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CONSTRUCTIONS OF CONTENTS AND MEASURES SATISFYING A PRESCRIBED SET OF INTEGRAL INEQUALITIES

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Abstract. Let Φ be a given set of real-valued functions on the set T and let $\beta: \Phi \to \bar{R}$ be a given functional with values in the extended real line $\bar{R} = [-\infty, \infty]$. The objective of the paper is to construct contents or measures μ with good regularity and smoothness properties satisfying the integral inequalities $\int^* \phi d\mu \leqslant \beta(\phi)$, where $\int^*$ denotes the upper integral.

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1. INTRODUCTION

Throughout this paper, let T denote a fixed non-empty set and if M is a given set, let 2^M and $2^{(M)}$ denote the set of all subsets of M and the set of all non-empty finite subsets of M, respectively, and let M^T denote the set of all functions from T into M. Moreover, let $N := \{1, 2, \ldots\}$ denote the set of all positive integers, $R := (-\infty, \infty)$ and $R_+ := [0, \infty)$ denote the *real line* and the *non-negative real line*, and $\bar{R} := [-\infty, \infty]$ and $\bar{R}_+ := [0, \infty]$ denote the *extended real line* and the *extended non-negative real line*, respectively.

The objective of this paper is to construct contents or measures on T satisfying a prescribed set of inequalities. More precisely, let $\Phi \subseteq R^T$ be a non-empty set and let $\beta: \Phi \to \bar{R}$ be a given functional on Φ. If $\Psi \subseteq \Phi$ is a given set and μ is a content on some algebra on T, we say that μ is a *lower Ψ-representation of β* if

$$\int^* \psi d\mu \leqslant \beta(\psi) \quad \text{for all } \psi \in \Psi,$$

where $\int^* f d\mu$ and $\int_* f d\mu$ denote the upper and lower μ-integrals of f whenever $f \in \bar{R}^T$ (see [6], Definition 2.4.2, pp. 329–330). The objective of this paper is to construct lower Ψ-representations μ of a given functional $\beta: \Phi \to \bar{R}$, where

$\Psi \subseteq \Phi$ is as large as possible and the content μ has good smoothness and/or regularity properties (for instance, a measure or a Radon measure). Lower representations of functionals have many applications to Riesz representations of linear functionals (see [1] and [2]), the moment problem, integral representations of kernels, construction of probability measures with given marginals satisfying a described set of integral inequalities (see [4]), and stochastic ordering.

In Theorem 3.4, we shall see that the existence of a lower Ψ-representation of β implies that β is non-negative Ψ-definite (see Section 3) and that the general lower representation problem reduces to the problem of finding lower representations of a certain increasing sublinear functional (the *positive Daniel functional* associated with β). In Theorems 4.2 and 4.3, we shall see that non-negative Ψ-definiteness implies the existence of contents or measures or Radon measures representing β on a reasonably large subset of Φ. In this section, we shall introduce some preliminary notions and results concerning sets of subsets and function spaces. In Section 2, we shall introduce some preliminary notions and results concerning set functions, and in Section 3 some preliminary notions and results concerning functionals are presented.

We extend the multiplication on $\boldsymbol{R}$ to $\bar{\boldsymbol{R}}$ as follows:

$$0\cdot(\pm\infty):=0, \quad a\cdot(\pm\infty):=\pm\infty \quad \text{if } 0<a\leqslant\infty$$

and

$$a\cdot(\pm\infty):=\mp\infty \quad \text{if } -\infty\leqslant a<0,$$

and we define

$$\frac{x}{y}:=x\cdot\frac{1}{y}, \quad \text{where } \frac{1}{0}:=\infty \text{ and } \frac{1}{\pm\infty}:=0.$$

Let $\dot{+}$ and $\underset{\cdot}{+}$ denote the following *upper* and *lower extensions* of the addition on $\boldsymbol{R}$:

$$a\dot{+}\infty = a\underset{\cdot}{+}\infty = (-\infty)\dot{+}\infty = \infty \quad \text{for all } -\infty<a\leqslant\infty,$$
$$b\dot{+}(-\infty) = b\underset{\cdot}{+}(-\infty) = (-\infty)\underset{\cdot}{+}\infty = -\infty \quad \text{for all } -\infty\leqslant b<\infty,$$

and we define

$$a\dot{-}b := a\dot{+}(-b) \quad \text{and} \quad a\underset{\cdot}{-}b := a\underset{\cdot}{+}(-b).$$

Note that $a\dot{+}b = a\underset{\cdot}{+}b$ if and only if $(a, b) \neq \pm(\infty, -\infty)$, and if so, we write $a+b := a\dot{+}b = a\underset{\cdot}{+}b$. Note that $a\dot{-}b = a\underset{\cdot}{-}b$ if and only if $(a, b)\neq\pm(\infty,\infty)$, and if so, we write $a-b := a\dot{-}b = a\underset{\cdot}{-}b$. If $(a_i)_{i\in I}\subseteq\bar{\boldsymbol{R}}_+$ is a family of extended non-negative numbers, we define

$$\sum_{i\in I} a_i := 0 \text{ if } I=\emptyset \quad \text{and} \quad \sum_{i\in I} a_i := \sup_{\pi\in 2^{(I)}} \sum_{i\in\pi} a_i \text{ if } I\neq\emptyset.$$

If $(b_i)_{i\in I}\subseteq\bar{\boldsymbol{R}}$, we define the *upper* and *lower sums* as follows:

$$\sum_{i\in I}^{*} b_i = \sum_{i\in I} b_i^+ \dot{-} \sum_{i\in I} b_i^-, \quad \sum_{i\in I}^{\circ} b_i = \sum_{i\in I} b_i^+ \underset{\cdot}{-} \sum_{i\in I} b_i^-.$$

We say that $(b_i)_{i\in I}$ is *summable* if $\sum^*_{i\in I} b_i = \sum^\circ_{i\in I} b_i$, and if so, we define

$$\sum_{i\in I} b_i := \sum^*_{i\in I} b_i = \sum^\circ_{i\in I} b_i.$$

Note that $(b_i)_{i\in I}$ is *summable* if and only if either $\sum_{i\in I} b_i^+ < \infty$ or $\sum_{i\in I} b_i^- < \infty$ and observe that

$$\sum^*_{1\leqslant i\leqslant n} b_i = b_1 \dot{+} \dots \dot{+} b_n \quad \text{and} \quad \sum^\circ_{1\leqslant i\leqslant n} b_i = b_1 \underset{\cdot}{+} \dots \underset{\cdot}{+} b_n.$$

Let $\mathscr{H} \subseteq 2^T$ be a given set. Then we say that $\mathscr{H}$ is *hereditary* if $\emptyset \in \mathscr{H}$ and $2^H \subseteq \mathscr{H}$ for all $H\in\mathscr{H}$, we say that $\mathscr{H}$ is *upwards directed* if $\mathscr{H} \neq \emptyset$ and for all $H_1, H_2 \in \mathscr{H}$ there exists $H\in\mathscr{H}$ satisfying $H \supseteq H_1 \cup H_2$, and we define

$$\bigcup \mathscr{H} := \bigcup_{H\in\mathscr{H}} H$$

and

$$\mathscr{H}^\diamond = \{A\subseteq T \mid \exists H \in \mathscr{H}\cup\{\emptyset\} \text{ so that } H \supseteq A\}, \qquad \mathscr{H}_\mathrm{C} = \{T\setminus H \mid H\in\mathscr{H}\},$$
$$\mathscr{F}(\mathscr{H}) = \{F\subseteq T \mid H\cap F\in\mathscr{H} \ \forall H\in\mathscr{H}\}, \qquad \mathscr{G}(\mathscr{H}) = \mathscr{F}(\mathscr{H})_\mathrm{C}.$$

Note that $\mathscr{H}^\diamond$ is the smallest hereditary set containing $\mathscr{H}$ and that $G\in\mathscr{G}(\mathscr{H})$ if and only if $H\setminus G\in\mathscr{H}$ for all $H\in\mathscr{H}$.

For a non-empty set I, we say that $\mathscr{H}$ is $(\bigcup I)$-*stable* if $\bigcup_{i\in I} H_i \in \mathscr{H}$ for every family $(H_i)_{i\in I} \subseteq \mathscr{H}$, and we denote by $\mathscr{H}_{(\cup I)}$ the smallest $(\bigcup I)$-stable set containing $\mathscr{H}$. We define $(\bigcap I)$-*stability* and $\mathscr{H}_{(\cap I)}$ similarly, and if I and J are non-empty sets, we denote by $\mathscr{H}_{(\cup I, \cap J)}$ the smallest $(\bigcup I, \bigcap J)$-stable set containing $\mathscr{H}$. We write

$$(\textstyle\bigcup f) := (\bigcup\{1,2\}) \quad \text{and} \quad (\bigcap f) := (\bigcap\{1,2\})$$

and

$$(\textstyle\bigcup c) := (\bigcup N) \quad \text{and} \quad (\bigcap c) := (\bigcap N).$$

We say that $\mathscr{H}$ is $(\bigcup \tau)$-*stable* or $(\bigcap\tau)$-*stable* if $\mathscr{H}$ is $(\bigcup I)$-stable or $(\bigcap I)$-stable for every non-empty set I and we denote by $\mathscr{H}_{(\cup\tau)}$ and $\mathscr{H}_{(\cap\tau)}$ the smallest $(\bigcup\tau)$-stable and $(\bigcap\tau)$-stable set containing $\mathscr{H}$, respectively. We say that $\mathscr{H}$ is a *set lattice* if $\mathscr{H}$ is a $(\bigcup f, \bigcap f)$-stable set containing $\emptyset$.

If $\Phi \subseteq \bar{R}^T$, we say that Φ is:

(i) *hereditary* if $\{f\in R^T \mid |f| \leqslant \phi\} \subseteq \Phi$ for all $\phi\in\Phi$;

(ii) *upper hereditary* if $\{f\in R^T \mid f \geqslant \phi\} \subseteq \Phi$ for all $\phi\in\Phi$;

(iii) a *positive cone* if $0\in\Phi$ and $a\phi\in\Phi$ for all $0 < a < \infty$ and all $\phi\in\Phi$;

(iv) a *cone* if Φ is a positive cone containing 0;

(v) a *convex cone* if Φ is a cone satisfying $\phi \dot{+} \psi \in \Phi$;

(vi) a *function algebra* if Φ is a linear subspace of R^T satisfying $\phi\cdot\psi\in\Phi$ for all $\phi, \psi\in\Phi$;

(vii) *rectilinear* if there exists a linear preordering $\leqslant$ on T such that ϕ is increasing on $(T, \leqslant)$ for all $\phi\in\Phi$.

We denote by $B(T)$ the set of all bounded real-valued functions on T and we define

$$\Phi^+ = \{\phi \in \Phi \mid \phi \geqslant 0\}, \quad \Phi^- = \{\phi \in \Phi \mid \phi \leqslant 0\}, \quad -\Phi = \{-\phi \mid \phi \in \Phi\},$$

$$\Phi_+ = \{f \in \boldsymbol{R}^T \mid f^+ \in \Phi\}, \quad \Phi_- := \{f \in \boldsymbol{R}^T \mid f^- \in \Phi\}.$$

If $\mathscr{H} \subseteq 2^T$ is a non-empty set, we define

$$W(T, \mathscr{H}) = \{f \in \boldsymbol{R}^T \mid \forall x < y \ \exists H \in \mathscr{H} \cup \{\emptyset, T\}\colon \{f > y\} \subseteq H \subseteq \{f > x\}\},$$

$$o(T, \mathscr{H}) = \{f \in \boldsymbol{R}^T \mid \{|f| > \delta\} \in \mathscr{H}^\diamond \ \forall \delta > 0\}.$$

Let T be a topological space. Then we denote by $\mathscr{G}(T)$, $\mathscr{F}(T)$ and $\mathscr{K}(T)$ the set of all open, all closed and all closed compact subsets of T, respectively, and by $\mathscr{B}(T)$ we denote the *Borel σ-algebra* on T; that is, the σ-algebra generated by $\mathscr{F}(T)$. Note that $\mathrm{Lsc}(T) := W(T, \mathscr{G}(T))$ and $\mathrm{Usc}(T) := W(T, \mathscr{F}(T))$ are the set of all lower and upper semicontinuous real-valued functions on T, and that $C(T) := \mathrm{Lsc}(T) \cap \mathrm{Usc}(T)$ is the set of all continuous real-valued functions on T.

1.1. LEMMA. *Let $\mathscr{H}, \mathscr{L} \subseteq 2^T$ be non-empty sets and let us define $\mathscr{K} := \mathscr{H} \cap \mathscr{L}^\diamond$. Then $\mathscr{F}(\mathscr{H})$ is a $(\bigcap f)$-stable set containing T, and $\mathscr{G}(\mathscr{H})$ is a $(\bigcup f)$-stable set containing $\emptyset$ and we have*:

(1) *If $A \subseteq T$ is a given set satisfying $A \supseteq \bigcup \mathscr{H}$, then $A \in \mathscr{F}(\mathscr{H})$ and we have $A \in \mathscr{G}(\mathscr{H})$ if and only if $\emptyset \in \mathscr{H}$.*

(2) *$\mathscr{H} \subseteq \mathscr{F}(\mathscr{H})$ if and only if $\mathscr{H}$ is $(\bigcap f)$-stable, and if I and J are non-empty sets, then we have*:

(a) *If $\mathscr{H}$ is $(\bigcup I)$-stable or $(\bigcap I)$-stable, then so is $\mathscr{F}(\mathscr{H})$.*

(b) *$\mathscr{F}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{H}_{(\cup I, \cap J)}) = \{F \subseteq T \mid F \cap H \in \mathscr{H}_{(\cup I, \cap J)}$ for all $H \in \mathscr{H}\}$.*

(3) *$\mathscr{F}(\mathscr{L}^\diamond) = 2^T$ and $\mathscr{F}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{K})$ and we have $\mathscr{H} = \mathscr{J} \cap \mathscr{H}^\diamond$ for every set $\mathscr{J} \subseteq 2^T$ satisfying $\mathscr{H} \subseteq \mathscr{J} \subseteq \mathscr{F}(\mathscr{H})$.*

(4) *$\bigcup \mathscr{K} \subseteq \bigcup \mathscr{L} = \bigcup \mathscr{L}^\diamond = \bigcup \mathscr{L}_{(\cup\tau)} = \{t \in T \mid \{t\} \in \mathscr{L}^\diamond\}$.*

(5) *Suppose that $\mathscr{H}$ is a set lattice. Then $\mathscr{F}(\mathscr{H})$ and $\mathscr{G}(\mathscr{H})$ are set lattices containing $\{\emptyset, T\}$ and if $\mathscr{L}$ is upwards directed, then $\mathscr{L}^\diamond$ and $\mathscr{K}$ are set lattices satisfying $\mathscr{K} \subseteq \mathscr{H} \subseteq \mathscr{F}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{K})$.*

(6) *$o(T, \mathscr{H})$ is a $\|\cdot\|_T$-closed hereditary cone and $o(T, \mathscr{H})$ is a linear space if and only if $\mathscr{H}$ is upwards directed.*

(7) *$o(T, \mathscr{H}) = \{f \in \boldsymbol{R}^T \mid \lim_{t \uparrow \mathscr{H}} |f(t)| = 0\}$, $\mathscr{H}^\diamond = \{A \subseteq T \mid 1_A \in o(T, \mathscr{H})\}$.*

Proof. (1) and (2). By the definition of $\mathscr{F}(\mathscr{K})$, we infer that $\mathscr{F}(\mathscr{H})$ is a $(\bigcap f)$-stable set containing T, and since $\mathscr{G}(\mathscr{K}) = \mathscr{F}(\mathscr{H})_{\mathrm{c}}$, we see that $\mathscr{G}(\mathscr{H})$ is a $(\bigcup f)$-stable set containing $\emptyset$. Let $A \supseteq \bigcup \mathscr{H}$ be given. Since $H \cap A = H$ for all $H \in \mathscr{H}$, we have $A \in \mathscr{F}(\mathscr{H})$, and since $\mathscr{H} \neq \emptyset$ and $H \setminus A = \emptyset$ for all $H \in \mathscr{H}$, we obtain $A \in \mathscr{G}(\mathscr{K})$ if and only if $\emptyset \in \mathscr{H}$, which proves (1), and (2) is evident.

(3) Since $\mathscr{L}^\diamond$ is hereditary, we have $\mathscr{F}(\mathscr{L}^\diamond) = 2^T$ and $\mathscr{F}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{K})$. So let $\mathscr{J} \subseteq 2^T$ be a given set satisfying $\mathscr{H} \subseteq \mathscr{J} \subseteq \mathscr{F}(\mathscr{H})$ and let $A \in \mathscr{J} \cap \mathscr{H}^\diamond$ be given. Since $\mathscr{H} \neq \emptyset$, there exists $H \in \mathscr{H}$ such that $A \subseteq H$, and since $A \in \mathscr{J} \subseteq \mathscr{F}(\mathscr{H})$, we have $A = A \cap H \in \mathscr{H}$. Hence $\mathscr{J} \cap \mathscr{H}^\diamond \subseteq \mathscr{H}$, and since $\mathscr{H} \subseteq \mathscr{J}$, the converse inclusion holds.

