|Y_{\infty -}| = \infty. \tag{3.3}$$

These are remarkable behaviors. If $A = 0$, then each of Y_t^+ and Y_t^- is the compensated sum of the jumps of $X^{(\mu)}$ in the union of some nonrandom time intervals with some nonrandom weights. For these behaviors it is essential that the Lévy measure is nonsymmetric and close to symmetric. Theorem 1.3 (ii) states that martingale compound Poisson processes with symmetric Lévy measures do not exhibit this kind of behaviors.

Proof of (3.1)–(3.3). Let the triplet of Y_t^p be $(A_t^p, \nu_t^p, \gamma_t^p)$ for $p = +, -, 0$. Then

$$A_t^p = \int_1^t s^{-2}\,1_{D^p}(s)\,A\,ds,$$

$$v_t^p(B) = \int_1^t 1_{D^p}(s)\,ds \int_{\boldsymbol{R}} 1_B(s^{-1}x)\,\nu(dx) \quad \text{for } B \text{ Borel with } B \not\ni 0,$$

$$\gamma_t^p = \int_1^t 1_{D^p}(s)\,h(s)\,ds.$$

Since A_t^p and $\int_{\boldsymbol{R}}(x^2 \wedge 1)\,\nu_t^p(dx)$ are bounded and increasing, $Y_t^p - \gamma_t^p$ is convergent in probability as $t \to \infty$. Since it is an additive process, $Y_t^p - \gamma_t^p$ is also convergent a.s. Since $\gamma_t^+ \to \infty$ and $\gamma_t^- \to -\infty$ (see the proof of Theorem 1.6), we obtain (3.1). We have convergence of Y_t^0 since $\gamma_t^0 = 0$. Convergence of Y_t comes from the fact that $\mu \in \mathfrak{D}(\Phi_{f_1})$. Recalling that $Y_t = Y_t^+ + Y_t^- + Y_t^0$, we obtain (3.2). In order to see (3.3), let $\nu_{\infty-}$ be the Lévy measure of $Y_{\infty-}$. Then

$$\int_{|x|>1} |x|\,\nu_{\infty-}(dx) = \int_1^\infty ds \int_{|s^{-1}x|>1} |s^{-1}x|\,\nu(dx)$$

$$= \int_{|x|>1} |x|\,\nu(dx) \int_1^{|x|} s^{-1}\,ds = \int_{|x|>1} |x| \log |x|\,\nu(dx),$$

which is infinite by virtue of Theorem 2.8 of [11] and of the fact that $\mu \notin \mathfrak{D}^0(\Phi_{f_1})$. Hence $E|Y_{\infty-}| = \infty$.

4. APPLICATIONS

The following results are consequences of Theorems 1.1 and 1.2.

PROPOSITION 4.1. *Let f_1 and f_2 be measurable and $|f_2| \leqslant |f_1|$. If $\mathfrak{D}(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_1})$ or if $\mathfrak{D}(\Phi_{f_2}) = \mathfrak{D}_{es}(\Phi_{f_2})$, then $\mathfrak{D}(\Phi_{f_1}) \subset \mathfrak{D}(\Phi_{f_2})$.*

Proof. In general we have (1.6). Hence it follows from Theorem 1.2 that if $\mathfrak{D}(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_1})$, then

$$\mathfrak{D}(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_1}) \subset \mathfrak{D}^0(\Phi_{f_2}) \subset \mathfrak{D}(\Phi_{f_2});$$

it follows from Theorem 1.1 that if $\mathfrak{D}(\Phi_{f_2}) = \mathfrak{D}_{es}(\Phi_{f_2})$, then

$$\mathfrak{D}(\Phi_{f_1}) \subset \mathfrak{D}_{es}(\Phi_{f_1}) \subset \mathfrak{D}_{es}(\Phi_{f_2}) = \mathfrak{D}(\Phi_{f_2}),$$

completing the proof. ∎

EXAMPLE 4.2. Let f_1 be a locally square-integrable function on $[0, \infty)$ satisfying $f_1(s) \asymp s^{-1/\alpha}$ as $s \to \infty$ with some $\alpha \in (0, 1) \cup (1, 2)$. Let $f_2(s)$ and $f_3(s)$ be measurable and satisfy $|f_2(s)| \leqslant |f_1(s)| \leqslant |f_3(s)|$. If $\alpha \in (0, 1)$, then $\mathfrak{D}(\Phi_{f_3}) \subset \mathfrak{D}(\Phi_{f_1}) \subset \mathfrak{D}(\Phi_{f_2})$. If $\alpha \in (1, 2)$, then $\mathfrak{D}(\Phi_{f_1}) \subset \mathfrak{D}(\Phi_{f_2})$.

