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LÉVY PROCESSES AND STOCHASTIC INTEGRALS IN BANACH SPACES

BY

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Dedicated to the memory of K. Urbanik

Abstract. We review infinite divisibility and Lévy processes in Banach spaces and discuss the relationship with notions of type and cotype. The Lévy–Itô decomposition is described. Strong, weak and Pettis-style notions of stochastic integral are introduced and applied to construct generalised Ornstein–Uhlenbeck processes.

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1. INTRODUCTION

This review article has several interlocking themes – Lévy processes, geometry and probability in Banach spaces, stochastic integration, stochastic evolution equations, Ornstein–Uhlenbeck processes and self-decomposability. We discuss each of these in turn.

Infinitely divisible distributions and Lévy processes, their dynamic counterpart, have been the subject of intense activity in recent years. This is partly because their path decomposition into a continuous Gaussian part plus an independent superposition of jumps of all possible sizes makes them ideally suited for modelling random phenomena which manifest discontinuity. This is particularly pertinent for models of the stock market and indeed this has become a major area of application for Lévy processes (see e.g. [11], [25], [19], Chapter 5 of [3] and Chapter 7 of [7]).

All of the above references use finite-dimensional (and often real-valued) Lévy processes. Many workers in the field have recognised that the complexity of the market is often captured more effectively using infinite-dimensional stochastic analysis, particularly in the realm of interest rates [9], and it is likely that Lévy processes will play a major role in this direction.

In this paper we review and survey results about Lévy processes in Banach spaces with particular emphasis on the stochastic integrals which are used to construct the stochastic evolution equations which are of use in finance. A key feature of probability in Banach spaces is the role of geometry. In the Hilbert space case, these finer aspects do not come into play as the underlying geometry is so well-behaved. After reviewing some basic facts about Lévy processes in Section 2, we describe the relationship between Lévy measures and the geometric notions of type and cotype in Section 3.

In Section 4, we describe the Lévy–Itô decomposition alluded to above, which gives the sample path structure of a generic Lévy process in terms of Gaussian and jump components. Following Dettweiler [13], we give an account of "strong" stochastic integration in Section 5. Geometric considerations again play a role in limiting the types of Banach spaces in which such integrals can be defined and despite the beautiful mathematics which so arises, this might be seen as a major drawback for stochastic evolution equations. In Section 6, we indicate how recent work on weaker types of stochastic integration can overcome this obstacle, as they are not tied to the Banach space geometry.

Lévy-driven stochastic evolution equations are introduced in Section 7. Most of the work on these has been in the case where the driving process is a Brownian motion [12]. One particular example which has seen some attention (at least in the Hilbert space context), and where the driving process is genuinely Lévy, leads to the infinite-dimensional Ornstein–Uhlenbeck process. Stationary solutions of this equation have the nice property that the random variables which comprise the process are operator self-decomposable in the sense of Urbanik [28].

Preliminaries. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t, t \geqslant 0), P)$ be a stochastic base wherein the filtration $(\mathscr{F}_t, t \geqslant 0)$ satisfies the usual hypotheses of completeness and right continuity. Let E be a separable Banach space with dual E'. Duality between E and E' is expressed by means of $\langle\cdot,\cdot\rangle$. $\mathscr{B}(E)$ will denote the σ-algebra of all Borel sets in E. The open ball in E of radius r and centered on x is denoted by $B_r(x)$. We write $B_1 := B_1(0)$. If E and F are Banach spaces, then $\mathscr{L}(E, F)$ denotes the linear space of all bounded linear operators from E to F. It is itself a Banach space with respect to the supremum norm. We write $\mathscr{L}(E) = \mathscr{L}(E, E)$.

If Y is an E-valued random variable defined on $(\Omega, \mathscr{F}, P)$, its law is denoted by p_Y. We write $X \stackrel{d}{=} Y$, whenever X and Y are E-valued random variables for which $p_X = p_Y$. If a sequence of probability measures $(\varrho_n, n \in N)$ converges weakly to ϱ, we write $\varrho_n \Rightarrow \varrho$ as $n \to \infty$.

$\mathscr{M}(E)$ is the set of all σ-finite Borel measures on E. For each $\mu \in \mathscr{M}(E)$, we define $\bar{\mu}(A) = \mu(-A)$. μ is symmetric if $\bar{\mu} = \mu$.