(5) follows from (1)–(3), and (4), (6) and (7) are evident. ∎

1.2. LEMMA. *Let $\mathscr{H} \subseteq 2^T$ be a given set, let us define $\mathscr{H}^* := \mathscr{H} \cup \{\emptyset, T\}$, and let $\Sigma(T, \mathscr{H})$ denote the set of all functions $f \in \mathbf{R}^T$ of the form*

$$f = a 1_T + b \sum_{i=1}^{n} 1_{H_i}$$

for some $a \in \mathbf{R}$, some $b \in \mathbf{R}_+$ and some $H_1, \ldots, H_n \in \mathscr{H}$ satisfying $H_1 \subseteq \ldots \subseteq H_n$. Then we have:

(1) *$W(T, \mathscr{H})$ is a $\|\cdot\|_T$-closed set and $W(T, \mathscr{H}_c) = -W(T, \mathscr{H})$.*

(2) *$B(T) \cap W(T, \mathscr{H}) = \mathrm{cl}_T \Sigma(T, \mathscr{H})$ and $1_A \in W(T, \mathscr{H})$ if and only if $A \in \mathscr{H}^*$.*

(3) *If $\varphi: \mathbf{R} \to \mathbf{R}$ is an increasing, continuous function, then $\varphi(f) \in W(T, \mathscr{H})$ for all $f \in W(T, \mathscr{H})$. In particular, $W(T, \mathscr{H})$ is a cone containing $a 1_T$, $a 1_T + f$, $f \wedge a$ and $f \vee a$ for all $a \in \mathbf{R}$ and all $f \in W(T, \mathscr{H})$.*

(4) *$\{f > a\} \in \mathscr{H}^*_{(\cup c)}$ and $\{f \geqslant a\} \in \mathscr{H}^*_{(\cap c)}$ for all $f \in W(T, \mathscr{H})$ and for all $a \in \mathbf{R}$.*

(5) *If $\mathscr{H}$ is $(\bigcup I)$-stable, then $\sup_{i \in I} f_i \in W(T, \mathscr{H})$ for every family $(f_i)_{i \in I} \subseteq W(T, \mathscr{H})$ satisfying $\sup_{i \in I} f_i(t) < \infty$ for all $t \in T$.*

(6) *If $\mathscr{H}$ is $(\bigcap I)$-stable, then $\inf_{i \in I} f_i \in W(T, \mathscr{H})$ for every family $(f_i)_{i \in I} \subseteq W(T, \mathscr{H})$ satisfying $\inf_{i \in I} f_i(t) > -\infty$ for all $t \in T$.*

(7) *If $(f_i)_{i \in I} \subseteq W(T, \mathscr{H})$ is a rectilinear family satisfying $\sum_{i \in I} |f_i(t)| < \infty$ for all $t \in T$, then $\sum_{i \in I} f_i \in W(T, \mathscr{H})$.*

(8) *If $\Phi \subseteq \mathbf{R}^T$ is a hereditary linear space, then Φ_- is an upper hereditary convex cone containing $\mathbf{R}^T_+$ and Φ_+ is a convex cone satisfying $\Phi_+ = -\Phi_-$ and we have*

$$\{f \in B(T) \mid 1_{\{f \neq 0\}} \in \Phi\} \subseteq \Phi = \Phi_+ \cap \Phi_-.$$

Proof. (1)–(7) follow from [6], Section 1.2, pp. 247–255, and (8) is an easy consequence of the definition of hereditariness. ∎

1.3. LEMMA. *Let $\mathscr{H} \subseteq 2^T$ be a set lattice and let $S(T, \mathscr{H})$ and $S_0(T, \mathscr{H})$ denote the linear space and the convex cone, respectively, generated by $\{1_H \mid H \in \mathscr{H} \cup \{T\}\}$. Then $W(T, \mathscr{H})$ is a cone containing $f \wedge h$ and $f \vee h$ for all $f, h \in W(T, \mathscr{H})$ and we have*

$$\mathrm{cl}_T S_0(T, \mathscr{H}) = B(T) \cap W(T, \mathscr{H}).$$

Let $\phi\colon \mathbf{R}^2 \to \mathbf{R}$ be uniformly continuous and increasing in each variable separately, let $k \geqslant 1$ be an integer and let $\psi\colon \mathbf{R}^k \to \mathbf{R}$ be continuous and increasing in each variable separately. Then we have:

(1) *$\phi(f, h) \in W(T, \mathscr{H})$ for all $f, h \in W(T, \mathscr{H})$ satisfying $\inf_{t\in T} f(t) > -\infty$ and $\inf_{t\in T} h(t) > -\infty$. In particular, $W^+(T, \mathscr{H})$ is a convex cone.*

(2) *If $\mathscr{H}$ is either $(\bigcup c)$-stable or $(\bigcap c)$-stable, then $W(T, \mathscr{H})$ is a convex cone and we have $\psi(f_1, \ldots, f_k) \in W(T, \mathscr{H})$ for all $f_1, \ldots, f_k \in W(T, \mathscr{H})$.*

(3) *If $\mathscr{H}$ is an algebra, then $W(T, \mathscr{H}) = -W(T, \mathscr{H})$ and $B(T) \cap W(T, \mathscr{H})$ is a $\|\cdot\|_T$-closed function algebra satisfying $B(T) \cap W(T, \mathscr{H}) = \mathrm{cl}_T S(T, \mathscr{H})$.*

(4) *If $\mathscr{H}$ is a σ-algebra, then $W(T, \mathscr{H})$ is the set of all real-valued $\mathscr{H}$-measurable functions.*

Proof. Let $f, h \in W(T, \mathscr{H})$ be given functions satisfying

$$a := \inf_{t\in T} f(t) > -\infty \quad \text{and} \quad b := \inf_{t\in T} h(t) > -\infty$$

and let us define

$$\mathscr{H}^* := \mathscr{H} \cup \{T\} \quad \text{and} \quad G(t) := \phi\big(f(t), h(t)\big)$$

for all $t \in T$. By Lemma 1.2 (3), we infer that $f - a1_T$ and $h - b1_T$ belong to $W^+(T, \mathscr{H})$, and since $G(t) = \phi_0\big(f(t)-a, h(t)-b\big)$, where $\phi_0(x, y) = \phi(x+a, y+b)$, we may without loss of generality assume that $f, h \in W^+(T, \mathscr{H})$. Let $\varepsilon > 0$ be given. Since ϕ is uniformly continuous, there exists $\delta > 0$ such that $|\phi(x, y) - \phi(u, v)| < \varepsilon$ for all $(x, y), (u, v) \in \mathbf{R}^2$ satisfying $|x-u| \leqslant 2\delta$ and $|y-v| \leqslant 2\delta$. Since $f \in W(T, \mathscr{H})$, there exist $F_0, F_1, \ldots \in \mathscr{H}^*$ satisfying

$$\{f \geqslant \delta(n+1)\} \subseteq F_n \subseteq \{f \geqslant \delta n\} \text{ for all } n \geqslant 0 \quad \text{and} \quad F_0 = T.$$

Then (F_n) is decreasing and $F_n \downarrow \emptyset$, and if we define $D_n := F_n \setminus F_{n+1}$ for all $n \geqslant 0$ and $f_\delta := \sum_{n=0}^{\infty} \delta n 1_{D_n}$, we see that (D_n) is a partition of T satisfying

$$F_n = \bigcup_{k\geqslant n} D_n \in \mathscr{H}^* \quad \text{for all } n \geqslant 0$$

and

$$0 \leqslant f_\delta = \delta \sum_{n=0}^{\infty} 1_{F_n} \leqslant f \leqslant f + 2\delta.$$

So by Lemma 1.2 (2) and (3) we have $\delta 1_{F_n} \in W(T, \mathscr{H})$, and since (F_n) is decreasing, it follows that $(\delta 1_{F_n} \mid n \geqslant 0)$ is rectilinear. Hence, by Lemma 1.2 (7), we have $f_\delta \in W(T, \mathscr{H})$. In the same manner, we see that there exists a disjoint partition (E_n) of T, satisfying

$$H_n := \bigcup_{k\geqslant n} E_n \in \mathscr{H}^* \text{ for all } n \geqslant 0 \quad \text{and} \quad h_\delta := \sum_{n=0}^{\infty} \delta n 1_{E_n} \in W(T, \mathscr{H})$$

and we obtain

$$0 \leqslant h_\delta = \delta \sum_{n=0}^{\infty} 1_{H_n} \leqslant h \leqslant h+2\delta.$$

Let us define $G_\delta := \phi(f_\delta, h_\delta)$ and let $x \in \mathbf{R}$ be given. Then we claim that there exists a finite set $\pi \subseteq \mathbf{N}_0^2$ satisfying

$$\text{(i)} \qquad \{G_\delta > x\} = \bigcup_{(n,m)\in\pi} F_n \cap H_m.$$

Proof of (i). If $\{G_\delta > x\} = \emptyset$, then (i) holds with $\pi = \emptyset$. So suppose that $\{G_\delta > x\} \neq \emptyset$ and let us define

$$\Lambda_x := \{(n, m) \in \mathbf{N}_0^2 \mid \phi(\delta n, \delta m) > x\}.$$

Since $(D_n \cap E_m)$ is a disjoint partition of T, we have

$$G_\delta = \sum_{n,m\geqslant 0} \phi(\delta n, \delta m) 1_{D_n \cap E_m},$$

and so we see that $\Lambda_x \neq \emptyset$ and

$$\{G_\delta > x\} = \bigcup_{(n,m)\in\Lambda_x} D_n \cap E_m.$$

Let us define

$$a_1 := \min\{n \mid \exists m \in \mathbf{N}_0\colon (n, m) \in \Lambda_x\}, \qquad a_2 := \min\{m \mid (a_1, m) \in \Lambda_x\},$$

$$b_2 := \min\{m \mid \exists\, n \in \mathbf{N}_0\colon (n, m) \in \Lambda_x\}, \qquad b_1 := \min\{n \mid (n, b_2) \in \Lambda_x\}.$$

Then $n \geqslant a_1$ and $m \geqslant b_2$ for all $(n, m) \in \Lambda_x$, and since $\Lambda_x \neq \emptyset$, we have $(a_1, a_2) \in \Lambda_x$ and $(b_1, b_2) \in \Lambda_x$. In particular, $a_1 \leqslant b_1$ and $b_2 \leqslant a_2$. Let π denote the set of all $(n, m) \in \Lambda_x$ satisfying $(n, m) \leqslant (b_1, a_2)$. Then π is a finite subset of Λ_x containing (a_1, a_2) and (b_1, b_2). Let A denote the set on the right-hand side of (i) with this choice of the set π. Let $t \in A$ be given. Then there exists $(n, m) \in \pi$ such that $t \in F_n \cap H_m$, and since $F_n = \bigcup_{k\geqslant n} D_k$ and $H_m = \bigcup_{k\geqslant m} E_k$, there exists a unique $(n', m') \in \mathbf{N}_0^2$ satisfying $(n', m') \geqslant (n, m)$ and $t \in D_{n'} \cap E_{m'}$. Moreover, since ϕ is increasing, we have $x < \phi(n, m) \leqslant \phi(n', m') = G_\delta(t)$. Hence $A \subseteq \{G_\delta > x\}$. Let $t \in \{G_\delta > x\}$ be given. Then there exists a unique $(n, m) \in \Lambda_x$ satisfying $t \in D_n \cap E_m$. If $(n, m) \in \pi$, then $t \in F_n \cap E_m \subseteq A$. So suppose that $(n, m) \notin \pi$. Then either $n > b_1$ or $m > a_2$, and we get $n \geqslant a_1$ and $m \geqslant a_2$. If $n > b_1$, we see that $(n, m) \geqslant (b_1, b_2) \in \pi$, and so $t \in D_n \cap E_m \subseteq F_{b_1} \cap H_{b_2} \subseteq A$. If $m > a_2$, we have $(n, m) \geqslant (a_1, a_2) \in \pi$, and so $t \in D_n \cap E_m \subseteq F_{a_1} \cap H_{a_2} \subseteq A$. Consequently, $\{G_\delta > x\} \subseteq A$, which completes the proof of (i).

Since $\mathscr{H}^*$ is a $(\bigcup f, \bigcap f)$-stable set containing F_n and H_m for all $n, m \geqslant 0$, it follows that $\{G_\delta > x\} \in \mathscr{H}^*$ for all $x \in \mathbf{R}$. In particular, we have $G_\delta \in W(T, \mathscr{H})$ and recall that $0 \leqslant f_\delta(t) \leqslant f(t) \leqslant f_\delta(t) + 2\delta$ and $0 \leqslant h_\delta(t) \leqslant h(t) \leqslant h_\delta(t) + 2\delta$ for all $t \in T$. Since ϕ is an increasing function satisfying $\phi(x+2\delta, y+2\delta) <$

$\phi(x, y)+\varepsilon$ for all $(x, y)\in \boldsymbol{R}_+^2$, we have

$$0 \leqslant G(t) \leqslant G_\delta(t) \leqslant \phi\big(f(t)+2\delta,\, h(t)+2\delta\big) \leqslant G(t)+\varepsilon.$$

Hence $\|G-G_\delta\|_T \leqslant \varepsilon$ and, by Lemma 1.2 (1), it follows that $W(T, \mathscr{H})$ is $\|\cdot\|_T$-closed. Since $G_\delta \in W(T, \mathscr{H})$, we have $G = \phi(f, h)\in W(T, \mathscr{H})$, which proves the last part of (3). Applying this to $\phi(x, y) := ax+by$, where $a, b\in \boldsymbol{R}_+$, we see that $W^+(T, \mathscr{H})$ is a convex cone. The remaining statements in the lemma follow easily from (1) and [6], Proposition 1.2.7, p. 254. ∎

1.4. Lemma. *Let T be a topological space and let $\mathscr{K} \subseteq 2^T$ be a non-empty set. Then we have:*

(1) *$\mathscr{K} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{K})$ if and only if $\mathscr{K} = \mathscr{K}(T)\cap \mathscr{K}^\diamond$, and if and only if $\mathscr{K} = \mathscr{C}\cap\mathscr{L}^\diamond$ for some sets $\mathscr{L}, \mathscr{C} \subseteq 2^T$ satisfying $\mathscr{C} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{C})$.*

If $\mathscr{K} = \mathscr{C}\cap\mathscr{L}^\diamond$ for some sets $\mathscr{L}, \mathscr{C} \subseteq 2^T$ satisfying $\mathscr{C} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{C})$, then $\mathscr{C}$ and $\mathscr{K}$ are $(\bigcap\tau)$-stable sets and we have:

(2) *$\bigcup\mathscr{K} \subseteq \bigcup\mathscr{L}$ and $\emptyset\in\mathscr{K} \subseteq \mathscr{C} \subseteq \mathscr{F}(T) \subseteq \mathscr{F}\big(\mathscr{K}(T)\big) \subseteq \mathscr{F}(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$.*

(3) *If $\mathscr{L}$ is upwards directed and $\mathscr{L} \subseteq \mathscr{G}(\mathscr{C})$, then*

(a) *$\mathscr{K} = \{K\in\mathscr{C} \mid K \subseteq \bigcup\mathscr{K}\} = \{K\in\mathscr{C} \mid K\subseteq \bigcup\mathscr{L}\}$.*

Proof. (1) Suppose that $\mathscr{K} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{K})$. By Lemma 1.1 (3), we have $\mathscr{K} = \mathscr{K}(T)\cap\mathscr{K}^\diamond$. Suppose that $\mathscr{K} = \mathscr{K}(T)\cap\mathscr{K}^\diamond$. Since $\mathscr{K}(T) \subseteq \mathscr{F}\big(\mathscr{K}(T)\big)$, we see that the last statement in (1) holds with $\mathscr{C} := \mathscr{K}(T)$ and $\mathscr{L} := \mathscr{K}^\diamond$. Suppose that $\mathscr{K} = \mathscr{C}\cap\mathscr{L}^\diamond$ for some non-empty sets $\mathscr{C}, \mathscr{L} \subseteq 2^T$ satisfying $\mathscr{C} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{C})$. By Lemma 1.1 (3), we have $\mathscr{F}(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$, and since $\mathscr{K} \subseteq \mathscr{C}$, we obtain $\mathscr{K} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{K})$, which completes the proof of (1).

(2) By (1), we have $\mathscr{K} = \mathscr{K}(T)\cap\mathscr{K}^\diamond$ and $\mathscr{C} = \mathscr{K}(T)\cap\mathscr{C}^\diamond$. Hence $\mathscr{K}$ and $\mathscr{C}$ are $(\bigcap\tau)$-stable sets containing $\emptyset$, and since $\mathscr{K}(T) \subseteq \mathscr{F}(T) \subseteq \mathscr{F}\big(\mathscr{K}(T)\big)$, we see that (2) follows from Lemma 1.1 (3) and (4).

(3) Let $K\in\mathscr{C}$ be a given set satisfying $K \subseteq \bigcup\mathscr{L}$. Then $\bigcap_{L\in\mathscr{L}}(K\backslash L) = \emptyset$, and since $\mathscr{L} \subseteq \mathscr{G}(\mathscr{C})$, we have $K\backslash L\in\mathscr{C} \subseteq \mathscr{K}(T)$ for all $L\in\mathscr{L}$. Since $\mathscr{L}$ is upwards directed, there exists $L_0\in\mathscr{L}$ such that $K\backslash L_0 = \emptyset$. Hence $K \subseteq L_0\in\mathscr{L}$, and since $K\in\mathscr{C}$, we have $K\in\mathscr{C}\cap\mathscr{L}^\diamond = \mathscr{K}$ for all $K\in\mathscr{C}$ satisfying $K\subseteq\bigcup\mathscr{L}$. Since $K \subseteq \bigcup\mathscr{K} \subseteq \bigcup\mathscr{L}$ for all $K\in\mathscr{K}$, we see that (3) holds. ∎

2. SET FUNCTIONS

Let $\mathscr{B}, \mathscr{H} \subseteq 2^T$ be non-empty sets and let $\beta\colon \mathscr{B}\to[0, \infty]$ be a set function. Then we define

$$\beta^*(A) = \inf_{B\in\mathscr{B}, B\supseteq A}\beta(B), \quad \beta_*(A) = \sup_{B\in\mathscr{B}, B\subseteq A}\beta(B) \quad \text{for all } A\subseteq T,$$

$$\beta^{\mathscr{H}}(A) = \inf_{H\in\mathscr{H}, H\supseteq A}\beta^*(H), \quad \beta_{\mathscr{H}}(A) = \sup_{H\in\mathscr{H}, H\subseteq A}\beta_*(H) \quad \text{for all } A\subseteq T,$$

$$\mathscr{H}_\beta^t = \{A\subseteq T \mid \inf_{H\in\mathscr{H}}\beta^*(A\backslash H) = 0\}$$

and we say that β is:

(i) *increasing* if $\beta(A) \leqslant \beta(B)$ for all $A, B \in \mathscr{B}$ satisfying $A \subseteq B$;

(ii) *inner $\mathscr{H}$-regular* if $\beta(B) = \beta_{\mathscr{H}}(B)$ for all $B \in \mathscr{B}$ or, equivalently, if $\beta_*(A) = \beta_{\mathscr{H}}(A)$ for all $A \subseteq T$;

(iii) *outer $\mathscr{H}$-regular* if $\beta(B) = \beta^{\mathscr{H}}(B)$ for all $B \in \mathscr{B}$ or, equivalently, if $\beta^*(A) = \beta^{\mathscr{H}}(A)$ for all $A \subseteq T$.

If $\mathscr{B}$ is a set lattice, we say that β is *supermodular* if $\beta(\emptyset) = 0$ and $\beta(A)+\beta(B) \leqslant \beta(A \cup B)+\beta(A \cap B)$ for all $A, B \subseteq T$.

If $\beta: 2^T \to [0, \infty]$ is a set function satisfying $\beta(\emptyset) = 0$, we denote by $\mathscr{M}_\beta$ the set of all *β-measurable sets* in the sense of Carathéodory (see [5], (1.22), p. 23); that is, $B \in \mathscr{M}_\beta$ if and only if $\beta(A) = \beta(A \cap B)+\beta(A \backslash B)$ for all $A \subseteq T$. Recall that $\mathscr{M}_\beta$ is an algebra such that the restriction of β to $\mathscr{M}_\beta$ is a content (see [5], (1.23), pp. 23–24).