Indeed, we have $\mathfrak{D}(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_1}) = \mathfrak{D}_{es}(\Phi_{f_1})$ if $\alpha \in (0, 1)$, and $\mathfrak{D}(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_1}) \subsetneq \mathfrak{D}_{es}(\Phi_{f_1})$ if $\alpha \in (1, 2)$ (Theorem 2.4 of [11]). Hence Proposition 4.1 applies.

PROPOSITION 4.3. *Let f be a locally square-integrable function on $[0, \infty)$ such that there are positive constants α, c_1, and c_2 satisfying*

$$\exp(-c_2 s^\alpha) \leqslant f(s) \leqslant \exp(-c_1 s^\alpha) \quad \text{for all large } s. \tag{4.1}$$

Then

$$\begin{aligned} \mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) = \mathfrak{D}_c(\Phi_f) = \mathfrak{D}_{es}(\Phi_f) \\ = \{\mu \in ID(\boldsymbol{R}^d): \int_{\boldsymbol{R}^d} (\log^+ |x|)^{1/\alpha} \mu(dx) < \infty\} \\ = \{\mu \in ID(\boldsymbol{R}^d): \int_{\boldsymbol{R}^d} (\log^+ |x|)^{1/\alpha} \nu(dx) < \infty\}, \end{aligned} \tag{4.2}$$

where ν is the Lévy measure of μ and $\log^+ u = (\log u) \vee 0$.

Obviously, in (4.2) we can use $(\log(1+|x|))^{1/\alpha}$ for $|x| > 1$ in place of $(\log^+ |x|)^{1/\alpha}$.

Proof of Proposition 4.3. Let $M = \{\mu \in ID(\boldsymbol{R}^d): \int_{\boldsymbol{R}^d} (\log^+ |x|)^{1/\alpha} \mu(dx) < \infty\}$. Then M takes the last expression in (4.2), which is a consequence of Theorem 25.3 of [8]. Let $f_j(s) = \exp(-c_j s^\alpha)$, $j = 1, 2$. Using Theorem 5.15 of [10] for these functions, we see that

$$\mathfrak{D}(\Phi_{f_j}) = \mathfrak{D}_c(\Phi_{f_j}) = \mathfrak{D}_{es}(\Phi_{f_j}) = M, \quad j = 1, 2.$$

Combined with Proposition 2.3 of this paper, the proof of that theorem also shows that $\mathfrak{D}^0(\Phi_{f_j}) = M$. Since $f_2(s) \leqslant f(s) \leqslant f_1(s)$ for all large s, it follows from Theorems 1.1 and 1.2 that

$$\mathfrak{D}^0(\Phi_{f_1}) \subset \mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}^0(\Phi_{f_2}), \quad \mathfrak{D}_{es}(\Phi_{f_1}) \subset \mathfrak{D}_{es}(\Phi_f) \subset \mathfrak{D}_{es}(\Phi_{f_2}).$$

Thus $\mathfrak{D}^0(\Phi_f) = \mathfrak{D}_{es}(\Phi_f) = M$. Using (1.6), we also have $\mathfrak{D}(\Phi_f) = \mathfrak{D}_c(\Phi_f) = M$. ∎

Theorem 5.15 of [10] deals with a function $f(s)$ such that

$$f(s) \asymp s^\beta \exp(-cs^\alpha) \quad \text{as } s \to \infty$$

with $\alpha > 0$, $\beta \in \boldsymbol{R}$, and $c > 0$. This function satisfies (4.1). Thus, if we show Theorem 5.15 of [10] only for $f(s) = \exp(-cs^\alpha)$, then the proof of our Proposition 4.3 is obtained and the rest of Theorem 5.15 of [10] is a consequence of our Proposition 4.3.