$\mathscr{M}_F(E)$ is the subset of $\mathscr{M}(E)$ comprising finite Radon measures. If $\mu \in \mathscr{M}_F(E)$, its characteristic function is the map $\hat{\mu}: E' \to \boldsymbol{C}$, where $\hat{\mu}(a) = \int_E e^{i\langle x,a\rangle} \mu(dx)$ for each $a \in E'$.

We will employ the notation $L^p(\Omega; E) := L^p(\Omega, \mathscr{F}, P; E)$ for each $p \geqslant 1$. Lebesgue measure on $\boldsymbol{R}^+$ is denoted by Leb, when convenient.

2. LÉVY MEASURES AND LÉVY PROCESSES

$\nu \in \mathscr{M}(E)$ is a *symmetric Lévy measure* if it is symmetric and satisfies

(i) $\nu(\{0\}) = 0$;

(ii) the mapping from E' to $\boldsymbol{R}$ given by

$$a \to \exp\{\int_E [\cos(\langle x, a\rangle) - 1]\, \nu(dx)\}$$

is the characteristic function of a measure in $\mathscr{M}_F(E)$.

$\nu \in \mathscr{M}(E)$ is a *Lévy measure* if $\nu + \bar{\nu}$ is a symmetric Lévy measure. We gather together some useful facts about Lévy measures in the next proposition. Full proofs can be found in [20], pp. 69–75 (see also [14], Section 3.4).

PROPOSITION 2.1. *If ν is a Lévy measure on E, then*:

(i) *For each $a \in E'$*,

$$\int_E |e^{i\langle x,a\rangle} - 1 - i\langle x, a\rangle 1_{B_1}(x)|\, \nu(dx) < \infty.$$

(ii) *The mapping from E' to $\boldsymbol{C}$ given by*

$$a \to \exp\{\int_E [e^{i\langle x,a\rangle} - 1 - i\langle x, a\rangle 1_{B_1}(x)]\, \nu(dx)\}$$

is the characteristic function of a probability measure in $\mathscr{M}_F(E)$.

(iii) $\nu(B_\delta(0)^c) < \infty$ *for all* $\delta > 0$.

(iv) $\sup_{\|a\| \leqslant 1} \int_{\|x\| \leqslant 1} |\langle x, a\rangle|^2 \nu(dx) < \infty$.

Let $X = (X(t), t \geqslant 0)$ be a Lévy process defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t, t \geqslant 0), P)$ and taking values in E. This means that:

1. Each X is adapted to the filtration $(\mathscr{F}_t, t \geqslant 0)$.

2. For each $0 \leqslant s < t \leqslant \infty$, $X(t) - X(s)$ is independent of $\mathscr{F}_s$ and its distribution depends only on $t - s$.

3. X has càdlàg paths with $X(0) = 0$ (a.s.).

4. $t \to X(t)$ is stochastically continuous.

We then have the Lévy–Khintchine formula: for each $t \geqslant 0$

$$E(e^{i\langle X(t),a\rangle}) = e^{-t\eta(a)}$$

for all $a \in E'$, where $\eta : E' \to C$ is a continuous, hermitian, negative definite function for which $\eta(0) = 0$, which takes the form

$$(2.1) \qquad \eta(a) = -i\langle m, a\rangle + \frac{1}{2}\langle Ra, a\rangle + \int_E [1 - e^{i\langle x, a\rangle} + i\langle x, a\rangle 1_{B_1}(x)]\, \nu(dx),$$

wherein $m \in E$, R is a positive symmetric linear operator from E' to E and ν is a Lévy measure. The triple (m, R, ν) is called the *characteristics* of X. It uniquely determines the law of each $X(t)$.

EXAMPLES. (1) Brownian motion with covariance R is the Lévy process B_R with characteristics $(0, R, 0)$. It has continuous sample paths (a.s.) and each $B_R(t)$ is centered Gaussian with

$$E\big(\langle B_R(s), a\rangle \langle B_R(t), b\rangle\big) = \langle Ra, b\rangle\, s \wedge t \quad \text{for each } a, b \in E'.$$

(2) If $(Y_n, n \in N)$ is a sequence of i.i.d. random variables with common law μ, and $(N(t), t \geqslant 0)$ is an independent Poisson process of intensity c, then X is a compound Poisson process, wherein each $X(t) := \sum_{i=1}^{N(t)} Y_i$. It has characteristics $\left(-c\int_{B_1} x\mu(dx), 0, c\mu\right)$.