Let $(T, \mathscr{B}, \mu)$ be a *content space*, that is, $\mathscr{B} \subseteq 2^T$ is an algebra and $\mu: \mathscr{B} \to [0, \infty]$ is a set function satisfying $\mu(\emptyset) = 0$ and $\mu(A \cup B) = \mu(A)+\mu(B)$ for all $A, B \in \mathscr{B}$ such that $A \cap B = \emptyset$. Then we define

$$\mathscr{B}^\circ := \{B \in \mathscr{B} \mid \mu(B) < \infty\} \quad \text{and} \quad \mu_\circ(A) = \mu_{\mathscr{B}^\circ}(A) \text{ for all } A \subseteq T.$$

We let $\mathscr{M}_\mu := \mathscr{M}_{\mu^*}$ denote the set of all *μ-measurable sets* and let $\bar{\mu}$ and $\bar{\mu}_\circ$ mean the restrictions of μ^* and $\mu_\circ$ to the algebra $\mathscr{M}_\mu$. We say that μ is *complete* if $\mathscr{B} = \mathscr{M}_\mu$, and we say that μ is *finitely founded* if $\mu(B) = \mu_\circ(B)$ for all $B \in \mathscr{B}$ or, equivalently, if $\mu_*(A) = \mu_\circ(A)$ for all $A \subseteq T$. If $\mathscr{B} \subseteq 2^T$ is an algebra, we denote by $M(T, \mathscr{B})$ the set of all contents on $(T, \mathscr{B})$, and by $M(T)$ the set of all contents defined on some algebra on T. We denote by $M_d(T, 2^T)$ the set of all *finite discrete measures* on $(T, 2^T)$, that is, the set of all $\mu \in M(T, 2^T)$ of the form $\mu(A) = \sum_{i=1}^n a_i 1_A(t_i)$ for all $A \subseteq T$ for some $t_1, \ldots, t_n \in T$ and some $a_1, \ldots, a_n \in \boldsymbol{R}_+$.

Let $\mu \in M(T)$ be a given content. Then we define $W(\mu) := W(T, \mathscr{M}_\mu)$ to be the set of all *weakly μ-measurable* functions and we denote by $L^1(\mu)$ the set of all *μ-integrable* functions $f \in \boldsymbol{R}^T$ (see [3], Definition III.2.17, p. 112). We denote by $L^*(\mu)$ the set of all $f \in \boldsymbol{R}^T$ satisfying $\int_* f d\mu = \int^* f d\mu$ and we define

$$\int_T f d\mu := \int_* f d\mu = \int^* f d\mu \quad \text{for all } f \in L^*(\mu).$$

Let $L(\mu) := W(\mu) \cap L^*(\mu)$ denote the set of all *μ-summable* functions.

Let $\mathscr{K} \subseteq 2^T$ be a set lattice. Then denote by $M_r(T, \mathscr{K})$ the set of all complete, inner $\mathscr{K}$-regular content spaces $(T, \mathscr{B}, \mu)$ satisfying $\mathscr{K} \subseteq \mathscr{B}$ and $\mu(K) < \infty$ for all $K \in \mathscr{K}$.

Let T be a topological space. Then we say that μ is a *Borel measure* if μ is a measure on some σ-algebra containing $\mathscr{B}(T)$ and we say that μ is a *Radon measure* if μ is an inner $\mathscr{K}(T)$-regular Borel measure. If $U \subseteq T$, we denote by $M_R(T \mid U)$ the set of all complete, finitely founded Radon measure spaces $(T, \mathscr{B}, \mu)$ satisfying $U \in \mathscr{B}$ and $\mu(T \backslash U) = 0$, and by $M_R^\circ(T \mid U)$ we denote the set of all $\mu \in M_R(T \mid U)$ which are *locally finite at U*; that is, $\mu^{\mathscr{G}(T)}(\{t\}) < \infty$ for all $t \in U$.

2.1. THEOREM. *Let $(T, \mathscr{B}, \mu)$ be a content space. Then $(T, M_\mu, \bar\mu)$ is a complete content space and $(T, \mathscr{M}_\mu, \bar\mu_\circ)$ is a complete, finitely founded content space satisfying*:

(1) $\mathscr{B} \subseteq \mathscr{M}_{\mu_*} \subseteq \mathscr{M}_\mu = \mathscr{M}_{\bar\mu} = \mathscr{M}_{\bar\mu_\circ} = \mathscr{M}_{\mu^*} = \mathscr{M}_{\mu_\circ}$.

(2) $\mu_\circ(A) = (\bar\mu_\circ)_*(A) \leqslant \mu_*(A) \leqslant \mu^*(A) = \bar\mu^*(A)$ *for all* $A \subseteq T$.

(3) $\bar\mu_\circ(B) = \mu_\circ(B) \leqslant \bar\mu(B) = \mu^*(B) = \mu_*(B) = \mu(B)$ *for all* $B \in \mathscr{B}$.

(4) $\int_* f\,d\mu = \int_* f\,d\bar\mu \leqslant \int_* f\,d\bar\mu_\circ \leqslant \int^* f\,d\bar\mu_\circ \leqslant \int^* f\,d\bar\mu = \int^* f\,d\mu$ *for all* $f \in \bar{R}^T$.

(5) $\int_* f\,d\mu = \int_* f^+\,d\mu \dot{-} \int^* f^-\,d\mu$ *and* $\int^* f\,d\mu = \int^* f^+\,d\mu \dot{-} \int_* f^-\,d\mu$ *for all* $f \in \bar{R}^T$.

(6) $\int^* f\,d\mu = \lim_{n\to-\infty} \int^* (f \vee n)\,d\mu$ *and* $\int_* f\,d\mu = \lim_{n\to\infty} \int_* (f \wedge n)\,d\mu$ *for all* $f \in \bar{R}^T$.

(7) $\int^* f\,d\mu = \int_0^\infty \mu^*(f > x)\,dx$ *and* $\int_* f\,d\mu = \int_0^\infty \mu_\circ(f > x)\,dx$ *for all* $f \in \bar{R}_+^T$.

(8) $L^1(\mu) = \{f \in L^*(\mu) \mid \int_T f\,d\mu \neq \pm\infty\} = \{f \in W(\mu) \mid \int^* |f|\,d\mu < \infty\}$.

(9) *If μ is finitely founded, then $\mathscr{M}_{\mu_*} = \mathscr{M}_\mu$ and we have $f \in L(\mu)$ if and only if $f \in W(\mu)$ and $\int_* f^+\,d\mu \wedge \int_* f^-\,d\mu < \infty$.*

Let $(\phi_i)_{i\in I} \subseteq \bar{R}_+^T$ be a family of non-negative functions and let us define $\phi_\pi(t) := \sum_{i\in\pi} \phi_i(t)$ for all $t \in T$ and all $\pi \subseteq I$. Then we have:

(10) $\sum_{i\in I} \int_* \phi_i\,d\mu \leqslant \int_* \phi_I\,d\mu$ *and* $\int^* \phi_\pi\,d\mu \leqslant \sum_{i\in\pi} \int^* \phi_i\,d\mu$ *for all* $\pi \in 2^{(I)}$.

(11) *Suppose that $(\phi_i)_{i\in I}$ is rectilinear. Then*

(a) $\sum_{i\in\pi} \int_* \phi_i\,d\mu = \int_* \phi_\pi\,d\mu$ *and* $\int^* \phi_\pi\,d\mu = \sum_{i\in\pi} \int^* \phi_i\,d\mu$ *for all* $\pi \in 2^{(I)}$, *and if* $\mu^*(\phi_I = \infty) = 0$, *then*

(b) $\sum_{i\in I} \int_* \phi_i\,d\mu = \int_* \phi_I\,d\mu$ *and* $\sum_{i\in I} \int^* \phi_i\,d\mu = \int^* \phi_I\,d\mu$.

Proof. (1)–(3) follow from [6], Lemma 2.3.4, p. 316; (4) and (5) follow from [6], Lemmas 2.4.3 and 2.4.4, pp. 330–331; (6) and (7) follow from [6], Theorem 2.4.6, p. 334; (8) and (9) follow from [6], Corollary 2.4.7, pp. 335–336; (10) follows from [6], Lemma 2.3.4, p. 316; and (11) follows from [6], Theorem 2.4.8, p. 336. ∎

2.2. THEOREM. *Let $(T, \mathscr{B}, \mu)$ be a content space. Let $\mathscr{K} \subseteq 2^T$ and $\mathscr{L} \subseteq \mathscr{M}_\mu$ be given sets. If*

$$\mathscr{B}^\circ := \{B \in \mathscr{B} \mid \mu(B) < \infty\} \quad \text{and} \quad \mathscr{L}^\circ := \{L \in \mathscr{L} \mid \mu^*(L) < \infty\},$$

then we have:

(1) *If μ is inner $\mathscr{K}$-regular, then $\emptyset \in \mathscr{K}$ and we have*:

(a) $\{B \subseteq T \mid B \cap K \in \mathscr{M}_{\mu_*}$ *for all* $K \in \mathscr{K}\} \subseteq \mathscr{M}_{\mu_*}$ *with equality if* $\mathscr{K} \subseteq \mathscr{M}_{\mu_*}$.

(b) *If* $\mathscr{K} \subseteq \mathscr{G}(\mathscr{L})$, *then* $\mu^*(A) = \mu^{\mathscr{L}}(A)$ *for all* $A \subseteq T$ *satisfying* $\mu^{\mathscr{L}}(A) < \infty$.

(c) $\mu \in M_r(T, \mathscr{K})$ *if and only if* $\mathscr{K} \subseteq \mathscr{B} = \mathscr{M}_{\mu_*}$ *and* $\mu(K) < \infty$ *for all* $K \in \mathscr{K}$.

(2) *If* $\mu^{\mathscr{L}}(B) < \infty$ *for all* $B \in \mathscr{B}^\circ$ *and* $f \in \boldsymbol{R}^T$ *is a given function, then the following three statements are equivalent*:

(a) $f \in W(\mu)$.

(b) *For every* $\varepsilon > 0$ *there exists* $h \in W(\mu)$ *so that* $\mu^*(|f-h| > \varepsilon) < \varepsilon$.

(c) *For every set* $L \in \mathscr{L}^\circ$ *there exists a countable set* $Q \subseteq \boldsymbol{R}$ *satisfying* $L \cap \{f > x\} \in \mathscr{M}_\mu$ *for all* $x \in \boldsymbol{R} \backslash Q$, *and for every* $\varepsilon > 0$ *there exists a function* $h \in W(\mu)$ *satisfying* $\mu^*(|f-h| > \varepsilon) < \infty$.

(3) *Suppose* $\mu(T) < \infty$. *If* $f \in \boldsymbol{R}^T$ *and* $\mathscr{G}$ *is a* $(\bigcap f)$*-stable set such that* μ *is outer* $\mathscr{G}$*-regular and* $f \leqslant \phi$ *for some* $\phi \in W(T, \mathscr{G})$, *then*

(a) $\int^* f d\mu = \inf\{\int^* h d\mu \mid h \in W(T, \mathscr{G}),\, h \geqslant f\}$.

Proof. (1) Since $\mu_{\mathscr{K}}(\emptyset) = \mu(\emptyset) = 0$, we see that $\emptyset \in \mathscr{K}$ and (1)(a) follows from [6], Lemma 2.3.3, p. 315. Suppose that $\mathscr{K} \subseteq \mathscr{G}(\mathscr{L})$. Let $A \subseteq T$ be a given set satisfying $\mu^{\mathscr{L}}(A) < \infty$ and let $\delta > 0$ be a given number. Since μ^* is increasing, we have $\mu^*(A) \leqslant \mu^{\mathscr{L}}(A) < \infty$ and there exists $L \in \mathscr{L}$ satisfying $L \supseteq A$ and $\mu^*(L) < \infty$. In particular, $\mu^*(L \backslash A) < \infty$, and since μ is inner $\mathscr{K}$-regular, there exists $K \in \mathscr{K}$ such that $K \subseteq L \backslash A$ and $\mu_*(L \backslash A) < \mu_*(K) + \delta$. Since $K \in \mathscr{K} \subseteq \mathscr{G}(\mathscr{L})$, we have $L_0 := L \backslash K \in \mathscr{L}$, and since $K \subseteq L \backslash A$, we obtain

$$A \subseteq L_0, \quad K \cap (L_0 \backslash A) = \emptyset \quad \text{and} \quad K \cup (L_0 \backslash A) = L \backslash A.$$

So, by superadditivity of μ_*, we get

$$\mu_*(L_0 \backslash A) + \mu_*(K) \leqslant \mu_*(L \backslash A) \leqslant \mu_*(K) + \delta,$$

and since $\mu_*(K) \leqslant \mu^*(L) < \infty$, we see that $\mu_*(L_0 \backslash A) < \delta$. Since $\mu^*(L_0) < \infty$ and $L_0 \in \mathscr{L} \subseteq \mathscr{M}_\mu$, we obtain $\mu_*(L_0) = \mu^*(L_0)$ (see [6], Lemma 2.3.4, p. 316), and so by [6], Definition 2.3.2, p. 315, we obtain

$$\mu^*(A) \leqslant \mu^{\mathscr{L}}(A) \leqslant \mu^*(L_0) = \mu_*(L_0) \leqslant \mu^*(A) + \mu_*(L_0 \backslash A) \leqslant \mu^*(A) + \delta.$$

Letting $\delta \downarrow 0$, we see that $\mu^*(A) = \mu^{\mathscr{L}}(A)$. Suppose that $\mu \in M_r(T, \mathscr{K})$. Then $\mathscr{K} \subseteq \mathscr{B} = \mathscr{M}_\mu$ and $\mu(K) < \infty$ for all $K \in \mathscr{K}$. Since μ is inner $\mathscr{K}$-regular, μ is finitely founded, and so by Theorem 2.1 (9) we have $\mathscr{K} \subseteq \mathscr{B} = \mathscr{M}_{\mu_*}$. Suppose that $\mathscr{K} \subseteq \mathscr{B} = \mathscr{M}_{\mu_*}$ and $\mu(K) < \infty$ for all $K \in \mathscr{K}$. Since μ is inner $\mathscr{K}$-regular, μ is finitely founded, and so by Theorem 2.1 (1) we have $\mathscr{B} = \mathscr{M}_\mu$. Hence μ is complete and $\mu = M_r(T, \mathscr{K})$, which completes the proof of (1).

(2)(a) $\Rightarrow$ (2)(b) is evident.

(2)(b) $\Rightarrow$ (2)(c). Suppose that (2)(b) holds. Then the last part of (2)(c) holds trivially. So let $L \in \mathscr{L}^\circ$ be given and let us define

$$R^L(x) = \mu^*(L \cap \{f > x\}) \quad \text{and} \quad R_L(x) = \mu_*(L \cap \{f > x\})$$

for all $x \in \boldsymbol{R}$. Then R^L and R_L are decreasing functions satisfying $0 \leqslant R_L(x) \leqslant R^L(x) \leqslant \mu^*(L) < \infty$ for all $x \in \boldsymbol{R}$. Let $\delta > 0$ be given. By (2)(b), there exists $h \in W(\mu)$ satisfying $\mu^*(|f-h| > \delta) < \delta$, and so there exists $B \in \mathscr{B}$ such that

$\mu(B) < \delta$ and $\{|f-h| > \delta\} \subseteq B$. Let $x \in \boldsymbol{R}$ be given. Since $h \in W(\mu)$, there exists $H \in \mathscr{M}_\mu$ satisfying $\{h > x+2\delta\} \subseteq H \subseteq \{h > x+\delta\}$. Since $|f(t)-h(t)| < \delta$ for all $t \in T \backslash B$, we have

$$\{f > x+3\delta\} \subseteq \{h > x+2\delta\} \cup B \subseteq H \cup B$$

and

$$H \subseteq \{h > x+2\delta\} \subseteq \{f > x\} \cup B.$$

Since $H, L \in \mathscr{M}_\mu$ and $\mu^*(L) < \infty$, we obtain $\bar{\mu}(L \cap H) = \mu_*(L \cap H)$ (see [6], Lemma 2.3.4, p. 316), and since $B \in \mathscr{B}$, we have

$$R^L(x+3\delta) \leqslant \bar{\mu}(L \cap H) + \mu(B) < \bar{\mu}(L \cap H) + \delta$$

and

$$\bar{\mu}(L \cap H) \leqslant R_G(x) + \mu(B) < R_L(x) + \delta.$$

Hence $R_L(x) \leqslant R^L(x) \leqslant R^L(x+3\delta) \leqslant R_L(x) + 2\delta$ for all $x \in \boldsymbol{R}$ and all $\delta > 0$. In particular, we see that $R_L(x) = R^L(x)$ for all $x \in \boldsymbol{R} \backslash Q$, where $Q \subseteq \boldsymbol{R}$ denotes the set of discontinuity points of R_L. Let $x \in \boldsymbol{R} \backslash Q$ be given. Then

$$\mu_*(L \cap \{f > x\}) = \mu^*(L \cap \{f > x\}) \leqslant \bar{\mu}(L) < \infty,$$

and so by [6], Lemma 2.3.4, p. 316, we have $L \cap \{f > x\} \in \mathscr{M}_\mu$, and since R_L is decreasing, Q is countable, which proves (2)(c).

(2)(c) $\Rightarrow$ (2)(a). Suppose that (2)(c) holds and let $x < y$ be given numbers. Let $\delta > 0$ be chosen such that $x+\delta < y-\delta$. By (2)(c), there exist $h \in W(\mu)$ and $B \in \mathscr{B}^\circ$ satisfying $\{|f-h| \geqslant \delta\} \subseteq B$, and since $\mu^{\mathscr{L}}(B) < \infty$, there exists $L \in \mathscr{L}^\circ$ such that $B \subseteq L$. So by (2)(c) there exists $u \in \boldsymbol{R}$ satisfying $x < u < y$ and $A_0 := L \cap \{f > u\} \in \mathscr{M}_\mu$. Since $h \in W(\mu)$ and $x+\delta < y-\delta$, there exists $H \in \mathscr{M}_\mu$ such that $\{h > y-\delta\} \subseteq H \subseteq \{h > x+\delta\}$, and since $x < u < y$ and $|f(t)-h(t)| < \delta$ for all $t \in A \backslash T$, we have

$$\begin{aligned}\{f > y\} &\subseteq (\{h > y-\delta\} \backslash L) \cup (\{f > y\} \cap L) \subseteq (H \backslash A) \cup A_0 \\ &\subseteq (\{h > x+\delta\} \backslash L) \cup \{f > u\} \subseteq \{f > x\}.\end{aligned}$$

Since $(H \backslash L) \cup A_0 \in \mathscr{M}_\mu$, we infer that $f \in W(\mu)$, which completes the proof of (2).