EXAMPLE 4.4. Let f be as in Proposition 4.3. If $f_2(s)$ and $f_3(s)$ are measurable and satisfy $|f_2(s)| \leqslant |f(s)| \leqslant |f_3(s)|$, then $\mathfrak{D}(\Phi_{f_3}) \subset \mathfrak{D}(\Phi_f) \subset \mathfrak{D}(\Phi_{f_2})$. Use Propositions 4.1 and 4.3.

Let $L_0(\boldsymbol{R}^d)$ be the class of selfdecomposable distributions on $\boldsymbol{R}^d$ and let $L_m(\boldsymbol{R}^d)$, $m = 1, 2, \ldots$, be the nested subclasses of $L_0(\boldsymbol{R}^d)$ studied by Urbanik [12], [13] and Sato [7]. The stochastic integral representation of $L_0(\boldsymbol{R}^d)$ given by Wolfe [15], [16], Jurek and Vervaat [4], and others is of the form Φ_f with $f(s) = e^{-s}$. Further, the representation of $L_m(\boldsymbol{R}^d)$ for $m = 1, 2, \ldots$ given by

Jurek [3] can be rewritten in the form Φ_f with $f(s) = \exp(-cs^{1/(m+1)})$. Hence we can apply Proposition 4.3 to those cases. Further applications related to [1] are in progress.

REFERENCES

[1] O. E. Barndorff-Nielsen, M. Maejima and K. Sato, *Some classes of infinitely divisible distributions admitting stochastic integral representations*, Bernoulli 12 (2006), pp. 1–33.

[2] A. Cherny and A. Shiryaev, *On stochastic integrals up to infinity and predictable criteria for integrability*, Lecture Notes in Math. No 1857, Springer, 2005, pp. 165–185.

[3] Z. J. Jurek, *The class $L_m(Q)$ of probability measures on Banach spaces*, Bull. Polish Acad. Sci. Math. 31 (1983), pp. 51–62.

[4] Z. J. Jurek and W. Vervaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrsch. Verw. Gebiete 62 (1983), pp. 247–262.

[5] S. Kwapień and W. A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple*, Birkhäuser, Boston 1992.

[6] B. Rajput and J. Rosinski, *Spectral representations of infinitely divisible processes*, Probab. Theory Related Fields 82 (1989), pp. 451–487.

[7] K. Sato, *Class L of multivariate distributions and its subclasses*, J. Multivariate Anal. 10 (1980), pp. 207–232.

[8] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge 1999.

[9] K. Sato, *Stochastic integrals in additive processes and application to semi-Lévy processes*, Osaka J. Math. 41 (2004), pp. 211–236.

[10] K. Sato, *Additive processes and stochastic integrals*, Illinois J. Math. 50 (2006) (Doob Volume), pp. 825–851.

[11] K. Sato, *Two families of improper stochastic integrals with respect to Lévy processes*, Alea, Latin American Journal of Probability and Mathematical Statistics 1 (2006), pp. 47–87. http://alea.impa.br/english/

[12] K. Urbanik, *Slowly varying sequences of random variables*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 20 (1972), pp. 679–682.

[13] K. Urbanik, *Limit laws for sequences of normed sums satisfying some stability conditions*, in: *Multivariate Analysis – III*, P. R. Krishnaiah (Ed.), Academic Press, New York, 1973, pp. 225–237.

[14] K. Urbanik and W. A. Woyczyński, *Random integrals and Orlicz spaces*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 15 (1967), pp. 161–169.

[15] S. J. Wolfe, *A characterization of certain stochastic integrals* in: *Tenth Conference on Stochastic Processes and Their Applications. Contributed Papers*, Stochastic Process. Appl. 12 (1982), p. 136.

[16] S. J. Wolfe, *On a continuous analogue of the stochastic difference equation $X_n = \varrho X_{n-1} + B_n$*, Stochastic Process. Appl. 12 (1982), pp. 301–312.

Ken-iti Sato
Hachiman-yama 1101-5-103
Tenpaku-ku
Nagoya, 468-0074 Japan
E-mail: ken-iti.sato@nifty.ne.jp

Received on 28.12.2005