Note that when E is a Hilbert space, the covariance operator of any E-valued Brownian motion is always trace class (see e.g. [12], p. 55).

3. TYPE, COTYPE AND LÉVY MEASURE

In Euclidean space, a Borel measure ν which has mass zero at the origin is a Lévy measure if and only if $\int (\|x\|^2 \wedge 1)\, \nu(dx) < \infty$. In a general Banach space, this may not be sufficient, but it is of interest to try to identify some types of spaces for which it is. This is intimately tied up with Banach space geometry – specifically the notions of "type" and "cotype". We give a rapid review of these ideas in this section. The paper [26] is an excellent reference for the main concepts. The connection with Lévy measures can be found in [6] (see also [5] and [1]).

Let $(\varepsilon_n, n \in N)$ be a Rademacher sequence, i.e. the ε_ns are i.i.d. symmetric Bernoulli random variables each having range $\{-1, 1\}$. E is said to be of *type* p if for every sequence $(x_n, n \in N)$ of elements of E

$$(3.1) \qquad \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p} < \infty \Rightarrow \sum_{n=1}^{\infty} \varepsilon_n x_n \text{ converges a.s.}$$

Every Banach space is of type 1 and only $E = \{0\}$ has type $p > 2$, so we assume that $1 < p \leqslant 2$. If $L' := L'(M, \mathscr{A}, \mu)$, where $(M, \mathscr{A}, \mu)$ is an arbitrary σ-finite measure space, then L^p is of type p for $1 < p \leqslant 2$ and of type 2 for $p > 2$.

A Banach space E is of *cotype* p if the implication in (3.1) is reversed. For cotype we can only have $p \geqslant 2$. Every Banach space is of cotype ∞ (where the left-hand side of (3.1) is interpreted as a supremum). For $1 \leqslant p \leqslant 2$, L^p is of cotype 2. Some useful relations between type and cotype are:

- If E is of type 2, then E' is of cotype 2.
- If E is of both type and cotype 2, then it is isometrically isomorphic to a Hilbert space.

A useful alternative characterisation of type and cotype is as follows: E is of type p iff there exists $C > 0$ such that, given any $n \in N$ and any set $\{X_1, \ldots, X_n\}$ of independent random variables in $L^p(\Omega; E)$,

$$E\left(\left\|\sum_{i=1}^{n} X_i\right\|^p\right) \leqslant C \sum_{i=1}^{n} E(\|X_i\|^p). \tag{3.2}$$

E is of cotype p iff the inequality in (3.2) is reversed.

Now let ν be a Borel measure on E and define $\gamma_p(x) := \|x\|^p \wedge 1$ for each $x \in E$ and each $p \geqslant 1$, so that $\int_E \gamma_p(x)\,\nu(dx) \in [0, \infty]$. In the following we will say that ν *integrates* γ_p if $\int_E \gamma_p(x)\,\nu(dx) < \infty$.

THEOREM 3.1 (Araujo and Giné). 1. *E is of cotype 2 iff every Lévy measure on E integrates γ_2.*

2. *E is of type p iff every Borel measure which integrates γ_p and assigns zero mass to $\{0\}$ is a Lévy measure.*

From this and the remarks made above we deduce the following

COROLLARY 3.1. *The following are equivalent:*
1. *ν is a Lévy measure iff ν integrates γ_2 and assigns zero mass to $\{0\}$.*
2. *E is isometrically isomorphic to a Hilbert space.*

Dynamical versions of the above results are due to Dettweiler [13].

THEOREM 3.2 (Dettweiler). 1. *If E has cotype p, then the Lévy measure of any Lévy process integrates γ_p.*

2. *If E is of type p and ν is a Borel measure which integrates γ_p and assigns zero mass to $\{0\}$, then there exists a Lévy process having Lévy measure ν.*