(3) Let $f \in \boldsymbol{R}^T$ be a given function satisfying $f \leqslant \phi$ for some $\phi \in W(T, \mathscr{G})$ and let m denote the infimum on the right-hand side of (3)(a). Then $\int^* f\,d\mu \leqslant m$. Hence we have the equality if $\int^* f\,d\mu = \infty$. So suppose that $\int^* f\,d\mu < \infty$ and let $c > \int^* f\,d\mu$ be a given number. By [6], Theorem 2.4.5, p. 331, there exists $a \in \boldsymbol{R}$ such that $\int^* (f \vee a)\,d\mu < c$, and since $\mu(T) < \infty$, there exists $\delta > 0$ such that

$$\delta(1+\mu(T)) + \int^* (f \vee a)\,d\mu < c.$$

Let us define $f_a := (f-a)^+$, $\psi := (\phi-a)^+$ and $f^a := \delta \sum_{i=0}^{\infty} 1_{A_i}$, where $A_i := \{f_a \geqslant i\delta\}$. Then $0 \leqslant f_a \leqslant f^a \leqslant f_a + \delta$ and, by Lemma 1.2 (2), we have $f_a \leqslant \psi \in W(T, \mathscr{G})$. Hence, there exists $G_i^* \in \mathscr{G}^* := \mathscr{G} \cup \{\emptyset, T\}$ such that

$$\{\psi \geqslant i\delta\} \subseteq G_i^* \subseteq \{\psi \geqslant (i-1)\delta\} \qquad \text{for all } i \geqslant 0.$$

Then (G_i^*) is decreasing, and since $f_a \leqslant \psi$ and ψ is finite, we obtain $A_i \subseteq G_i^*$ and $G_i^* \downarrow \emptyset$. Since μ is finite and outer $\mathscr{G}$-regular, there exists $G^i \in \mathscr{G}^*$ such that $G^i \supseteq A_i$ and $\mu^*(G^i) \leqslant 2^{-i-1} + \mu^*(A_i)$ for all $i \geqslant 0$. Let us define

$$G_i := \bigcap_{j=0}^{i} (G_i^* \cap G^j).$$

Since (A_i) and (G_i^*) are decreasing and $A_i \subseteq G_i^* \cap G^i$, we see that (G_i) is decreasing and $A_i \subseteq G_i \subseteq G_i^* \cap G^i$ for all $i \geqslant 0$. Hence

$$h := \delta \sum_{i=0}^{\infty} 1_{G_i} \geqslant f^a \quad \text{and} \quad \mu^*(G_i) < 2^{-i-1} + \mu^*(A_i),$$

and since $\mathscr{G}^*$ is $(\bigcap f)$-stable and $G_i \downarrow \emptyset$, we have $G_i \in \mathscr{G}^*$ for all $i \in I$ and $0 \leqslant h(t) < \infty$ for all $t \in T$. By Lemma 1.2 (2), it follows that $1_{G_i} \in W(T, \mathscr{G})$, and since (G_i) is decreasing, $(\delta 1_{G_i})$ is rectilinear. Hence, by Lemma 1.2 (2) and (7), we see that h and $h+a$ belong to $W(T, \mathscr{G})$. Note that $f \leqslant f \vee a = f_a + a \leqslant f^a + a \leqslant h + a$ and $f^a \leqslant f_a + \delta$. So by Theorem 2.1 (11) and rectilinearity of (1_{A_i}) and (1_{H_i}) we have

$$\begin{aligned}
\int^* f\,d\mu \leqslant m \leqslant \int^* (h+a)\,d\mu &= a\mu(T) + \int^* h\,d\mu = a\mu(T) + \delta \sum_{i=0}^{\infty} \mu^*(H_i) \\
&\leqslant a\mu(T) + \delta \sum_{i=0}^{\infty} \mu^*(A_i) + \delta \sum_{i=0}^{\infty} 2^{-i-1} \leqslant a\mu(T) + \delta + \int^* f^a\,d\mu \\
&\leqslant a\mu(T) + \delta\big(1+\mu(T)\big) + \int^* f_a\,d\mu = \delta\big(1+\mu(T)\big) + \int^* (f \vee a)\,d\mu < c
\end{aligned}$$

for all $c > \int^* f\,d\mu$. Letting $c \downarrow \int^* f\,d\mu$, we see that $m = \int^* f\,d\mu$. ∎

2.3. THEOREM. *Let $\mathscr{K}$ be a $(\bigcap f)$-stable set and let $(T, \mathscr{B}, \mu)$ be a content space satisfying $\mu \in M_r(T, \mathscr{K})$. Then μ is a complete, finitely founded content and if $U \subseteq T$ is a given set satisfying $U \supseteq \bigcup \mathscr{K}$, then we have*:

(1) $\emptyset \in \mathscr{K} \subseteq \mathscr{F}(\mathscr{K}) \subseteq \mathscr{B} = \mathscr{M}_\mu = \mathscr{M}_{\mu_*} = \{B \subseteq T \mid B \cap K \in \mathscr{B}$ *for all* $K \in \mathscr{K}\}$.

(2) $\mu_{\mathscr{K}}(A) = \mu_\circ(A) = \mu_*(A) \leqslant \mu^*(A) \leqslant \mu^{\mathscr{L}}(A)$ *for all* $A \subseteq T$ *and for all* $\mathscr{L} \subseteq 2^T$.

(3) $U \in \mathscr{F}(\mathscr{K}) \cap \mathscr{G}(\mathscr{K})$ *and we have* $\mu(T \setminus U) = 0$ *and* $\mu(T) = \mu(U)$.

(4) *If* $\mathscr{L} \subseteq \mathscr{B}$ *and* $\mathscr{K} \subseteq \mathscr{G}(\mathscr{L})$, *then* $\mu^*(A) = \mu^{\mathscr{L}}(A)$ *for all* $A \subseteq T$ *satisfying* $\mu^{\mathscr{L}}(A) < \infty$. *In particular*, $\mu^*(A) = \mu^{\mathscr{G}(\mathscr{K})}(A)$ *for all* $A \subseteq T$ *satisfying* $\mu^{\mathscr{G}(\mathscr{K})}(A) < \infty$.

(5) *If* $\mathscr{K}_{(\cap c)} \subseteq \mathscr{B}$ *and* $\mu(K_n) \to 0$ *for every decreasing sequence* $(K_n) \subseteq \mathscr{K}$ *satisfying* $K_n \downarrow \emptyset$, *then* $(T, \mathscr{B}, \mu)$ *is a measure space.*

Proof. (1) follows from Theorem 2.1 (9) and Theorem 2.2 (1); (2) is evident and (3) is implied by Lemma 1.1 (1) and inner $\mathscr{K}$-regularity of μ. Since

$\mathscr{G}(\mathscr{G}(\mathscr{K})) = \mathscr{F}(\mathscr{K})$, we see that (4) follows from (1) and Theorem 2.2 (1), and (5) is implied by [6], Theorem 2.4.7, pp. 319–320.

2.4. THEOREM. *Let T be a topological space and let $\mathscr{K} \subseteq \mathscr{K}(T)$ be a non-empty set satisfying $\mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{K})$. Then $\mathscr{K}$ is a $(\bigcap\tau)$-stable set containing $\emptyset$ and if $(T, \mathscr{B}, \mu)$ is a given content space, then we have:*

(1) *If $\mu \in M_r(T, \mathscr{K})$, then $\mu \in M_R(T \mid U)$ for all $U \supseteq \bigcup \mathscr{K}$.*

(2) *If $\mu \in M_R(T \mid U)$, then $\mu^*(A) = \mu^{\mathscr{G}(T)}(A)$ for all $A \subseteq T$ satisfying $\mu^{\mathscr{G}(T)}(A) < \infty$.*

(3) *If $\mu \in M_R^\circ(T \mid U)$, then $\mu^*(C) = \mu^{\mathscr{G}(T)}(C) < \infty$ for every compact set $C \subseteq U$.*

Proof. (1) Suppose that $\mu \in M_r(T, \mathscr{K})$ and let $U \subseteq T$ be a given set such that $U \supseteq \bigcup \mathscr{K}$. By Lemma 1.4, we infer that $\mathscr{K}$ is a $(\bigcap\tau)$-stable set satisfying $\mathscr{F}(T) \subseteq \mathscr{F}(\mathscr{K})$. So, by Theorem 2.3 (1) and (5), we see that μ is a complete, finitely founded Borel measure satisfying $U \in \mathscr{B}$ and $\mu(T \backslash U) = 0$, and since $\mathscr{K} \subseteq \mathscr{K}(T)$ and μ is inner $\mathscr{K}$-regular, $\mu \in M_R(T \mid U)$.

(2) Let us define $\mathscr{C} := \{K \in \mathscr{K}(T) \mid K \subseteq U, \mu(K) < \infty\}$. Let $B \in \mathscr{B}$ and $a < \mu(B)$ be given. Since $U \in \mathscr{B}$ and $\mu(T \backslash U) = 0$, we have $U \cap B \in \mathscr{B}$ and $\mu(B) = \mu(U \cap B) > a$. Since μ is finitely founded, there exists $B_0 \in \mathscr{B}$ satisfying $B_0 \subseteq U \cap B$ and $a < \mu(B_0) < \infty$, and since μ is inner $\mathscr{K}(T)$-regular, there exists $K \in \mathscr{K}(T)$ such that $K \subseteq B_0$ and $\mu(K) > a$. From the relations $\mu(B_0) < \infty$ and $B_0 \subseteq U \cap B$ we see that $K \in \mathscr{C}$. Hence μ is inner $\mathscr{C}$-regular, and since μ is complete and $\mathscr{C} \subseteq \mathscr{K}(T) \subseteq \mathscr{B}$, we have $\mu \in M_r(T, \mathscr{C})$. Since $\mathscr{C} \subseteq \mathscr{F}(T) = \mathscr{G}(\mathscr{G}(T))$, the assertion (2) follows from Theorem 2.3 (4).

(3) Suppose that $\mu \in M_R^\circ(T \mid U)$. Since $\mu^{\mathscr{G}(T)}(\{t\}) < \infty$ for all $t \in U$, there exist open sets G_t satisfying $t \in G_t$ and $\mu(G_t) < \infty$ for all $t \in U$. Let $C \subseteq U$ be a compact set. Since $(G_t \mid t \in U)$ is an open covering of C, there exists a finite set $\pi \subseteq U$ satisfying

$$C \subseteq G := \bigcup_{t \in \pi} G_t.$$

Hence $\mu^{\mathscr{G}(T)}(C) \leqslant \mu(G) \leqslant \sum_{t \in \pi} \mu(G_t) < \infty$, and so we see that (3) follows from (2). ∎

3. FUNCTIONALS

Let $\Phi \subseteq \bar{R}^T$ be a non-empty set and let $\beta: \Phi \to \bar{R}$ be a given functional. Then we denote by $\beta^\circ(v) := -\beta(-v)$ the *dual functional* of β for all $v \in -\Phi$, and we say that β is *increasing* if $\beta(\phi) \leqslant \beta(\psi)$ for all $\phi, \psi \in \Phi$ satisfying $\phi \leqslant \psi$. Let

$$\operatorname{dom} \beta := \{\phi \in \Phi \mid |\beta(\phi)| < \infty\}$$

denote the *finite domain* of β, let

$$\operatorname{dom}^* \beta := \{\phi \in \Phi \mid \beta(\phi) < \infty\} \quad \text{and} \quad \operatorname{dom}_* \beta := \{\phi \in \Phi \mid \beta(\phi) > -\infty\}$$

denote the *upper* and *lower finite domains* of β, respectively; moreover, let $\mathscr{L}_\beta$ denote the set of all $A \subseteq T$ satisfying $1_A \in \operatorname{dom}^* \beta$. If Φ is a convex cone, we say that β is:

(i) *subadditive* if $\beta(\phi \dot{+} \psi) \leqslant \beta(\phi) \dot{+} \beta(\psi)$ for all $\phi, \psi \in \Phi$;

(ii) *positively homogeneous* if $\beta(0) < \infty$ and $\beta(a\phi) = a\beta(\phi)$ for all $\phi \in \Phi$ and all $0 < a < \infty$;

(iii) *sublinear* if $\beta(0) = 0$ and β is subadditive and positively homogeneous.

If $\eta: \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$ is a functional and $\mathscr{K} \subseteq 2^T$, we denote by $\mathscr{K}_\eta^t$ the set of all $A \subseteq T$ satisfying $\inf_{K \in \mathscr{K}} \eta(1_{A \setminus K}) = 0$ and we define

$$L^1(\eta) = \{f \in \boldsymbol{R}^T \mid \eta(f) = \eta^\circ(f) \neq \pm\infty\}, \quad L^\#(\beta) = \{f \in \boldsymbol{R}^T \mid \eta(|f|) < \infty\},$$

$$L^u(\eta) = \{f \in \boldsymbol{R}^T \mid \lim_{n \to \infty} \eta(T^n |f|) = 0\}, \quad \mathfrak{L}^{\mathscr{K}}(\eta) = o(T, \mathscr{K}_\eta^t) \cap L^u(\eta),$$

where $T^n h := (h \wedge n^{-1}) + (h - n)^+$ denote *upper truncation* of h for all $h \in \boldsymbol{R}_+^T$ and all $n \geqslant 1$. If $\eta: \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$ is an increasing sublinear functional, then $\|f\|_\eta := \eta(|f|)$ is a seminorm on $\boldsymbol{R}^T$ and if $\Phi \subseteq \boldsymbol{R}^T$, we denote by $\mathrm{cl}_\eta \Phi$ the $\|\cdot\|_\eta$-closure of Φ.

Let $\Phi \subseteq \bar{\boldsymbol{R}}^T$ be a non-empty set, let $\beta: \Phi \to \bar{\boldsymbol{R}}$ be a given functional, and let $\Psi \subseteq \Phi$ be a non-empty set. Then denote by β^Ψ and β_Ψ the *upper* and *lower* Ψ*-envelopes* of η; that is:

$$\beta^\Psi(f) = \inf\{\beta(\psi) \mid \psi \in \Psi, \psi \geqslant f\}, \quad \beta_\Psi(f) = \sup\{\beta^\circ(\upsilon) \mid \upsilon \in -\Psi, \upsilon \leqslant f\}$$

for all $f \in \boldsymbol{R}^T$. Note that β^Ψ is an increasing functional on $\boldsymbol{R}^T$ with dual functional β_Ψ. If $\mathscr{H} \subseteq 2^T$ is a given set satisfying $E := \{1_H \mid H \in \mathscr{H}\} \subseteq \Phi$, we define $\beta^{\mathscr{H}}(A) := \beta^E(1_A)$ and $\beta_{\mathscr{H}}(A) := \beta_E(1_A)$ for all $A \subseteq T$. We denote by $\beta^{\langle\Psi\rangle}$ and $\beta_{\langle\Psi\rangle}$ the *positive upper* and *lower Daniel* Ψ*-functionals*, respectively; that is:

$$\beta^{\langle\Psi\rangle}(f) = \inf\{\overset{*}{\sum_{\psi\in\pi}} x(\psi)\beta(\psi) \mid \pi \in 2^{(\Psi)}, x \in \boldsymbol{R}_+^\pi, \sum_{\psi\in\pi} x(\psi)\psi \geqslant f\},$$

$$\beta_{\langle\Psi\rangle}(f) = \sup\{\overset{\circ}{\sum_{\psi\in\pi}} x(\upsilon)\beta^\circ(\upsilon) \mid \pi \in 2^{(-\Psi)}, x \in \boldsymbol{R}_+^\pi, \sum_{\upsilon\in\pi} x(\upsilon)\upsilon \geqslant f\}$$

for all $f \in \boldsymbol{R}^T$. Note that $\beta^{\langle\Psi\rangle}$ is an increasing, positively homogeneous, subadditive functional on $\boldsymbol{R}^T$ with dual functional $\beta_{\langle\Psi\rangle}$ and that we have

$$(3.1) \qquad \beta^{\langle\Psi\rangle}(\psi) \leqslant \beta^\Psi(\psi) \leqslant \beta(\psi) \text{ for all } \psi \in \Psi \quad \text{and} \quad \beta^{\langle\Psi\rangle}(f) \leqslant \beta^\Psi(f) \text{ for all } f \in \boldsymbol{R}^T.$$

We say that β is *non-negative* $\langle\Psi\rangle$*-definite* if $\beta^{\langle\Psi\rangle}(0) \geqslant 0$ and we define

$$H_\beta^\Psi = \{f \in \boldsymbol{R}^T \mid \overset{*}{\sum_{s\in\pi}} x(\psi)\beta(\psi) > -\infty \text{ for all } \pi \in 2^{(\Psi)} \text{ and for all } x \in \boldsymbol{R}_+^\pi : \sum_{s\in\pi} x(\psi)\psi \geqslant f\}.$$

Note that H_β^Ψ is an upper hereditary, positive cone containing $\operatorname{dom}_* \beta^{\langle\Psi\rangle}$.

3.1. LEMMA. *Let $\Phi \subseteq \boldsymbol{R}^T$ be a convex cone, let $\beta\colon \Phi \to \bar{\boldsymbol{R}}$ be a given functional, and let us define $\Lambda := \Phi \cap (-\Phi)$. Then Λ is a linear subspace of $\boldsymbol{R}^T$ and we have*:

(1) *If β is positively homogeneous, then either $\beta(0) = 0$ or $\beta(0) = -\infty$ and we have $\beta^{\Phi}(1_{\{\Phi \geqslant a\}}) \leqslant a^{-1}\beta(\phi)$ for all $\phi \in \Phi^+$ (Markov's inequality).*

(2) *If β is subadditive and π is a non-empty finite set, then*:

(a) *$\sum_{i\in\pi}\phi_i \in \Phi$ and $\beta\big(\sum^*_{i\in\pi}\phi_i\big) \leqslant \sum^*_{i\in\pi}\beta(\phi_i)$ for all $(\phi_i)_{i\in\pi} \subseteq \Phi$;*

(b) *$\sum_{i\in\pi}\psi_i \in -\Phi$ and $\sum^{\circ}_{i\in\pi}\beta^{\circ}(\psi_i) \leqslant \beta^{\circ}\big(\sum^{\circ}_{i\in\pi}\psi_i\big)$ for all $(\psi_i)_{i\in\pi} \subseteq -\Phi$;*

(c) *$\phi+\psi \in \Phi$ and $\beta(\phi) \dot{+} \beta^{\circ}(\psi) \leqslant \beta(\phi+\psi)$ for all $\phi \in \Phi$ and for all $\psi \in \Lambda$;*

(d) *$\phi+\psi \in -\Phi$ and $\beta^{\circ}(\phi+\psi) \leqslant \beta(\phi) \dot{+} \beta^{\circ}(\psi)$ for all $\phi \in \Lambda$ and for all $\psi \in -\Phi$.*

(3) *If β is subadditive and $\beta(0) \geqslant 0$, then $\beta^{\circ}(\phi) \leqslant \beta(\phi)$ for all $\phi \in \Lambda$.*

(4) *If β is positively homogeneous and subadditive, then*

(a) *β is sublinear if and only if $\beta(0) \geqslant 0$ if and only if $\beta(0) > -\infty$, and if and only if there exists $\phi \in \Phi$ such that $\beta(\phi) \neq \pm\infty$.*

(5) *Suppose that β is increasing and positively homogeneous and let us define*

$$\mathscr{V} := \{A \subseteq T \mid 1_A \in \Psi\}.$$

If $\mathscr{H} \subseteq 2^T$ is a given set containing $\{\varnothing, T\}$ such that $\Psi \subseteq W(T, \mathscr{H})$ and $1_H \in \Phi$ for all $H \in \mathscr{H}$, then

(a) *$\beta^{\mathscr{H}}(A) = \beta^{\mathscr{H}\cap\mathscr{L}_\beta}(A) \leqslant \beta^{\Psi}(1_A) \leqslant \beta^{\mathscr{V}}(A)$ for all $A \subseteq T$.*

Proof. (1) Since β is positively homogeneous, we have $a\beta(0) = \beta(0) < \infty$ for all $0 < a < \infty$. Hence $\beta(0)$ is either 0 or $-\infty$. Let $\phi \in \Phi^+$ and $0 < a < \infty$ be given. Then $1_{\{\phi \geqslant a\}} \leqslant a^{-1}\phi$, and since $a^{-1}\phi \in \Phi$ and β is positively homogeneous, we have

$$\beta^{\Phi}(1_{\{\phi \geqslant a\}}) \leqslant \beta(a^{-1}\phi) = a^{-1}\beta(\phi).$$

(2) Since β is subadditive, (2)(a) follows by induction on the number of elements in π, and since $-\sum^*_{i\in\pi}u_i = \sum^{\circ}_{i\in\pi}(-u_i)$, we see that (2)(b) follows from (2)(a). Let $\phi \in \Phi$ and $\psi \in \Lambda$ be given. Since Φ is a convex cone containing ϕ and $\pm\psi$, we have $\phi+\psi \in \Phi$, and since $\phi = (\phi+\psi)-\psi$ and β is subadditive, we obtain

$$\beta(\phi) \leqslant \beta(\phi+\psi) \dot{+} \beta(-\psi) = \beta(\phi+\psi) \dot{-} \beta^{\circ}(\psi),$$

and so we see that $\beta(\phi)+\beta^{\circ}(\psi) \leqslant \beta(\phi+\psi)$, which proves (2)(c). In the same manner, we prove that (2)(d) holds.