4. THE LÉVY–ITÔ DECOMPOSITION

Let $X = (X(t), t \geqslant 0)$ be an E-valued Lévy process with characteristics (b, R, ν). The jump at time t is $\Delta X(t) := X(t) - X(t-)$. We obtain a Poisson random measure N on $\mathbf{R}^+ \times (E - \{0\})$, which has intensity measure $\mathrm{Leb} \otimes \nu$, by the prescription

$$N(t, A) := \#\{0 \leqslant s \leqslant t,\ \Delta X(t) \in A\}$$

for each $A \in \mathscr{B}(E)$. The associated compensator is denoted by $\tilde{N}$, so

$$\tilde{N}(dt, dx) = N(dt, dx) - dt\,\nu(dx).$$

We say that $A \in \mathscr{B}(E)$ is *bounded below* if $0 \notin \bar{A}$. Arguing as in Section 2.3 of [3], we see that if A is bounded below, for each $t \geqslant 0$

$$Y_A(t) := \sum_{0 \leqslant s \leqslant t} \Delta X(s) 1_{\{\Delta X(s) \in A\}} = \int_A xN(t, dx) \text{ is finite a.s.}$$

In fact, $(Y_A(t), t \geqslant 0)$ is a compound Poisson process.

If A is bounded below and $A \subseteq B_r(0)$ for some $r > 0$, we may define

$$Z_A(t) := Y_A(t) - \int_A x\nu(dx) = \int_A x\tilde{N}(t, dx).$$

Following arguments due to Dettweiler [13], we let $(A_n, n \in N)$ be a sequence of Borel sets in B_1 such that each $A_n^c := B_1 - A_n$ is bounded below and $A_n \downarrow \{0\}$ as $n \to \infty$ (e.g. we may take each $A_n = B_{1/(n+1)}(0)$). Each

$$Z_{A_n^c}(t) = Z_{A_1^c}(t) + \sum_{j=1}^{n-1} Z_{A_{j+1}^c - A_j^c}(t)$$

is a sum of independent random variables. Then for each $a \in E'$

$$\Phi_{n,t}(a) := E\left(\exp\{i\langle Z_{A_n^c}(t), a\rangle\}\right) = \exp\{t \int_{A_n^c} (e^{i\langle y,a\rangle} - 1 - i\langle y, a\rangle)\, \nu(dy)\}$$

$$\to \exp\{t \int_{B_1} (e^{i\langle y,a\rangle} - 1 - i\langle y, a\rangle)\, \nu(dy)\} =: \Phi_t(a)$$

as $n \to \infty$. By a slight variation of the proof of Proposition 2.1 (ii), we see that Φ_t is the characteristic function of an infinitely divisible probability measure $\varrho(t)$. By the Banach space version of Glivenko's theorem, $p_{Z_{A_n^c}(t)} \Rightarrow \varrho(t)$ as $n \to \infty$, and hence, by the Itô–Nisio theorem, $Z_{A_n^c}(t)$ converges a.s. to a random variable $Z(t)$ whose law is $\varrho(t)$. We write

$$Z(t) = \int_{B_1} x\tilde{N}(t, dx),$$

and note that it is independent of the choice of approximating sequence.

We then obtain

THEOREM 4.1 (Lévy–Itô decomposition). *If $X = (X(t), t \geqslant 0)$ is an E-valued Lévy process with characteristics (b, R, ν), there exists a Brownian motion B_R with covariance R and an independent Poisson random measure N on $R^+ \times (E - \{0\})$, with intensity measure* Leb$\otimes\nu$ *so that, for each $t \geqslant 0$,*

$$X(t) = bt + B_R(t) + \int_{B_1} x\tilde{N}(t, dx) + \int_{B_1^c} xN(t, dx).$$

An alternative proof of the Lévy–Itô decomposition is given in the recent paper [2] which is based on a different approach to defining the compensated integral. Let $f: E \to E$ be a simple function, i.e. $f(x) = \sum_{j=1}^n c_j 1_{A_j}$ for some $n \in N$, $A_1, \ldots, A_n$ disjoint Borel sets which are bounded below and $c_1, \ldots, c_n \in E$. Let $f: E \to E$ be measurable with $\int_E \|f(x)\|^p \nu(dx) < \infty$ for some

$p \geqslant 1$. We say that f is *p-approximable* if there exists a sequence $(f_n, n \in N)$ of simple functions such that

1. $(f_n, n \in N)$ converges to f pointwise ν a.e.;
2. $\lim_{n\to\infty} \int_E \|f(x) - f_n(x)\|^p \nu(dx) = 0$.