(3) Let $\phi \in \Lambda$ be given. Since $\phi - \phi = 0$ and β is subadditive, we have $\beta(0) \leqslant \beta(\phi) \dot{-} \beta^{\circ}(\phi)$, and since $\beta(0) \geqslant 0$, we get $\beta^{\circ}(\phi) \leqslant \beta(\phi)$.

(4) The first two equivalences in (4) follow from (1), and if β is sublinear, then $0 \in \Lambda$ and $\beta(0) = 0 \neq \pm\infty$. Suppose that $\phi \in \Phi$ and $\beta(\phi) \neq \pm\infty$. Since

β is positively homogeneous and subadditive, we have $\beta(0) < \infty$ and $\beta(\phi) \leqslant \beta(\phi) \dot{+} \beta(0)$, and since $\beta(\phi)$ is finite, we obtain $\beta(0) > -\infty$, which completes the proof of (4).

(5) Let $A \subseteq T$ be given. The first equality and the last inequality in (5)(a) are evident. So let us show that $\beta^{\mathscr{H}}(A) \leqslant \beta^{\Psi}(1_A)$. If $\beta^{\Psi}(1_A) = \infty$, this is evident. So suppose that $\beta^{\Psi}(1_A) < \infty$ and let $a > \beta^{\Psi}(1_A)$ be a given number. Then there exist $0 < \delta < 1$ and $\psi \in \Psi$ such that $\psi \geqslant 1_A$ and $\beta(\psi) < a\delta$. Since $\{\emptyset, T\} \subseteq \mathscr{H}$ and $\psi \in \Psi \subseteq W(T, \mathscr{H})$, there exists $H \in \mathscr{H}$ satisfying $\{\psi \geqslant 1\} \subseteq H \subseteq \{\psi > \delta\}$. Since $\psi \geqslant 1_A$, we have $A \subseteq H$ and $\delta 1_H \leqslant \psi$, and since Φ is a convex cone containing 1_H and β is increasing and positively homogeneous, we obtain $\delta\beta(1_H) = \beta(\delta 1_H) \leqslant \beta(\psi) < a\delta$. Consequently, $\beta^{\mathscr{H}}(A) \leqslant \beta(1_H) < a$. Letting $a \downarrow \beta^{\Psi}(1_A)$, we see that $\beta^{\mathscr{H}}(A) \leqslant \beta^{\Psi}(1_A)$ for all $A \subseteq T$, which completes the proof of (5). ∎

3.2. Lemma. *Let $\Psi \subseteq \Phi \subseteq \boldsymbol{R}^T$ be non-empty sets, let $\beta\colon \Phi \to \overline{\boldsymbol{R}}$ be a given functional and let Υ and Θ denote the convex cones generated by $\Psi \cap \operatorname{dom} \beta$ and $\Psi \cap \operatorname{dom}^* \beta$, respectively. Then we have*:

(1) *$\beta^{\Psi}(f) \leqslant \beta^{\Lambda}(f)$ for all $f \in \boldsymbol{R}^T$ and for all $\Lambda \subseteq \Phi$ satisfying $\Lambda \cap \operatorname{dom}^* \beta \subseteq \Psi$.*

(2) *If $E \subseteq \boldsymbol{R}^T$ is an upper hereditary set, then $\beta^{\Lambda}(h) \leqslant \beta^{\Psi}(h)$ for all $h \in E$ and all $\Lambda \subseteq \Phi$ satisfying $\Lambda \supseteq \Psi \cap E \cap \operatorname{dom}^* \beta$.*

(3) *$\beta^{\langle\Psi\rangle}(f) = (\beta^{\langle\Psi\rangle})^{\Theta}(f)$ for all $f \in \boldsymbol{R}^T$ and $\beta^{\langle\Psi\rangle}(f) = (\beta^{\langle\Psi\rangle})^{\Upsilon}(f)$ for all $f \in H^{\Psi}_{\beta}$.*

(4) *$\beta^{\langle\Psi\rangle}$ is sublinear if and only if β is non-negative $\langle\Psi\rangle$-definite if and only if $\beta^{\langle\Psi\rangle}(0) > -\infty$, and if and only if $\beta^{\langle\Psi\rangle}(f) \neq \pm\infty$ for some $f \in \boldsymbol{R}^T$.*

(5) *If Ψ is a convex cone and β is positively homogeneous and subadditive on Ψ, then $\beta^{\langle\Psi\rangle}(f) = \beta^{\Psi}(f)$ for all $f \in \overline{\boldsymbol{R}}^T$.*

Proof. (1) is evident. So let $E \subseteq \boldsymbol{R}^T$ be an upper hereditary set and let $\Lambda \subseteq \Phi$ be a given set satisfying $\Lambda \subseteq \Psi \cap E \cap \operatorname{dom}^* \beta$. Let $h \in E$ be given and let us show that $\beta^{\Lambda}(h) \leqslant \beta^{\Psi}(h)$. If $\beta^{\Psi}(h) = \infty$, this is evident. So suppose that $\beta^{\Psi}(h) < \infty$ and let $a > \beta^{\Psi}(h)$ be given. Then there exists $\psi \in \Psi$ satisfying $\psi \geqslant h$ and $\beta(\psi) < a$. Since E is an upper hereditary set containing h, we have $\psi \in \Psi \cap E \cap \operatorname{dom}^* \beta \subseteq \Lambda$, and so we obtain $\beta^{\Lambda}(h) \leqslant \beta(\psi) < a$. Letting $a \downarrow \beta^{\Psi}(h)$, we see that $\beta^{\Lambda}(h) \leqslant \beta^{\Psi}(h)$.

(3) Let $f \in \boldsymbol{R}^T$ be given. Since $\Upsilon \subseteq \Theta$ and $\beta^{\langle\Psi\rangle}$ is increasing, we have $\beta^{\langle\Psi\rangle}(f) \leqslant (\beta^{\langle\Psi\rangle})^{\Theta}(f) \leqslant (\beta^{\langle\Psi\rangle})^{\Upsilon}(f)$. Hence, if $\beta^{\langle\Psi\rangle}(f) = \infty$, we have equality. So suppose that $\beta^{\langle\Psi\rangle}(f) < \infty$ and let $a > \beta^{\langle\Psi\rangle}(f)$ be given. Then there exist $\pi \in 2^{(\Psi)}$ and $x \in \boldsymbol{R}^{\pi}_{+}$ satisfying

$$v := \sum_{\psi\in\pi} x(\psi)\psi \geqslant f \quad \text{and} \quad q := \sideset{}{^*}\sum_{\psi\in\pi} x(\psi)\beta(\psi) < a.$$

Then

$$\upsilon = \sum_{\psi \in \tau} x(\psi)\psi \quad \text{and} \quad q = \sum_{\psi \in \tau}^{*} x(\psi)\beta(\psi) < a,$$

where $\tau := \{\psi \in \pi \mid x(\psi) \neq 0\}$. Since $x \in \boldsymbol{R}_+^\pi$, we have $\tau \subseteq \Psi \cap \mathrm{dom}^*(\beta)$, and if $f \in H_\beta^\Psi$, we get

$$-\infty < \sum_{\psi \in \pi}^{*} x(\psi)\beta(\psi) < \infty,$$

and so $\tau \subseteq \Psi \cap \mathrm{dom}(\beta)$. Hence $\upsilon \in \Theta$, and if $f \in H_\beta^\Psi$, then $\upsilon \in \Upsilon$. By the definition of $\beta^{\langle\Psi\rangle}$, we have $\beta^{\langle\Psi\rangle}(\upsilon) \leqslant q < a$. Hence $(\beta^{\langle\Psi\rangle})^\Theta(f) \leqslant \beta^{\langle\Psi\rangle}(\upsilon) < a$, and if $f \in H_\beta^\Psi$, we get $(\beta^{\langle\Psi\rangle})^\Upsilon(f) \leqslant \beta^{\langle\Psi\rangle}(\upsilon) < a$. Letting $a \downarrow \beta^{\langle\Psi\rangle}(f)$, we obtain (2).

(4) follows from Lemma 3.1 (4).

(5) Let $f \in \boldsymbol{R}^T$ be given. By Lemma 3.1, we get $\beta^{\langle\Psi\rangle}(f) \leqslant \beta^\Psi(f)$. Consequently, if $\beta^{\langle\Psi\rangle}(f) = \infty$, we have equality. So suppose that $\beta^{\langle\Psi\rangle}(f) < \infty$ and let $a > \beta^{\langle\Psi\rangle}(f)$ be given. Then there exist $\pi \in 2^{(\Psi)}$ and $x \in \boldsymbol{R}_+^\pi$ satisfying

$$\upsilon := \sum_{\psi \in \pi} x(\psi)\psi \geqslant f \quad \text{and} \quad \sum_{\psi \in \pi}^{*} x(\psi)\beta(\psi) < a.$$

Since Ψ is a convex cone containing π, we have $\upsilon \in \Psi$ and $\beta^\Psi(f) \leqslant \beta(\upsilon)$. Let us define $\tau := \{\psi \in \pi \mid x(\psi) > 0\}$. If $\tau = \varnothing$, we get

$$\upsilon = 0 \leqslant f \quad \text{and} \quad 0 = \sum_{\psi \in \pi}^{*} x(\psi)\beta(\psi) < a,$$

and Lemma 3.1 (1) and positive homogeneity of β imply $\beta^\Psi(f) \leqslant \beta(0) < a$. If $\tau \neq \varnothing$, we have $\upsilon = \sum_{\psi \in \tau}^{*} x(\psi)\psi \geqslant f$, and since β is positively homogeneous and subadditive on Ψ, we obtain

$$\beta^\Psi(f) \leqslant \beta(\upsilon) \leqslant \sum_{\psi \in \pi}^{*} x(\psi)\beta(\psi) < a.$$

Hence $\beta^{\langle\Psi\rangle}(f) \leqslant \beta^\Psi(f) < a$. Letting $a \downarrow \beta^{\langle\Psi\rangle}(f)$, we see that $\beta^{\langle\Psi\rangle}(f) = \beta^\Psi(f)$. ∎

3.3. LEMMA. *Let $\eta: \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$ be an increasing sublinear functional, let $\mathscr{C} \subseteq 2^T$ be a set lattice, and let us define $\mathscr{K} := \mathscr{C} \cap \mathscr{L}_\eta$. Then $\mathscr{L}_\eta$ and $\mathscr{C}_\eta^t \subseteq 2^T$ are hereditary set lattices and we have:*

(1) *$\boldsymbol{L}^u(\eta)$ and $\boldsymbol{L}^*(\eta)$ are hereditary $\|\cdot\|_\eta$-closed linear spaces and $\boldsymbol{L}^1(\eta)$ is a $\|\cdot\|_\eta$-closed linear space such that the restriction of η to $\boldsymbol{L}^1(\eta)$ is finite and linear.*

(2) *$\boldsymbol{L}^u(\eta) \subseteq \boldsymbol{L}^*(\eta) \subseteq o(T, \mathscr{L}_\eta)$, $\mathscr{L}_\eta = \{A \subseteq T \mid 1_A \in \boldsymbol{L}^u(\eta)\}$.*

(3) *$\boldsymbol{L}^*(\eta) = \{f \in \boldsymbol{R}^T \mid \eta(f) \vee \eta(f^-) < \infty\} = \{f \in \boldsymbol{R}^T \mid \eta(-f) \vee \eta(f^+) < \infty\}$.*

(4) *$\mathscr{C} \subseteq \mathscr{C}_\eta^t$ and $\mathscr{L}_\eta$ and $\mathscr{C}_\eta^t$ are hereditary set lattices and $o(T, \mathscr{C}_\eta^t)$ and $\mathfrak{L}^\mathscr{C}(\eta)$ are hereditary $\|\cdot\|_\eta$-closed linear spaces.*

(5) *$\mathcal{K}$ is a set lattice satisfying $\mathcal{K} \subseteq \mathcal{C} \subseteq \mathcal{F}(\mathcal{C}) \subseteq \mathcal{F}(\mathcal{K})$ and*

(a) $\mathcal{K}^t_\eta = \{A \subseteq T \mid 1_A \in \mathfrak{L}^{\mathcal{K}}(\eta)\} \subseteq \mathcal{L}_\eta$,

(b) $\{f \in B(T) \mid \{f \neq 0\} \in \mathcal{K}^t_\eta\} \subseteq \mathfrak{L}^{\mathcal{K}}(\eta)$.

(6) *If $\Phi \subseteq \boldsymbol{R}^T$ is a non-empty set and Θ denotes the convex cone generated by Φ, then η^Θ and $\eta^{\langle\Phi\rangle}$ are increasing sublinear functionals and*

(a) $\eta(f) \leqslant \eta^\Theta(f) \leqslant \eta^{\langle\Phi\rangle}(f) \leqslant \eta^\Phi(f)$ *for all $f \in \boldsymbol{R}^T$ with equality if $f \in \Phi$.*

(7) *Let $\mathbb{L}$ denote the set of all $f \in \boldsymbol{R}^T$ satisfying $\lim_{n\to\infty} \eta((f^- - n)^+) = 0$ and let us define $W := W(T, \mathcal{G}(\mathcal{C}))$. Then $\mathbb{L}$ is an upper hereditary convex cone containing $\mathbb{L}^u_-(\eta)$ and η^W is sublinear on $\mathbb{L}$, and $\eta^W(1_A) = \eta^{\mathcal{G}(\mathcal{C})}(A)$ for all $A \subseteq T$.*

Proof. (1) and (2) follow from [6], Lemma 2.1.5 and Theorem 2.1.6, pp. 293–295, and since $|f| = 2f^- + f = 2f^+ - f$, we see that (3) follows from subadditivity of η.

(4) Since η is increasing and sublinear, $\mathcal{L}_\eta$, $\mathcal{C}^t_\eta$ are hereditary set lattices such that $\mathcal{C} \subseteq \mathcal{C}^t_\eta$. Let $f \in \mathrm{cl}_\eta\, o(T, \mathcal{K}^t_\eta)$ and $\varepsilon, \delta > 0$ be given and let us define $F := \{|f| > \delta\}$. Then there exists $h \in o(T, \mathcal{K}^t_\eta)$ satisfying $\eta(|f-h|) < \varepsilon\delta/2$, and if we define $H := \{|h| > \delta/2\}$ and $G := \{|f-h| > \delta/2\}$, then $F \subseteq H \cup G$. Since $h \in o(T, \mathcal{K}^t_\eta)$, there exists $K \in \mathcal{K}^t_\eta$ such that $\eta(1_{H\setminus K}) < \varepsilon/2$, and by Lemma 3.1 (1) we have

$$\eta(1_G) \leqslant \frac{2}{\delta}\eta(|f-h|) < \frac{\varepsilon}{2}.$$

Hence, by sublinearity of η we obtain

$$\eta(1_{F\setminus K}) \leqslant \eta(1_G) + \eta(1_{H\setminus K}) < \varepsilon,$$

and so $F = \{|f| > \delta\} \in \mathcal{K}^t_\eta$ for all $\delta > 0$; that is, $f \in o(T, \mathcal{K}^t_\eta)$. Hence $o(T, \mathcal{K}^t_\eta)$ is $\|\cdot\|_\eta$-closed, and so (4) follows from (1).

(5) By (4) and Lemma 1.1 (2) and (3), we see that $\mathcal{K} \subseteq \mathcal{C} \subseteq \mathcal{F}(\mathcal{C}) \subseteq \mathcal{F}(\mathcal{K})$ and the first equality in (5)(a) follows from (2) and Lemma 1.1 (7). Let $A \in \mathcal{K}^t_\eta$ be given. Then there exists $K \in \mathcal{K}$ such that $\eta(1_{A\setminus K}) < 1$, and since $K \in \mathcal{L}_\eta$, we have $\eta(1_A) \leqslant \eta(1_K) + \eta(1_{A\setminus K}) < \infty$, which proves the last inclusion in (5)(a). By (4), we see that $\mathfrak{L}^{\mathcal{K}}(\eta)$ is a hereditary linear space, and so (5)(b) follows from (5)(a) and Lemma 1.2 (8).

(6) Since η is increasing and sublinear, η^Θ is an increasing sublinear functional satisfying $\eta \leqslant \eta^\Theta$. Let $f \in \boldsymbol{R}^T$ be given. By Lemma 3.1, we have $\eta^{\langle\Phi\rangle}(f) \leqslant \eta^\Phi(f)$. So let us show that $\eta^\Theta(f) \leqslant \eta^{\langle\Phi\rangle}(f)$. If $\eta^{\langle\Phi\rangle}(f) = \infty$, this is evident. Suppose that $\eta^{\langle\Phi\rangle}(f) < \infty$ and let $a > \eta^{\langle\Phi\rangle}(f)$ be given. Then there exist $\pi \in 2^{(\Phi)}$ and $x \in \boldsymbol{R}^\pi_+$ satisfying

$$v := \sum_{\phi\in\pi} x(\phi)\phi \geqslant f \quad \text{and} \quad \sideset{}{^*}\sum_{\phi\in\pi} x(\phi)\eta(\phi) < a.$$

Then $\upsilon \in \Theta$, and by sublinearity of η we have

$$\eta^{\Theta}(f) \leqslant \eta(\upsilon) \sum_{\phi \in \pi}^{*} x(\phi)\eta(\phi) < a.$$

Letting $a \downarrow \eta^{\langle \Phi \rangle}(f)$, we see that $\eta^{\Theta}(f) \leqslant \eta^{\langle \Phi \rangle}(f)$, and since η is increasing, $\eta(\phi) = \eta^{\Phi}(\phi)$ for all $\phi \in \Phi$, which completes the proof of (6).