Now suppose that f is p-approximable by $(f_n, n \in N)$. For any $B \in \mathscr{B}(E - \{0\})$, define

$$\int_B f(x)\tilde{N}(t, dx) := \lim_{n\to\infty} \int_B f_n(x)\tilde{N}(t, dx),$$

whenever the right-hand side exists in $L^p(\Omega, \mathscr{F}, P)$ and the limit is independent of the choice of approximating sequence. In this case, $\int_B f(x)\tilde{N}(t, dx)$ is called the *strong p-integral* of f. In [2], the Lévy–Itô decomposition is established wherein $\int_{B_1} x\tilde{N}(t, dx)$ is a strong 2-integral, but such a decomposition only holds under the constraint that either γ_1 is ν-integrable or E is type 2 and γ_2 is ν-integrable.

Note that both Dettweiler [13] and Albeverio and Rüdiger [2] obtain the Lévy–Itô decomposition for the more general class of *additive processes*, i.e. those which have independent but not necessarily stationary increments.

5. STRONG STOCHASTIC INTEGRATION

In this section, we closely follow the beautiful survey paper by Dettweiler [13]. Let $X = (X(t), t \geqslant 0)$ be an E-valued Lévy process and let F be another separable Banach space. Fix $T > 0$. $\mathscr{P}$ will denote the predictable σ-algebra on $[0, T] \times \Omega$. We fix a subspace Υ of $\mathscr{L}(E, F)$ and equip Υ with a norm such that its embedding into $\mathscr{L}(E, F)$ is continuous. Let $\mathscr{E}(\Upsilon)$ denote the space of all simple predictable mappings from $\mathbf{R}^+ \times \Omega$ into $\mathscr{E}(\Upsilon)$, so $\Phi \in \mathscr{E}(\Upsilon)$ if there is a partition $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ and Υ-valued random variables $\Phi_0, \ldots, \Phi_n$ with each Φ_j being $\mathscr{F}_{t_j}$-measurable $(1 \leqslant j \leqslant n)$ such that

$$\Phi = \sum_{j=0}^{n} \Phi_j 1_{]t_j, t_{j+1}]}.$$

For such a process we define its *strong stochastic integral* via the usual prescription:

$$I_T(\Phi) := \int_0^T \Phi(s)\,dX(s) := \sum_{j=0}^{n} \Phi_j\big(X(t_{j+1}) - X(t_j)\big).$$

The process X is said to be *L^p-primitive*, where $1 \leqslant p < \infty$, if there exists a measure λ on $(\Omega \times [0, T], \mathscr{P})$ and a constant $C > 0$ such that

$$E\Big(\Big\|\int_0^T \Phi(s)\,dX(s)\Big\|_F^p\Big) \leqslant C \int_{\Omega\times[0,T]} \|\Phi(\omega, s)\|_\Upsilon^p \,\lambda(d\omega, ds)$$

for all $T \geqslant 0$. Since $\mathscr{E}(\Upsilon)$ is dense in $\mathscr{H}^p(\lambda; T) := L^p(\Omega \times [0, T], \mathscr{P}, \lambda)$, we see that I_T extends by continuity to a bounded map from $\mathscr{H}^p(\lambda; T)$ into $L^p(\Omega, \mathscr{F}, P)$, and this gives us the required extension of the strong stochastic integral.

The requirement that X be L^p-primitive imposes geometric constraints on the Banach space F and probabilistic constraints on the process X. To appreciate these, we need some definitions.

The Banach space F is said to be *p-uniformly smooth* if there exists $K > 0$ such that

$$\|x+y\|^p + \|x-y\|^p \leqslant 2\|x\|^p + K\|y\|^p \quad \text{for all } x, y \in F,$$

and it is *p-smoothable* if it is p-uniformly smooth with respect to an equivalent norm. In fact, we must take $1 < p \leqslant 2$. The p-smoothable property is equivalent to a number of other interesting geometric Banach space properties (see e.g. Theorem 18.7 in [26], p. 91). The spaces L^r, $r > 1$, are all p-smoothable for some $1 < p \leqslant 2$.