(7) By sublinearity of η we see that $\mathbb{L}$ is an upper hereditary convex cone, and since $(f^- - n)^+ \leqslant T^n f^-$, it follows that $L^u_-(\eta) \subseteq \mathbb{L}$. Let $f_1, f_2 \in \mathbb{L}$ be given and let us show that

$$\eta^W(f_1 + f_2) \leqslant \eta^W(f_1) + \eta^W(f_2).$$

If $\eta^W(f_1) = \infty$ or $\eta^W(f_2) = \infty$, this is evident. Suppose that $\eta^W(f_1) < \infty$ and $\eta^W(f_2) < \infty$ and let $a_1 > \eta^W(f_1)$ and $a_2 > \eta^W(f_2)$ be given numbers. Then there exist $\phi_1, \phi_2 \in W$ such that $\phi_i \geqslant f_i$ and $\eta(\phi_i) < a_i$ for $i = 1, 2$, and since $\mathbb{L}$ is an upper hereditary set containing f_i, we have $\phi_i \in \mathbb{L}$. Since $(\phi_i \vee (-n)) = \phi_i + (\phi_i^- - n)^+$, there exists an integer $n \geqslant 1$ such that $\eta(\psi_i) < a_i$, where $\psi_i := \phi_i \vee (-n)$ for $i = 1, 2$. By Lemma 1.2 (3), we have $\psi_1, \psi_2 \in W$, and since $\psi_i \geqslant -n$ and $\mathscr{G}(\mathscr{C})$ is a set lattice, we get $\psi := \psi_1 + \psi_2 \in W$ by Lemma 1.3 (1). Since $\psi \geqslant \phi_1 + \phi_2 \geqslant f_1 + f_2$, we have

$$\eta^W(f_1 + f_2) \leqslant \eta(\psi) \leqslant \eta(\psi_1) + \eta(\psi_2) < a_1 + a_2.$$

Letting $a_i \downarrow \eta^W(f_i)$, we see that

$$\eta^W(f_1 + f_2) \leqslant \eta^W(f_1) \dot{+} \eta^W(f_2) \quad \text{for all } f_1, f_2 \in \mathbb{L}.$$

Since W is a cone, η^W is sublinear on $\mathbb{L}$ and by Lemma 3.1 (5), we get $\eta^W(1_A) = \eta^{\mathscr{G}(\mathscr{C})}(A)$ for all $A \subseteq T$. ∎

3.4. THEOREM. *Let $\Phi \subseteq \boldsymbol{R}^T$ be a non-empty set and let $\beta := \Phi \to \bar{\boldsymbol{R}}$ be a given functional. Let $\Psi \subseteq \Phi$ be a non-empty set and let $\mu \in M(T)$ be a given content. Then the following three statements are equivalent:*

(1) *μ is a lower Ψ-representation of β.*

(2) *β is non-negative Ψ-definite and $\beta^{\langle \Psi \rangle}$ is an increasing sublinear functional on $\boldsymbol{R}^T$ satisfying*

$$\beta_{\langle \Psi \rangle}(f) \leqslant \int_* f\, d\mu \leqslant \int^* f\, d\mu \leqslant \beta^{\langle \Psi \rangle}(f) \quad \textit{for all } f \in \boldsymbol{R}^T.$$

(3) *μ is a lower Ψ-representation of $\beta^{\langle \Psi \rangle}$.*

Let $\eta := \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$ be an increasing sublinear functional and let $\Upsilon \subseteq \boldsymbol{R}^T$ be a non-empty set such that μ is a lower Υ-representation of η. Then $\eta^{\langle \Upsilon \rangle}$ is an increasing sublinear functional on $\boldsymbol{R}^T$ and we have:

(4) $\eta_{\Upsilon}(f) \leqslant \eta_{\langle \Upsilon \rangle}(f) \leqslant \int_* f\, d\mu \leqslant \int^* f\, d\mu \leqslant \eta^{\langle \Upsilon \rangle}(f) \leqslant \eta^{\Upsilon}(f)$ *for all* $f \in \boldsymbol{R}^T$.

(5) *Let* $\mathbf{L} \subseteq L^*(\eta)$ *be a hereditary linear space and let* $\Lambda \subseteq R^T$ *be a non-empty set satisfying* $\Theta := \Lambda \cap \mathbf{L}_- \cap L^*(\eta) \subseteq \Upsilon$. *Then*

(a) $\int^* f\,d\mu \leqslant \eta^\Upsilon(f) \leqslant \eta^\Theta(f) = \eta^\Lambda(f)$ *for all* $f \in \mathbf{L}_-$

and if $\eta^\Lambda(f) \leqslant \eta(f)$ *for all* $f \in \mathbf{L}_-$, *then* μ *is a lower* $\mathbf{L}_-$*-representation of* η *and we have*

(b) $-\infty < \int_* f\,d\mu \leqslant \int^* f\,d\mu \leqslant \eta(f) = \eta^V(f)$ *for all* $f \in \mathbf{L}_-$ *and for all* $V \supseteq \Theta$.

Proof. The implication "(2) $\Rightarrow$ (3)" is evident and "(3) $\Rightarrow$ (1)" follows from Lemma 3.1. Suppose that (1) holds. Let $f \in R^T$ be given and let us show that $\int^* f\,d\mu \leqslant \beta^{\langle\Psi\rangle}(f)$. If $\beta^{\langle\Psi\rangle}(f) = \infty$, this is evident. Suppose that $\beta^{\langle\Psi\rangle}(f) < \infty$ and let $a > \beta^{\langle\Psi\rangle}(f)$ be given. Then there exist $\pi \in 2^{(\Psi)}$ and $x \in R_+^\pi$ satisfying

$$\upsilon := \overset{*}{\sum_{\psi\in\pi}} x(\psi)\psi \geqslant f \quad \text{and} \quad \overset{*}{\sum_{\psi\in\pi}} x(\psi)\beta(\psi) < a.$$

Since the upper μ-integral is increasing and sublinear and $\int^* \psi d\mu \leqslant \beta(\psi)$ for all $\psi \in \pi$, we have

$$\int^* f\,d\mu \leqslant \int^* \upsilon d\mu \leqslant \overset{*}{\sum_{\psi\in\pi}} x(\psi)\int^* \psi d\mu \leqslant \overset{*}{\sum_{\psi\in\pi}} x(\psi)\beta(\psi) < a.$$

Letting $a \downarrow \beta^{\langle\Psi\rangle}(f)$, we obtain $\int^* f\,d\mu \leqslant \beta^{\langle\Psi\rangle}(f)$ for all $f \in R^T$ and applying this on $-f$, we infer that the inequalities in (2) hold. In particular, $\beta^{\langle\Psi\rangle}(0) \geqslant 0$, and so by Lemma 3.2 (2) we see that β is non-negative $\langle\Psi\rangle$-definite and that $\beta^{\langle\Psi\rangle}$ is an increasing sublinear functional on R^T.

(4) follows from the equivalence of (1) and (2). Since $\mathbf{L} \subseteq L^*(\eta)$, we obtain $\mathbf{L}_- \subseteq L_-^*(\eta)$, and so by Lemma 3.3 (3) we have $\mathbf{L}_- \cap L^*(\eta) = \mathbf{L}_- \cap \operatorname{dom}^*\eta$. Hence $\Theta = \Lambda \cap \mathbf{L}_- \cap \operatorname{dom}^*\eta \subseteq \Upsilon$, and by Lemma 1.2 (8) we infer that $\mathbf{L}_-$ is an upper hereditary convex cone. So by Lemma 3.2 (1) and (2) we have $\eta^\Upsilon(f) \leqslant \eta^\Theta(f) = \eta^\Lambda(f)$ for all $f \in \mathbf{L}_-$. Hence (5)(a) follows from (4). Suppose that $\eta^\Lambda(f) \leqslant \eta(f)$ for all $f \in \mathbf{L}_-$ and let $V \subseteq R^T$ be a given set satisfying $V \supseteq \Theta$. By (5)(a), we see that μ is a lower $\mathbf{L}_-$-representation of η. Let $f \in \mathbf{L}_-$ be given. Since $R_+^T \subseteq \mathbf{L}_- \subseteq L_-^*(\eta)$, we have $\int^* f^-\,d\mu \leqslant \eta(f^-) < \infty$, and since η is increasing, we obtain $\eta(f) \leqslant \eta^V(f) \leqslant \eta^\Theta(f) = \eta^\Lambda(f) = \eta(f)$. Consequently, (5)(b) follows from (5)(a). ■

4. LOWER REPRESENTATIONS OF SUBLINEAR FUNCTIONALS

By Theorem 3.4 we infer that μ is a lower Ψ-representation of β if and only if μ is a lower Ψ-representation of $\beta^{\langle\Psi\rangle}$ and if and only if $\beta^{\langle\Psi\rangle}$ is an increasing sublinear functional on R^T such that μ is a lower R^T-representation of $\beta^{\langle\Psi\rangle}$. Therefore, we see that it suffices to solve the lower representation problem for an increasing sublinear functional $\eta : R^T \to \bar{R}$. This will be done in Theorems 4.2 and 4.3 below but first we need the following *sandwich theorem*:

4.1. Theorem. *Let $\mathcal{K} \subseteq 2^T$ be a set lattice and let $\mathcal{G} \subseteq \mathcal{G}(\mathcal{K})$ be a non-empty set. Let $\beta: 2^T \to [0, \infty]$ be a supermodular set function and let $\varrho: \mathcal{G} \to [0, \infty]$ be a given set function satisfying*

(1) *$\beta(K) \leqslant \beta(K \backslash G) + \varrho(G)$ for all $K \in \mathcal{K}$ and for all $G \in \mathcal{G}$.*

Then $\beta_{\mathcal{K}}$ is an increasing supermodular set function and we have:

(2) *$\beta_{\mathcal{K}}(A \cup B) \leqslant \beta_{\mathcal{K}}(A) + \varrho^{\mathcal{G}}(B)$ for all $A, B \subseteq T$.*

(3) *If $\mathcal{D} \subseteq 2^T$ is non-empty and linearly ordered by inclusion, then there exists a content $\mu \in M(T, 2^T)$ satisfying*

(a) *$\beta_{\mathcal{K}}(A) \leqslant \mu(A) \leqslant \varrho^{\mathcal{G}}(A)$ for all $A \subseteq T$ and $\mu(D) = \beta_{\mathcal{K}}(D)$ for all $D \in \mathcal{D}$.*

(4) *There exists an inner $\mathcal{K}$-regular content space $(T, \mathcal{B}, \mu)$ satisfying:*

(a) *$\mathcal{K} \subseteq \mathcal{F}(\mathcal{K}) \subseteq \mathcal{B} = \mathcal{M}_{\mu_*} = \{B \subseteq T \mid K \cap B \in \mathcal{B}$ for all $K \in \mathcal{K}\}$,*

(b) *$\mu(T) = \beta_{\mathcal{K}}(T)$ and $\beta_{\mathcal{K}}(A) \leqslant \mu_*(A) \leqslant \mu^*(A) \leqslant \varrho^{\mathcal{G}}(A)$ for all $A \subseteq T$.*

Proof. In the literature, there exists a series of "sandwich theorems" under various conditions on β and ϱ; see [1], [2], [4], [7], and [8]. However, I have not found a version which fits to our objective and I have chosen to give a self-contained proof.

Since $\mathcal{K}$ is a set lattice and β supermodular, we see that $\beta_{\mathcal{K}}$ is an increasing submodular set function.

(2) Let $A, B \subseteq T$, $K \in \mathcal{K}$ and $G \in \mathcal{G}$ be given sets satisfying $K \subseteq A \cup B$ and $B \subseteq G$. Then $K \backslash G \subseteq A$, and since $G \in \mathcal{G} \subseteq \mathcal{G}(\mathcal{K})$, we have $K \backslash G \in \mathcal{K}$. Hence $\beta(K \backslash G) \leqslant \beta_{\mathcal{K}}(A)$, and so by (1) we obtain $\beta(K) \leqslant \beta_{\mathcal{K}}(A) + \varrho(G)$. Taking infimum over G and supremum over K, we see that $\beta_{\mathcal{K}}(A \cup B) \leqslant \beta_{\mathcal{K}}(A) + \varrho^{\mathcal{G}}(B)$.

Let $D_1 \subseteq \ldots \subseteq D_n \subseteq T$ be given sets and let $\Sigma \subseteq 2^T$ be a finite σ-algebra containing $D_1, \ldots, D_n$. Let us define $D_0 := \emptyset$ and let $\mathcal{S}$ denote the set of all non-empty atoms of the σ-algebra Σ. Then $\mathcal{S}$ is a finite disjoint partition of T and we denote by k the number of elements in $\mathcal{S}$. Since $\beta_{\mathcal{K}}$ is increasing and $\beta_{\mathcal{K}}(\emptyset) = 0$, there exists a unique integer $0 \leqslant r \leqslant n$ such that $\beta_{\mathcal{K}}(D_v) < \infty$ if $0 \leqslant v \leqslant r$ and $\beta_{\mathcal{K}}(D_v) = \infty$ if $r < v \leqslant n$. Let us define $\mathcal{S}^0 := \{S \in \mathcal{S} \mid S \subseteq D_r\}$ and $\mathcal{S}_0 := \mathcal{S} \backslash \mathcal{S}^0$ and let p and $m = k - p$ denote the number of elements in $\mathcal{S}^0$ and $\mathcal{S}_0$, respectively. Since $D_0 \subseteq D_1 \subseteq \ldots \subseteq D_r$ and $D_v \in \Sigma$, there exists an enumeration $S_1, \ldots, S_p$ of $\mathcal{S}^0$ and integers $0 = p_0 \leqslant p_1 \leqslant \ldots \leqslant p_r = p$ satisfying $D_v = S^{p_v}$ for all $0 \leqslant v \leqslant r$, where $S^0 := \emptyset$ and $S^i := S_1 \cup \ldots \cup S_i$ for $1 \leqslant i \leqslant p$. If $m \geqslant 1$, we choose $S_{p+1} \in \mathcal{S}_0$ such that

$$\beta_{\mathcal{K}}(S^p \cup S_{p+1}) = \min_{S \in \mathcal{S}_0} \beta_{\mathcal{K}}(S^p \cup S)$$

and we define

$$S^{p+1} := S^p \cup S_{p+1} \quad \text{and} \quad \mathcal{S}_1 := \mathcal{S}_0 \backslash \{S_{p+1}\}.$$

If $m \geqslant 2$, we choose $S_{p+2} \in \mathscr{S}_1$ such that

$$\beta_{\mathscr{K}}(S^{p+1} \cup S_{p+2}) = \min_{S \in \mathscr{S}_1} \beta_{\mathscr{K}}(S^{p+1} \cup S)$$

and we define

$$S^{p+2} := S^{p+1} \cup S_{p+2} \quad \text{and} \quad \mathscr{S}_2 := \mathscr{S}_1 \setminus \{S_{p+2}\}.$$

Proceeding similarly we obtain an enumeration $S_1, \ldots, S_k$ of $\mathscr{S}$ and integers $0 = p_0 \leqslant p_1 \leqslant \ldots \leqslant p_r = p$ satisfying

(i) $D_\nu = S^{p_\nu}$ and $\beta_{\mathscr{K}}(D_\nu) < \infty$ for all $0 \leqslant \nu \leqslant r$,

(ii) $\beta_{\mathscr{K}}(D_\nu) = \infty$ for all $r < \nu \leqslant n$,

(iii) $\beta_{\mathscr{K}}(S^i) = \min_{i \leqslant \nu \leqslant k} \beta_{\mathscr{K}}(S^{i-1} \cup S_\nu)$ for all $p < i \leqslant k$,

where $S^0 := \emptyset$ and $S^i := S_1 \cup \ldots \cup S_i$ for $1 \leqslant i \leqslant k$.

Let $u_i \in S_i$ be an arbitrary but fixed element in S_i for all $1 \leqslant i \leqslant k$ and let us define $c_i := \beta_{\mathscr{K}}(S^i) - \beta_{\mathscr{K}}(S^{i-1})$ for $1 \leqslant i < q$ and $c_i := \infty$ for $q \leqslant i \leqslant k$, where

$$q := \inf\{0 \leqslant i \leqslant k \mid \beta_{\mathscr{K}}(S^i) = \infty\}$$

with the convention $\inf\emptyset := k+1$. Since $\beta_{\mathscr{K}}$ is increasing and $\beta_{\mathscr{K}}(S^p) < \infty$, we have $p < q \leqslant k+1$, and if we define $\mu(A) := \sum_{i=0}^{k} c_i 1_A(u_i)$ for all $A \subseteq T$, then μ is a discrete measure on $(T, 2^T)$ and we claim that the following holds:

(iv) $\varrho^{\mathscr{G}}(S_i) = \beta_{\mathscr{K}}(S^i) = \infty$ for all $q \leqslant i \leqslant k$, $\beta_{\mathscr{K}}(S^i) < \infty$ for all $0 \leqslant i < q$;

(v) $\mu(S^i) = \beta_{\mathscr{K}}(S^i)$ for all $i = 0, 1, \ldots, k$;

(vi) $\beta_{\mathscr{K}}(A) \leqslant \mu(A)$ for all $A \in \Sigma$;

(vii) $\mu(D_\nu) = \beta_{\mathscr{K}}(D_\nu)$ for all $\nu = 0, 1, \ldots, n$;

(viii) $\mu(A) \leqslant \varrho^{\mathscr{G}}(A)$ for all $A \in \Sigma$.

Proof of (iv). Since $\beta_{\mathscr{K}}$ is increasing, $\beta_{\mathscr{K}}(S^i) < \infty$ for all $0 \leqslant i < q$ and $\beta_{\mathscr{K}}(S^i) = \infty$ for all $q \leqslant i \leqslant k$. Let $q \leqslant i \leqslant k$ be given. By (iii), we have $\beta_{\mathscr{K}}(S^q) \leqslant \beta_{\mathscr{K}}(S^{q-1} \cup S_i)$, and so by (2) we obtain

$$\infty = \beta_{\mathscr{K}}(S^q) \leqslant \beta_{\mathscr{K}}(S^{q-1} \cup S_i) \leqslant \beta_{\mathscr{K}}(S^{q-1}) + \varrho^{\mathscr{G}}(S_i).$$

Since $\beta_{\mathscr{K}}(S^{q-1}) < \infty$, we get $\varrho^{\mathscr{G}}(S^i) = \infty = \beta(S^i)$, which completes the proof of (iv).

Proof of (v). Let $0 \leqslant i \leqslant k$ be given. If $i < q$, we have $c_j = \beta_{\mathscr{K}}(S^j) - \beta_{\mathscr{K}}(S^{j-1})$ for all $1 \leqslant j \leqslant i$, and since $\beta_{\mathscr{K}}(S^0) = \beta_{\mathscr{K}}(\emptyset) = 0$ and $\mu(S^i) = \sum_{1 \leqslant j \leqslant i} c_i$, we see that $\mu(S^i) = \beta_{\mathscr{K}}(S^i)$. If $i \geqslant q$, we have $c_i = \infty$ and $\beta_{\mathscr{K}}(S^i) = \infty$, and since $\mu(S^i) \geqslant c_i$, we infer that $\beta_{\mathscr{K}}(S^i) = \infty = \mu(S^i)$. Thus $\mu(S^i) = \beta_{\mathscr{K}}(S^i)$ for all $1 \leqslant i \leqslant k$.

Proof of (vi). Let Π denote the set of all subsets of $\{1, \ldots, k\}$ and let us define $S_\pi := \bigcup_{i \in \pi} S_i$ for all $\pi \in \Pi$ with the convention $S_\emptyset := \emptyset$. Since Σ is a finite σ-algebra, we have $\Sigma := \{S_\pi \mid \pi \in \Pi\}$. Hence, we must show that $\beta_{\mathscr{K}}(S_\pi) \leqslant \mu(S_\pi)$ for all $\pi \in \Pi$. We shall do this by induction on the number of elements in π. Let

Π_j denote the set of all $\pi \in \Pi$ with exactly j elements for $j = 0, 1, \ldots, k$. If $\pi \in \Pi_0$, we have $\pi = \emptyset$ and $S_\pi = \emptyset$, and so $\mu(S_\pi) = 0 = \beta_{\mathscr{K}}(S_\pi)$ for all $\pi \in \Pi_0$. Let $1 \leqslant j \leqslant k$ be a given integer satisfying $\mu(S_\pi) \geqslant \beta_{\mathscr{K}}(S_\pi)$ for all $\pi \in \Pi_{j-1}$ and let us show that $\mu(S_\pi) \geqslant \beta_{\mathscr{K}}(S_\pi)$ for all $\pi \in \Pi_j$. So let $\pi \in \Pi_j$ be given. Since $j \geqslant 1$, we have $\pi \neq \emptyset$, and let $i := \max \pi$ denote the largest element in π. If $i \geqslant q$, we have $\mu(S_\pi) \geqslant c_i = \infty$, and so $\mu(S_\pi) \geqslant \beta_{\mathscr{K}}(S_\pi)$. Then suppose that $i < q$. Since $\tau := \pi \backslash \{i\} \in \Pi_{j-1}$, we have $\mu(S_\tau) \geqslant \beta_{\mathscr{K}}(S_\tau)$, and since $i < q$ and $S_\pi \subseteq S^i$, we have $\beta_{\mathscr{K}}(S_\pi) \leqslant \beta_{\mathscr{K}}(S^i) < \infty$. Observe that $S^i = S^{i-1} \cup S_\pi$ and $S_\tau = S^{i-1} \cap S_\pi$. Hence, by supermodularity of $\beta_{\mathscr{K}}$, we obtain

$$\beta_{\mathscr{K}}(S^{i-1}) + \beta_{\mathscr{K}}(S_\pi) \leqslant \beta_{\mathscr{K}}(S^i) + \beta_{\mathscr{K}}(S_\tau) \leqslant \beta_{\mathscr{K}}(S^i) + \mu(S_\tau).$$

Since $S_\pi = S_\tau \cup S_i$ and $S_\tau \cap S_i = \emptyset$, we have $\mu(S_\pi) = \mu(S_\tau) + \mu(S_i) = \mu(S_\tau) + c_i$, and since $i < q$, we get $c_i = \beta_{\mathscr{K}}(S^i) - \beta(S^{i-1}) < \infty$. Consequently,

$$\begin{aligned} \beta_{\mathscr{K}}(S^i) + \beta_{\mathscr{K}}(S_\pi) &= \beta_{\mathscr{K}}(S^{i-1}) + \beta_{\mathscr{K}}(S_\pi) + c_i \\ &\leqslant \beta_{\mathscr{K}}(S^i) + \mu(S_\tau) + c_i = \beta_{\mathscr{K}}(S^i) + \mu(S_\pi), \end{aligned}$$

and since $\beta_{\mathscr{K}}(S^i) < \infty$, we infer that $\beta_{\mathscr{K}}(S_\pi) \leqslant \mu(S_\pi)$ for all $\pi \in \Pi_j$. Thus, by induction on j, we see that $\beta_{\mathscr{K}}(S_\pi) \leqslant \mu(S_\pi)$ for all $\pi \in \Pi$, and since $\Sigma = \{S_\pi \mid \pi \in \Pi\}$, we have proved (vi).

Proof of (vii). Let $0 \leqslant v \leqslant n$ be given. If $v \leqslant r$, we have $D_v = S^{p_v}$, and so by (v) we obtain $\beta_{\mathscr{K}}(D_v) = \mu(D_v)$. If $v > r$, we get $\beta_{\mathscr{K}}(D_v) = \infty$, and so by (vi) we have $\beta_{\mathscr{K}}(D_v) = \infty = \mu(D_v)$, which completes the proof of (vii).

Proof of (viii). Let $A \in \Sigma$ be given. If $\varrho^{\mathscr{G}}(A) = \infty$ or $A = \emptyset$, then (viii) holds trivially. So suppose that $A \neq \emptyset$ and $\varrho^{\mathscr{G}}(A) < \infty$. Then there exists $\pi \in \Pi$ such that $\pi \neq \emptyset$ and $A = S_\pi$. Let $i = \max \pi$ denote the largest integer in π. Since $\varrho^{\mathscr{G}}$ is increasing and $S_i \subseteq S_\pi$, we have $\varrho^{\mathscr{G}}(S_i) \leqslant \varrho^{\mathscr{G}}(S_\pi) = \varrho^{\mathscr{G}}(A) < \infty$, and so, by (iv), we see that $i < q$. By (2) and (vi), we obtain

$$\beta_{\mathscr{K}}(S^i) \leqslant \beta_{\mathscr{K}}(S^i \backslash S_\pi) + \varrho^{\mathscr{G}}(S_\pi) \leqslant \mu(S^i \backslash S_\pi) + \varrho^{\mathscr{G}}(S_\pi),$$

and by (v) we get $\mu(S^i) = \beta_{\mathscr{K}}(S^i) < \infty$. Since μ is a measure and $A = S_\pi \subseteq S^i$, the following relations hold:

$$\mu(A) = \mu(S_\pi) = \mu(S^i) - \mu(S^i \backslash S_\pi) = \beta_{\mathscr{K}}(S^i) - \mu(S^i \backslash S_\pi) \leqslant \varrho^{\mathscr{G}}(S_\pi) = \varrho^{\mathscr{G}}(A),$$

which proves (viii).

(3) Let $\mathscr{D} \subseteq 2^T$ be a non-empty set such that $\mathscr{D}$ is linearly ordered by inclusion. Let $A \subseteq T$ and $D \in \mathscr{D}$ be given and let M_A^D denote the set of all $\mu \in M(T, 2^T)$ satisfying $\beta_{\mathscr{K}}(A) \leqslant \mu(A) \leqslant \varrho^{\mathscr{G}}(A)$ and $\mu(D) = \beta_{\mathscr{K}}(D)$. By Tychonov's theorem, $[0, \infty]^{2^T}$ is a compact Hausdorff space in the product topology and observe that M_A^D is a closed subset of $[0, \infty]^{2^T}$ for all $A \subseteq T$ and all $D \in \mathscr{D}$. Let $A_1, \ldots, A_k \subseteq T$ and $D_1, \ldots, D_n \in \mathscr{D}$ be given sets. Since $\mathscr{D}$ is linearly ordered by inclusion, we may choose the enumeration of the D_i's such that

$D_1 \subseteq \ldots \subseteq D_n$. Applying (i)–(viii) with $\Sigma := \sigma(A_1, \ldots, A_k, D_1, \ldots, D_n)$, we see that

$$\bigcap_{i=1}^{n} \bigcap_{j=1}^{k} M_{A_j}^{D_i} \neq \emptyset,$$

and since M_A^D is closed and compact in $[0, \infty]^{2^T}$, we have

$$\bigcap_{A \subseteq T} \bigcap_{D \in \mathscr{D}} M_A^D \neq \emptyset.$$

Thus, there exists a content $\mu \in M(T, 2^T)$ satisfying (3)(a).

(4) Let M denote the set of all $\mu \in M(T, 2^T)$ satisfying $\mu(T) = \beta_{\mathscr{K}}(T)$ and $\beta_{\mathscr{K}}(A) \leqslant \mu(A) \leqslant \varrho^{\mathscr{G}}(A)$ for all $A \subseteq T$. Let $\preceq$ denote the preordering on M given by $\mu_1 \preceq \mu_2$ if and only if $\mu_1(K) \leqslant \mu_2(K)$ for all $K \in \mathscr{K}$. Let $(\Gamma, \leqslant)$ be a given net and let $(\mu_\gamma)_{\gamma \in \Gamma} \subseteq M$ be a Γ-net such that (μ_γ) is increasing with respect to preordering $\preceq$. Note that M is a closed subset of the compact Hausdorff space $[0, \infty]^{2^T}$, and so there exist $\mu \in M$ and a subnet of (μ_γ) which converge to μ in the product topology on $[0, \infty]^{2^T}$. Since (μ_γ) is $(\preceq)$-increasing, $(\mu_\gamma(K))$ is increasing for all $K \in \mathscr{K}$. In particular,

$$\sup_{\gamma \in \Gamma} \mu_\gamma(K) = \lim_{\gamma \uparrow \Gamma} \mu_\gamma(K)$$

and, consequently,

$$\mu(K) = \sup_{\gamma \in \Gamma} \mu_\gamma(K) \quad \text{for all } K \in \mathscr{K}.$$

Hence $\mu_\gamma \preceq \mu$ for all $\gamma \in \Gamma$. Applying (3) with $\mathscr{D} := \{\emptyset, T\}$, we see that $M \neq \emptyset$. So by Zorn's lemma there exists a maximal element $\theta \in M$ for the preordering $\preceq$, that is:

(ix) $\theta \in M$ and if $\vartheta \in M$ and $\vartheta(K) \geqslant \theta(K)$ for all $K \in \mathscr{K}$, then $\vartheta(K) = \theta(K)$ for all $K \in \mathscr{K}$

and we claim that

(x) $\theta(K_1) = \theta(K_1 \cap K_2) + \theta_{\mathscr{K}}(K_1 \setminus K_2)$ for all $K_1, K_2 \in \mathscr{K}$.

Proof of (x). Let $K_1, K_2 \in \mathscr{K}$ be given and let us define $D := K_1 \setminus K_2$. Let $K \in \mathscr{K}$ and $G \in \mathscr{G}$ be given. Since $\theta \in M$ and $G \in \mathscr{G}$, we have $\theta(G) \leqslant \varrho^{\mathscr{G}}(G) \leqslant \varrho(G)$, and since θ is a content,

$$\theta(K) \leqslant \theta(K \setminus G) + \theta(G) \leqslant \theta(K \setminus G) + \varrho(G).$$

Hence θ is a supermodular set function such that the pair (θ, ϱ) satisfies condition (1). So by (3) applied to $(\beta, \varrho) := (\theta, \varrho)$ and the set $\mathscr{D} := \{\emptyset, D, T\}$, there exists a content $\nu \in M(T, 2^T)$ satisfying $\theta_{\mathscr{K}}(A) \leqslant \nu(A) \leqslant \varrho^{\mathscr{G}}(A)$ for all $A \subseteq T$, $\nu(D) = \theta_{\mathscr{K}}(D)$ and $\nu(T) = \theta_{\mathscr{K}}(T)$. Since $\theta \in M$, we have $\theta(T) = \beta_{\mathscr{K}}(T)$ and $\beta_{\mathscr{K}}(A) \leqslant \theta(A) \leqslant \varrho^{\mathscr{G}}(A)$ for all $A \subseteq T$. Hence $\beta_{\mathscr{K}}(A) \leqslant \theta_{\mathscr{K}}(A) \leqslant \nu(A) \leqslant \varrho^{\mathscr{G}}(A)$ for all $A \subseteq T$ and $\nu(T) = \theta_{\mathscr{K}}(T) \leqslant \theta(T) = \beta_{\mathscr{K}}(T) \leqslant \nu(T)$; that is, $\nu(T) = \beta_{\mathscr{K}}(T)$, and so $\nu \in M$. From the relation $\nu(K) \geqslant \theta(K)$ for all $K \in \mathscr{K}$ we infer that $\nu(K) = \theta(K)$ for all $K \in \mathscr{K}$ by (ix). Since K_1 and $K_1 \cap K_2$ belong to $\mathscr{K}$ and $\nu(D) = \theta_{\mathscr{K}}(D)$, we have

$$\theta(K_1) = \nu(K_1) = \nu(K_1 \cap K_2) + \nu(D) = \theta(K_1 \cap K_2) + \theta_{\mathscr{K}}(D),$$

which proves (x).

Let $\mathscr{B} := \mathscr{M}_{\theta_{\mathscr{K}}}$ denote the set of all $\theta_{\mathscr{K}}$-measurable sets and let μ denote the restriction of $\theta_{\mathscr{K}}$ to $\mathscr{M}_{\theta_{\mathscr{K}}}$. By [5], (1.23), pp. 23–24, we see that $(T, \mathscr{B}, \mu)$ is a content space and, by (x) and [5], (1.24) and (25), pp. 26–28, applied to the restriction of θ to $\mathscr{K}$, we have $\mathscr{K} \subseteq \mathscr{B}$. Let $A \subseteq T$ and $a < \mu_*(A)$ be given. Then there exists $B \in \mathscr{B}$ such that $B \subseteq A$ and $\theta_{\mathscr{K}}(B) = \mu(B) > a$. Hence, there exists $K \in \mathscr{K}$ such that $K \subseteq B$ and $\theta(K) > a$. Since $K \subseteq B \subseteq A$ and $K \in \mathscr{K} \subseteq \mathscr{B}$, we obtain $a < \theta(K) = \mu(K) \leqslant \mu_{\mathscr{K}}(A) \leqslant \mu_*(A)$. Letting $a \uparrow \mu_*(A)$, we see that $\mu_*(A) = \mu_{\mathscr{K}}(A)$, and since $\mu(C) = \theta(C)$ for all $C \in \mathscr{K}$, we have $\mu_*(A) = \mu_{\mathscr{K}}(A) = \theta_{\mathscr{K}}(A)$ for all $A \subseteq T$. Hence $(T, \mathscr{B}, \mu)$ is an inner $\mathscr{K}$-regular content space satisfying $\mathscr{B} = \mathscr{M}_{\mu_*}$, and so we see that (4)(a) follows from Theorem 2.2 (1)(a). Since $\theta \in M$, we have $\mu_*(A) = \theta_{\mathscr{K}}(A) \geqslant \beta_{\mathscr{K}}(A)$ for all $A \subseteq T$ and $\beta_{\mathscr{K}}(T) \leqslant \mu(T) \leqslant \theta(T) = \beta_{\mathscr{K}}(T)$. Let $A \subseteq T$ and $G \in \mathscr{G}$ be a given set satisfying $G \supseteq A$. Since $G \in \mathscr{G} \subseteq \mathscr{G}(\mathscr{K})$ and $\theta \in M$, we obtain $\varrho(G) \geqslant \varrho^{\mathscr{G}}(G) \geqslant \theta(G)$, and by (4)(a) we have $G \in \mathscr{B}$. From the property $\theta \in M$ we infer that $\mu^*(A) \leqslant \mu(G) = \theta_{\mathscr{K}}(G) \leqslant \theta(G) \leqslant \varrho(G)$. Taking infimum over G, we see that $\mu^*(A) \leqslant \varrho^{\mathscr{G}}(A)$, which completes the proof of (4)(b). ∎

4.2. Theorem. *Let $\eta\colon \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$ be an increasing sublinear functional and let $E(T)$ denote the set of all $f \in \boldsymbol{R}^T$ such that*

$$\sup_{t\in T} f(t) < \infty \quad \textit{and} \quad \inf_{t\in\{f>0\}} f(t) > 0.$$

Then there exist a content $\lambda \in M(T, 2^T)$, an increasing sublinear functional $\xi\colon \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$, and a net $(\mu_\pi)_{\pi\in\Pi} \subseteq M_d(T, 2^T)$ satisfying:

(1) $\eta^\circ(f) \leqslant \xi(f) = \xi^\circ(f) = \lim_{\pi\uparrow\Pi} \int_T f\, d\mu_\pi \leqslant \eta(f)$ *for all* $f \in \boldsymbol{R}^T$.

(2) $\lambda(A) = \xi(1_A)$ *for all* $A \subseteq T$, *and λ is a lower $\boldsymbol{L}^u_-(\xi)$-representation of ξ and η.*

(3) $\eta^\circ(f) \leqslant \int^* f\, d\lambda = \xi(f) \leqslant \eta(f)$ *for all* $f \in \boldsymbol{L}^u(\xi) \cup (\boldsymbol{L}^u_-(\xi) \cap E(T))$.

(4) $\boldsymbol{L}^*(\eta) \subseteq \boldsymbol{L}^*(\xi) \subseteq \boldsymbol{L}^1(\lambda)$, $\boldsymbol{L}^u(\eta) \subseteq \boldsymbol{L}^u(\xi)$, $\boldsymbol{L}^1(\eta) \subseteq \boldsymbol{L}^1(\xi)$.

Proof. (1) Let us define $\Theta := \{\theta \in \boldsymbol{R}^T \mid \eta(\theta) < \infty\}$ and let $\boldsymbol{R}^\Theta$ be equipped with its product topology. Let Υ denote the set of all $\upsilon \in \boldsymbol{R}^\Theta$ for which there exists $\mu \in M_d(T, 2^T)$ satisfying $\upsilon(\theta) \geqslant \int_T \theta\, d\mu$ for all $\theta \in \Theta$. Since η is an increasing sublinear functional, we see that Θ and Υ are convex cones. Let $n \geqslant 1$ be a given integer and let us define $\eta_n(\theta) := \eta(\theta)$ if $\theta \in \Theta$ and $\eta(\theta) > -\infty$, and $\eta_n(\theta) := -n$ if $\theta \in \Theta$ and $\eta(\theta) = -\infty$. Then $\eta_n \in \boldsymbol{R}^\Theta$ and we claim that $\eta_n \in \operatorname{cl}\Upsilon$, where cl denotes the closure operation for the product topology on $\boldsymbol{R}^\Theta$.