Let π_T be a partition of $[0, T]$ taking the form $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$. We define the p-integrable variation $\mathrm{Var}^{(p)}(X; \pi_T)$ of X with respect to this partition to be

$$\mathrm{Var}^{(p)}(X; \pi_T) := \sum_{i=0}^{n} E\left(\|X(t_{i+1}) - X(t_i)\|^p\right), \quad \text{where } p > 1,$$

so $\mathrm{Var}^{(p)}(X; \pi_T) \in [0, \infty]$. We say that X is of *p-integrable variation* if

$$\mathrm{Var}_T^{(p)}(X) := \sup_{\pi_T} \mathrm{Var}^{(p)}(X; \pi_T) < \infty \quad \text{for all } T > 0.$$

- Let E be a Hilbert space and X be a Lévy process having the Lévy–Itô decomposition $X(t) = B_R(t) + \int_{B_1} x\tilde{N}(t, dx)$ for all $t \geqslant 0$. X has integrable 2-variation, in fact

$$\mathrm{Var}_T^{(2)}(X) = \left(\mathrm{tr}(R) + \int_{B_1} \|x\|^2 \nu(dx)\right) T.$$

- If E has cotype p and X is centred with $E\left(\|X(t)\|^p\right) < \infty$, then X is of integrable p-variation [13].

We have the following beautiful result which shows the importance of these ideas in establishing L^p-primitivity.

THEOREM 5.1 (Dettweiler). *If F is p-smoothable and X is of integrable p-variation for some $1 < p \leqslant 2$, then X is L^p-primitive for every Υ with $\lambda = P \otimes \alpha$, where, for each $0 \leqslant s < t < \infty$,*

$$\alpha(]s, t]) := \liminf_{\pi_t} \mathrm{Var}^{(p)}(X; \pi_t) - \liminf_{\pi_s} \mathrm{Var}^{(p)}(X; \pi_s).$$

If we drop the condition that F be p-smoothable, Dettweiler has shown that there is no natural theory of strong stochastic integration for any Υ

which is sufficiently rich so as to contain all finite rank operators. For full details see Theorem 3.2 in [13], p. 70.

In the above discussion, we have given a "holistic" construction of the integral $\int_0^T \Phi(s)\,dX(s)$. An alternative approach is to make a direct use of the Lévy–Itô decomposition and try to define

$$(5.1)\quad \int_0^T \Phi(s)\,dX(s)$$

$$= b\int_0^T \Phi(s)\,ds + \int_0^T \Phi(s)\,dB_R(s) + \int_0^T \int_{B_1} \Phi(s)\,x\tilde{N}(ds, dx) + \int_0^T \int_{B_1^c} \Phi(s)\,xN(ds, dx).$$

The first term in (5.1) is a (pathwise) Bochner integral (provided Φ is sufficiently regular) and the last term is just a random finite sum, in fact

$$\int_0^T \int_{B_1^c} \Phi(s)\,xN(ds, dx) := \sum_{0 \leqslant s \leqslant t} \Phi(s)\,\Delta X(s)\,1_{\{\Delta X(s) \in B_1^c\}}.$$

The difficulty is then reduced to the definition of the two middle terms and these will be subject to the constraints described above.

The most intensive development of this approach has been in the case where $E = F$ is a Hilbert space and $\Upsilon = \mathscr{L}(E)$ (see [4], [27], [17], [18]). In this case, we may consider more general stochastic integrals which are of the form $\int_0^t \int_{B_1} \Phi(s, x)\,M(ds, dx)$, where M is the martingale-valued measure defined on $[0, T] \times B_1$ given by

$$(5.2)\qquad M(]0, t], A) = B_R(t)\,\delta_0(A) + \int_{A-\{0\}} x\tilde{N}(ds, dx)$$

for each $0 < t \leqslant T$ and each $A \in \mathscr{B}(B_1)$. It is shown in [4] that this stochastic integral can be defined, by a natural extension from simple functions, to the real Hilbert space of all $\mathscr{P}\otimes\mathscr{B}(B_1)$-measurable Φ for which

$$E\Big(\int_0^T \int_{B_1} \mathrm{tr}\big(\Phi(s, x)\,T(x)\,\Phi(s, x)^*\big)\,\nu(dx)\,ds\Big) < \infty,$$

where

$$T(x) = \begin{cases} R & \text{if } x = 0, \\ \langle x, \cdot\rangle x & \text{if } x \neq 0. \end{cases}$$

This recipe coincides with that of [12] for strong stochastic integrals with respect to Brownian motion if $\Phi(s, x) \equiv 0$, whenever $x \neq 0$.

Finally, we remark that Rüdiger [24] has extended the work of [2] to define strong p-integrals $\int_0^t \int_{E-\{0\}} f(x, s)\,\tilde{N}(ds, dx)$ for suitable predictable mappings $f: E \times \boldsymbol{R}^+ \times \Omega \to F$, where E and F are separable Banach spaces.