Suppose that $\eta_n \notin \mathrm{cl}\,\Upsilon$. Since $\Upsilon \subseteq \boldsymbol{R}^\Theta$ is a convex cone, $\mathrm{cl}\,\Upsilon$ is a closed convex subset of $\boldsymbol{R}^\Theta$, and so by the Hahn–Banach theorem (see [3], Theorem V.2.10, p. 417) there exists a continuous linear functional $F: \boldsymbol{R}^\Theta \to \boldsymbol{R}$ satisfying

$$F(\eta_n) < c := \inf_{\upsilon \in \Upsilon} F(\upsilon).$$

Since Υ is a convex cone and $c > -\infty$, we have $c = 0$, and since F is a continuous linear functional, there exist $\pi \in 2^{(\Theta)}$ and $x \in \boldsymbol{R}^\pi$ such that

$$F(\varrho) = \sum_{\theta \in \pi} x(\theta)\varrho(\theta) \quad \text{for all } \varrho \in \boldsymbol{R}^\Theta.$$

Let $\vartheta \in \pi$ be given and let us define $\varrho_\vartheta(\vartheta) := 1$ and $\varrho_\vartheta(\theta) := 0$ for $\theta \in \Theta \setminus \{\vartheta\}$. Then $F(\varrho_\vartheta) = x(\vartheta)$, and since $\varrho_\vartheta \geqslant 0$, we have $\varrho_\vartheta \in \Upsilon$. Hence $x(\vartheta) = F(\varrho_\vartheta) \geqslant c = 0$ for all $\vartheta \in \pi$. Let $t \in T$ be given and let us define $\kappa_t(\theta) := \theta(t)$ for all $\theta \in \Theta$ and $\phi := \sum_{\theta \in \pi} x(\theta)\theta$. Since $\kappa_t(\theta) = \int_T \theta d\delta_t$, we have $\kappa_t \in \Upsilon$ and $\phi(t) = F(\kappa_t) \geqslant c = 0$ for all $t \in T$, and since $x(\theta) \geqslant 0$ for all $\theta \in \pi$ and η is an increasing sublinear functional, we obtain

$$0 = c > F(\eta_n) = \sum_{\theta \in \pi} x(\theta)\eta_n(\theta) \geqslant \sideset{}{^*}\sum_{\theta \in \pi} x(\theta)\eta(\theta) \geqslant \eta(\phi) \geqslant 0,$$

which is impossible. Hence $\eta_n \in \mathrm{cl}\,\Upsilon$ for all $n \geqslant 1$. Let $\bar{\Upsilon}$ denote the closure of Υ in $\bar{\boldsymbol{R}}^\Theta$ (with respect to the product topology). Then $\mathrm{cl}\,\Upsilon = \bar{\Upsilon} \cap \boldsymbol{R}^\Theta$, and since $\eta_n \in \mathrm{cl}\,\Upsilon$ and $\eta_n(\theta) \to \eta(\theta)$ for all $\theta \in \Theta$, we have $\eta_\Theta \in \bar{\Upsilon}$, where η_Θ denotes the restriction of η to the set Θ. Hence, there exist a net $(\Gamma, \leqslant)$ and a Γ-net $(\varrho_\gamma)_{\gamma \in \Gamma} \subseteq \Upsilon$ such that $\eta(\theta) = \lim_{\gamma \uparrow \Gamma} \varrho_\gamma(\theta)$ for all $\theta \in \Theta$, and since $\varrho_\gamma \in \Upsilon$, there exist measures $\nu_\gamma \in M_d(T, 2^T)$ such that $\int_T \theta d\nu_\gamma \leqslant \varrho_\gamma(\theta)$ for all $\theta \in \Theta$. Hence, we have

(i) $\limsup_{\gamma \uparrow \Gamma} \int_T \theta d\nu_\gamma \leqslant \lim_{\gamma \uparrow \Gamma} \varrho_\gamma(\theta) = \eta(\theta)$ for all $\theta \in \Theta$

and by Thychonov's theorem there exists a net $(\Pi, \preceq)$ and a subnet $(\nu_{\kappa(\pi)})_{\pi \in \Pi}$ of (ν_γ) such that the limit

$$\xi(f) := \lim_{\pi \uparrow \Pi} \int_T f\, d\nu_{\kappa(\pi)}$$

exists in $\bar{\boldsymbol{R}}$ for all $f \in \boldsymbol{R}^T$. Since $L^1(\nu) = \boldsymbol{R}^T$ for all $\nu \in M_d(T, 2^T)$, we see that ξ is an increasing sublinear functional satisfying $\xi(f) = \xi^\circ(f)$ for all $f \in \boldsymbol{R}^T$ and by (i) we have $\xi(\theta) \leqslant \eta(\theta)$ for all $\theta \in \Theta$. Since $\eta(f) = \infty$ for all $f \in \boldsymbol{R}^T \setminus \Theta$, we have $\xi(f) \leqslant \eta(f)$ for all $f \in \boldsymbol{R}^T$. Applying this to $-f$, we infer that $\eta^\circ(f) \leqslant \xi(f) \leqslant \eta(f)$ for all $f \in \boldsymbol{R}^T$, and so we see that (1) holds with $\mu_\pi := \nu_{\kappa(\pi)}$.

(2)–(4). Let us define $\lambda(A) := \xi(1_A)$ for all $A \subseteq T$. Since $\xi = \xi^\circ$, it follows that ξ is additive on $\boldsymbol{R}_+^T$, and so we see that λ is a content on $(T, 2^T)$. Let $f \in \boldsymbol{L}^*(\xi)$ be given. By Theorem 2.1 (7), we have

$$\int^* |f|\, d\mu = \int_0^\infty \lambda(|f| > x)\, dx$$

and by [6], Theorem 2.3.9, p. 310, and additivity of ξ on $\boldsymbol{R}_+^T$, we obtain

$$\int_0^\infty \lambda(|f| > x)\,dx \leqslant \xi(|f|) < \infty.$$

Since $\mathcal{M}_\lambda = 2^T$, we get $W(\lambda) = \boldsymbol{R}^T$, and so by Theorem 2.1 (8) we have $f \in L^1(\lambda)$; that is, $\boldsymbol{L}^*(\xi) \subseteq L^1(\mu)$ and the remaining inclusions in (4) follow from (1). Let $f \in \boldsymbol{L}_-^u(\xi)$ be given. Then $f^- \in \boldsymbol{L}^u(\xi) \subseteq L^1(\lambda)$, and so we see that $\int_* f\,d\lambda > -\infty$. By Theorem 2.1 (7), we have

$$\int^* f^\pm\,d\lambda = \int_0^\infty \lambda(f^\pm > x)\,dx,$$

and since $f^- \in L^1(\lambda)$, we get $\int^* f^-\,d\lambda = \int_* f^-\,d\lambda < \infty$. So by [6], Theorem 2.3.9, p. 310, and Theorem 2.2.6, pp. 294–295, and by additivity of ξ, we have

$$\int_* f^-\,d\lambda = \xi(f^-) < \infty \quad \text{and} \quad \int^* f^+\,d\lambda \leqslant \xi(f^+).$$

Since $\xi = \xi^\circ$ and $0 \leqslant \xi(f^-) < \infty$, the following relations hold:

$$\xi(f) \leqslant \xi(f^+) + \xi(-f^-) = \xi(f^+) - \xi(f^-) = \xi(f^+) + \xi^\circ(-f^-) \leqslant \xi(f).$$

Hence $\xi(f) = \xi(f^+) - \xi(f^-)$, and so by Theorem 2.1 (5) we have

$$\int^* f\,d\lambda = \int^* f^+\,d\lambda - \int^* f^-\,d\lambda \leqslant \xi(f^+) - \xi(f^-) = \xi(f).$$

Thus, by (1), we see that λ is a lower $\boldsymbol{L}_-^u(\xi)$-representation of ξ and η. In particular, (3) holds if $\int^* f\,d\lambda = \infty$ or if $f \in \boldsymbol{L}^u(\xi)$. So suppose that $f \in \boldsymbol{L}_-^u(\xi) \cap E(T)$ and that $\int^* f\,d\lambda < \infty$. Then $\int^* f\,d\lambda \leqslant \xi(f)$, and since $f \in E(T)$, there exist positive numbers $b > a > 0$ such that $a1_A \leqslant f^+ \leqslant b1_A$, where $A := \{f > 0\}$. Since $a > 0$ and $\int^* f^+\,d\lambda < \infty$, we obtain $\xi(1_A) = \lambda(A) < \infty$ and $\xi(f^+) \leqslant b\lambda(A) < \infty$. Hence, by Lemma 3.3, we have $f^+ \in \boldsymbol{L}^u(\xi)$, and so

$$-f \in \boldsymbol{L}_-^u(\xi) \quad \text{and} \quad \xi(f) = -\xi^\circ(-f) = -\xi(-f) \leqslant -\int^*(-f)\,d\mu = \int_* f\,d\mu,$$

which completes the proof of (2)–(4). ∎

4.3. THEOREM. *Let $\eta\colon \boldsymbol{R}^T \to \bar{\boldsymbol{R}}$ be an increasing sublinear functional, let $\mathcal{K} \subseteq 2^T$ be a given set lattice and let us define $W := W(T, \mathcal{G}(\mathcal{K}))$. Then there exists an inner $\mathcal{K}$-regular content space $(T, \mathcal{B}, \mu)$ satisfying:*

(1) *$\mathcal{K} \subseteq \mathcal{F}(\mathcal{K}) \subseteq \mathcal{B} = \mathcal{M}_{\mu_*} = \{B \subseteq T \mid K \cap B \in \mathcal{B}$ for all $K \in \mathcal{K}\}$.*

(2) *$\eta_{\mathcal{K}}(A) \leqslant \mu_*(A) \leqslant \mu^*(A) \leqslant \eta^{\mathcal{G}(\mathcal{K})}(A) = \eta^W(1_A)$ for all $A \subseteq T$.*

(3) *If $\eta^W(1_K) < \infty$ for all $K \in \mathcal{K}$, then μ is finitely founded and $\mu \in M_r(T, \mathcal{K})$.*

(4) *μ is a lower $(W \cap \boldsymbol{L}_-^*(\mu) \cap \mathfrak{L}_-^{\mathcal{K}}(\eta))$-representation of η and we have $A \in \mathcal{F}(\mathcal{K}) \cap \mathcal{G}(\mathcal{K})$ and $\mu(T \setminus A) = 0$ for all $A \subseteq T$ such that $A \supseteq \bigcup \mathcal{K}$.*

(5) *If μ is finitely founded, then μ is a lower $(W \cap \mathfrak{L}_-^{\mathscr{K}}(\xi))$-representation of η and μ is a lower $\mathfrak{L}_-^{\mathscr{K}}(\xi)$-representation of η^W.*

(6) *If μ is finitely founded, $\mathscr{K}_{(\cap c)} \subseteq \mathscr{B}$ and $\mu(K_n) \to 0$ for every decreasing sequence $(K_n) \subseteq \mathscr{K}$ such that $K_n \downarrow \emptyset$ and $\mu(K_1) < \infty$, then $(T, \mathscr{B}, \mu)$ is a measure space.*

Moreover, if $\Lambda \subseteq \boldsymbol{R}^T$ is a non-empty set satisfying

$$\Theta := \Lambda \cap \mathfrak{L}_-^{\mathscr{K}}(\eta) \cap \boldsymbol{L}^*(\eta) \subseteq W$$

and $\eta^\Lambda(f) \leqslant \eta(f)$ for all $f \in \mathfrak{L}_-^{\mathscr{K}}(\eta)$, then μ is a lower $\mathfrak{L}_-^{\mathscr{K}}(\eta)$-representation of η and we have:

(7) $-\infty < \int_* f\,d\mu \leqslant \int^* f\,d\mu \leqslant \eta(f) = \eta^V(f)$ *for all $f \in \mathfrak{L}_-^{\mathscr{K}}(\eta)$ and for all $V \supseteq \Theta$.*

(8) *If $\mathscr{K} \subseteq \mathscr{L}_\eta$, then μ is finitely founded and $\mu \in M_r(T, \mathscr{K})$.*

(9) *Let T be a topological space such that $\mathscr{K} \subseteq \mathscr{L}_\eta$ and $\mathscr{K} \subseteq \mathscr{K}(T) \subseteq \mathscr{F}(\mathscr{K})$. Then $\bigcup \mathscr{K} \subseteq \bigcup \mathscr{L}_\eta$ and we have:*

(a) *$\mu \in M_R(T \mid U)$ for all $U \subseteq T$ such that $U \supseteq \bigcup \mathscr{K}$.*

(b) *If $\Theta^+ \subseteq \mathrm{Lsc}(T)$, then $\bigcup \mathscr{L}_\eta$ is open and $\mu \in M_R^\circ(T \mid U)$ for all $U \subseteq T$ such that $\bigcup \mathscr{K} \subseteq U \subseteq \bigcup \mathscr{L}_\eta$.*

Proof. (1)–(3). Let $\lambda \in M(T, 2^T)$ be the content from Theorem 4.2. Then λ is an increasing modular set function on $(T, 2^T)$ and the triple $(\beta, \varrho, \mathscr{G}) := (\lambda, \lambda, \mathscr{G}(\mathscr{K}))$ satisfies condition (1) in Theorem 4.1. So by Theorem 4.1 (4) there exists an inner $\mathscr{K}$-regular content space $(T, \mathscr{B}, \mu)$ satisfying (1) and $\lambda_{\mathscr{K}}(A) \leqslant \mu_*(A) \leqslant \mu^*(A) \leqslant \lambda^{\mathscr{G}(\mathscr{K})}(A)$ for all $A \subseteq T$. Hence (2) follows from Theorem 4.2 and Lemma 3.1 (5), and (3) follows from (1), (2) and Theorem 2.2 (1)(c).

(4) Let $f \in W \cap \boldsymbol{L}_-^*(\mu) \cap \mathfrak{L}_-^{\mathscr{K}}(\eta)$ be a given function and let $y > x > 0$ and $\varepsilon > 0$ be given numbers. By Lemma 1.2, we have $f^+ \in W(T, \mathscr{G}(\mathscr{K}))$ and $f^- \in W(T, \mathscr{F}(\mathscr{K}))$, and since $\{\emptyset, T\} \in \mathscr{G}(\mathscr{K}) \cap \mathscr{F}(\mathscr{K})$, there exists $G \in \mathscr{G}(\mathscr{K})$ and $F \in \mathscr{F}(\mathscr{K})$ satisfying

$$\{f^+ > y\} \subseteq G \subseteq \{f^+ > x\} \quad \text{and} \quad \{f^- > y\} \subseteq F \subseteq \{f^- > x\}.$$

By Theorem 4.2, we have $\mathscr{K}_\eta^t \subseteq \mathscr{K}_\lambda^t$, and since $f^- \in o(T, \mathscr{K}_\eta^t)$, there exists $K \in \mathscr{K}$ such that $\lambda(\{f^- > x\} \setminus K) < \varepsilon$. Since $F \in \mathscr{F}(\mathscr{K})$, we have $C := F \cap K \in \mathscr{K}$, $C \subseteq \{f^- > x\}$ and $\lambda(F \setminus C) < \varepsilon$. From the inequalities $\mu(G) \leqslant \lambda(G)$ and $\lambda(C) \leqslant \mu(C)$ we infer that

$$\mu^*(f^+ > y) \leqslant \mu(G) \leqslant \lambda(G) \leqslant \lambda(f^+ > x),$$

$$\lambda(f^- > y) \leqslant \lambda(F) = \lambda(C) + \lambda(F \setminus C) \leqslant \mu(C) + \varepsilon \leqslant \mu^*(f^- > x) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we see that $\mu^*(f^+ > y) \leqslant \lambda(f^+ > x)$ and $\lambda(f^- > y) \leqslant \mu^*(f^- > x)$ for all $y > x > 0$. Since $x \curvearrowright \mu^*(h > x)$ and $x \curvearrowright \lambda(h > x)$ are decreasing, there exists a countable set $Q \subseteq (0, \infty)$ such that $\mu^*(f^+ > x) \leqslant \lambda(f^+ > x)$ and $\lambda(f^- > x) \leqslant \mu^*(f^- > x)$ for all $x \in (0, \infty)\backslash Q$. Hence, by Theorem 2.1,

$$\int^* f^+ \, d\mu \leqslant \int^* f^- \, d\lambda \quad \text{and} \quad \int^* f^- \, d\lambda \leqslant \int^* f^- \, d\mu,$$

and since $f^- \in L^*(\mu)$, we have

$$\int_* f^- \, d\lambda \leqslant \int^* f^- \, d\lambda \leqslant \int_* f^- \, d\mu.$$

So, by Theorem 2.1,

$$\int^* f \, d\mu = \int^* f^+ \, d\mu \dot{-} \int_* f^- \, d\mu \leqslant \int^* f^+ \, d\lambda \dot{-} \int_* f^- \, d\lambda = \int^* f \, d\lambda,$$

and since $f \in L^u_-(\eta)$ and λ is a lower $L^u_-(\eta)$-representation of η, we obtain $\int^* f \, d\mu \leqslant \eta(f)$ for all $f \in W \cap L^*_-(\mu) \cap \mathfrak{L}^{\mathscr{K}}_-(\eta)$, which proves the first statement in (4). Let $A \subseteq T$ be a given set satisfying $A \supseteq \bigcup \mathscr{K}$. By Lemma 1.1 (1), we have $A \in \mathscr{F}(\mathscr{K}) \cap \mathscr{G}(\mathscr{K})$, and since μ is inner $\mathscr{K}$-regular and $\emptyset$ is the only set in $\mathscr{K}$ which is contained in $T\backslash A$, we get $\mu(T\backslash A) = 0$.

(5) By (1), we have $W \subseteq W(\mu)$, and so by Theorem 2.1 and finite foundedness of μ we see that $\phi^- \in L(\mu)$ for all $\phi \in W$. Hence, by (4) we infer that μ is a lower $(W \cap \mathfrak{L}^{\mathscr{K}}_-(\xi))$-representation of η, and so by Theorem 3.4 (5) it follows that μ is a lower $\mathfrak{L}^{\mathscr{K}}_-(\xi)$-representation of η^W.

(6) Let us define $\mathscr{C} := \{K \in \mathscr{K} \mid \mu(K) < \infty\}$. Since μ is finitely founded and inner $\mathscr{K}$-regular, μ is inner $\mathscr{C}$-regular and by (1) and Theorem 2.1 (9) we see that μ is complete. Hence $\mu \in M_r(T, \mathscr{C})$, and so (6) follows from Theorem 2.3 (5).

(7) follows from Theorem 3.4 (5), and since $\boldsymbol{R}^T_+ \subseteq \mathfrak{L}^{\mathscr{K}}_-(\eta)$, we see that (8) is implied by (3) and (7) with $V := W$.

(9) By (8), we have $\mu \in M_r(T, \mathscr{K})$, and so we see that (9)(a) follows from Theorem 2.4 (1). Suppose that $\Theta^+ \subseteq \mathrm{Lsc}(T)$ and let $t \in \bigcup \mathscr{L}_\eta$ be given. By (7), we have $\eta^\Theta(1_{\{t\}}) = \eta(1_{\{t\}}) < \infty$. Consequently, there exists $\theta \in \Theta$ such that $\theta \geqslant 1_{\{t\}}$ and $\eta(\theta) < \infty$. Since $\theta(t) \geqslant 1$ and $\theta \in \Theta^+ \subseteq \mathrm{Lsc}(T)$, we see that $G := \{\theta > \frac{1}{2}\}$ is an open set containing t, and by Lemma 3.1 (1) we have $\eta(1_G) \leqslant 2\eta(\theta) < \infty$. Hence $\eta^{\mathscr{G}(T)}(\{t\}) \leqslant \eta(1_G) < \infty$, and so $G \subseteq \bigcup \mathscr{L}_\eta$. In particular, $\bigcup \mathscr{L}_\eta$ is open and by Lemma 1.1 (2) we have $\mathscr{G}(T) \subseteq \mathscr{G}(\mathscr{K})$. So, by (2), $\mu^{\mathscr{G}(T)}(\{t\}) \leqslant \eta^{\mathscr{G}(T)}(\{t\}) < \infty$ for all $t \in \bigcup \mathscr{L}_\eta$. Hence (9)(b) follows from (9)(a). ∎

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