6. WEAK AND PETTIS-STYLE STOCHASTIC INTEGRATION

In this section, we continue to deal with stochastic integration with respect to martingale-valued measures of the form (5.2), but these are now taking values in a separable Banach space E. We denote by $\imath$ the isometric embedding of E into E'' given by $\imath(x)(a) = \langle x, a\rangle$ for each $x \in E$, $a \in E'$. Generalising Section 2 of [4], we define a field $(Q_x, x \in E)$ of positive, symmetric operators from E' to E by the prescription

$$Q_x = \begin{cases} R & \text{if } x = 0, \\ \imath(x)(\cdot)x & \text{if } x \neq 0. \end{cases}$$

Now we extend the construction of [21]. Let $F = \big(F(t, x), t \geqslant 0, x \in B_1\big)$ be $\mathscr{P}\otimes\mathscr{B}(B_1)$-measurable $\mathscr{L}(E)$-valued random variables. We say that F is *Q-weakly L^2* if for each $x \in B_1$, $a \in E'$, $T > 0$

1. $t \to F(t, x) Q_x F(t, x)^* a$ is weakly measurable,
2. $\int_0^T \int_{B_1} \langle F(t, x) Q_x F(t, x)^* a, a\rangle \nu(dx)\, dt < \infty$.

Let $\mathscr{A} = \mathscr{A}_0 \cup \{0\}$, where $\mathscr{A}_0 = \{A \in \mathscr{B}(B_1); 0 \notin \bar{A}\}$ and let $\mathscr{S}(T)$ be the space of all F for which

$$F = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} F_{ij} 1_{]t_i, t_{i+1}]} 1_{A_j},$$

where $N_1, N_2 \in N$, $0 = t_0 < t_1 < \ldots < t_{N_1+1} = T$, $A_0, \ldots, A_{N_2}$ are disjoint sets in $\mathscr{A}$ and each F_{ij} is a bounded $\mathscr{F}_{t_i}$-measurable $\mathscr{L}(E)$-valued random variable. For $F \in \mathscr{S}(T)$, and for each $a \in E'$, we define

$$\int_0^T \int_{B_1} \langle F(t, x) M(dt, dx), a\rangle := \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \langle F_{ij} M(t_{i+1} - t_i, A_j), a\rangle. \tag{6.1}$$

Standard calculations lead to

$$E\Big(\Big|\int_0^T \int_{B_1} \langle F(t, x) M(dt, dx), a\rangle\Big|^2\Big) = \int_0^T \int_{B_1} \langle F(t, x) Q_x F(t, x)^* a, a\rangle \nu(dx)\, dt,$$

and this enables us to extend the construction of the *weak stochastic integral* $\int_0^T \int_{B_1} \langle F(t, x) M(dt, dx), a\rangle$ to arbitrary F which are Q-weakly L^2 by a standard limiting argument.

We say that an F which is Q-weakly L^2 is *stochastically Pettis integrable* if there exists an E-valued stochastic process $\big(Y(t), t \geqslant 0\big)$ such that

$$\langle Y(t), a\rangle = \int_0^T \int_{B_1} \langle F(t, x) M(dt, dx), a\rangle \quad \text{for each } a \in E'.$$

The stochastic Pettis integral has been introduced and studied in some detail by van Neerven and Weis [21] (see also [23]) in the case where M

reduces to a Brownian motion. It generalises the original construction of Pettis [22] to a random framework and appears to be independent of geometric constraints.

7. LÉVY-DRIVEN ORNSTEIN–UHLENBECK PROCESSES

In recent years there has been a great deal of interest in *stochastic evolution equations* of the form

$$dY(t) = JY(t)\,dt + B\big(Y(t)\big)\,dM(t) \tag{7.1}$$

whose solution, subject to the initial condition $Y(0) = Y_0$ (a.s.), is a stochastic process $Y = \big(Y(t),\, t \geqslant 0\big)$ taking values in a Banach space E. Here J is the infinitesimal generator of a C_0 semigroup of linear operators on E, $M = \big(M(t),\, t \geqslant 0\big)$ is a semimartingale taking values in another separable Banach space F, and B is a suitable Lipschitz mapping from E to $\mathscr{L}(F, E)$. Of course, solutions to (7.1) may not exist in general. The most extensive studies have been in the case where E and F are Hilbert spaces and M is a Brownian motion or a cylindrical Wiener process in the sense of [12], pp. 96–98. Recently there has begun to be some interest in the generalisation to Lévy processes (but still in a Hilbert space context); see e.g. [4], [17], [18], [27]. One of the simplest examples of (7.1), which has itself been the object of extensive study, is

$$dY(t) = JY(t)\,dt + B\,dX(t). \tag{7.2}$$

Here we have taken B to be a fixed operator in $\mathscr{L}(F, E)$ and $X = \big(X(t),\, t \geqslant 0\big)$ to be an F-valued Lévy process. The equation (7.2) has a *weak solution* if we can find Y for which

$$\langle Y(t), a\rangle = \langle Y_0, a\rangle + \int_0^t \langle Y(s), J^* a\rangle\, ds + \langle BX(t), a\rangle$$

for all $a \in \mathrm{Dom}(J^*)$, $t \geqslant 0$. Necessary and sufficient conditions are known for a unique weak solution to (7.2) to exist when F is a Hilbert space and X is an F-valued Brownian motion or, more generally, a cylindrical Wiener process (see [8], [21]). The construction of [21] utilises the stochastic Pettis integral to construct this weak solution in a direct and appealing way.

From now on we will take $E = F$ to be a Hilbert space and X to be a general Lévy process. In this case, there is a unique weak solution to (7.2) and it coincides with the *mild solution* given by

$$Y(t) = S(t)\,Y_0 + \int_0^t S(t-s)\,dX(s). \tag{7.3}$$

The stochastic integral appearing in (7.3) is sometimes called a *stochastic convolution*. It is constructed in a strong sense in [10] and [4], in the former case holistically and in the latter case via the Lévy–Itô decomposition as outlined in Section 4 above. The process $Y = (Y(t),\ t \geqslant 0)$ as given in (7.3) is called a *Lévy-driven Ornstein–Uhlenbeck process*. It clearly generalises the well-known real-valued process obtained by taking $E = \boldsymbol{R}$ and each $S(t) = e^{-\lambda t}$ for some $\lambda > 0$ (see e.g. Section 4.3.5 of [4] and references therein). Y is a Markov process. If $(S(t),\ t \geqslant 0)$ is "stable", i.e. $\lim_{t\to\infty} S(t)u = 0$ for all $u \in E$, then Y is strictly stationary if and only if (see [10])

(A) $\lim_{t\to\infty} \int_0^t S(t)\, b dx$ exists;

(B) $\int_0^\infty \operatorname{tr}(S(t) R S(t)^*)\, dt < \infty$;

(C) $\int_0^\infty \int_{E-\{0\}} (\|S(r)x\|^2 \wedge 1)\, \nu(dx)\, dr < \infty$;

(D) $\lim_{t\to\infty} \int_0^t \int_{E-\{0\}} S(r)x\, [1_{B_1}(S(r)(x)) - 1_{B_1}(x)]\, \nu(dx)\, ds$ exists.

In the exponentially stable case where, for all $t \geqslant 0$, $\|S(t)\| \leqslant Ce^{-\lambda t}$ for some $C \geqslant 1$, $\lambda > 0$:

- (A) and (B) always hold.
- $\int_0^\infty S(t)\, dX(t)$ exists in distribution iff $\int_0^\infty \int_{B_1^c} S(u) x N(du, dx)$ exists in distribution.

A sufficient condition for stationarity is $\int_{\|x\| \geqslant 1} \log(1 + \|x\|)\, \nu(dx) < \infty$ and it is well-known that this is also necessary when $\dim(E) < \infty$ (see e.g. Theorem 4.3.17 in [3] and references therein). We say that an E-valued random variable Z is *operator self-decomposable* if and only if for each $t > 0$ there exists a random variable Z_t which is independent of Z and for which

$$Z \overset{d}{=} S(t)Z + Z_t. \tag{7.4}$$

From (7.3) we see that Y_0 is operator self-decomposable with each $Z_t \overset{d}{=} \int_0^t S(t-s)\, dX(s)$ if and only if Y is strictly stationary. Indeed, in this case we have from (7.3)

$$Y(0) \overset{d}{=} Y(t) = S(t)\, Y(0) + Z_t.$$

We remark that operator self-decomposability for Banach space valued random variables was first introduced by Urbanik in [28] (see also [15], [16]).

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