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A WARNING ABOUT AN INDEPENDENCE PROPERTY RELATED TO RANDOM BROWNIAN SCALING

BY

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Abstract. In this note, which develops a part of our paper [2], we consider independence properties between Brownian motion, after Brownian scaling on a random interval (a, b), and the length $(b-a)$ of the interval. We indicate three examples for which the Brownian scaled process is independent of the corresponding length. On the other hand, we discuss a case where this independence property does not hold and investigate further results for that example.

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Consider a real-valued Brownian motion $(B_t, t \geqslant 0)$, and the families of zeros:

$$g_t = \sup\{s \leqslant t : B_s = 0\}, \quad d_t = \inf\{s > t : B_s = 0\}, \quad t \geqslant 0.$$

We also consider, for random times a, b such that $0 \leqslant a < b$, the random Brownian scaling operation:

$$B_u^{[a,b]} \equiv \frac{1}{\sqrt{b-a}} B_{a+u(b-a)}, \quad u \leqslant 1.$$

In a number of cases of pairs (a, b), it turns out that $B_u^{[a,b]}$ is independent of $(b-a)$. This is well known in the following cases (i)–(iii):

(i) $a = 0$, $b = g_t$.
Then $B^{[0,g_t]}$ is a standard Brownian bridge independent of g_t.

(ii) $a = g_t$, $b = t$.
Then $B^{[g_t,t]}$ is independent of g_t, and hence of $t - g_t \equiv b-a$. Moreover, $|B^{[g_t,t]}|$ is a standard Brownian meander $(m_u, u \leqslant 1)$.

We recall, for the sequel of this paper, that

(1) $$(m_u^2, u \leqslant 1) \sim (b_u^2 + R_u^2, u \leqslant 1),$$

where $(b_u, u \leqslant 1)$ is a standard Brownian bridge, independent of $(R_u, u \leqslant 1)$, a standard 2-dimensional BES process.

(iii) $a = g_t$, $b = d_t$.

Then $B^{[g_t, d_t]}$ is independent of both g_t and d_t; furthermore, $|B^{[g_t, d_t]}|$ is a standard BES(3)-bridge.

Thus, given these 3 examples, it would seem quite plausible that in general $B^{[a,b]}$ is always independent of $(b-a)$.

In the remainder of this note we show that this is not a general fact, more precisely, we shall express $B^{[0,d_t]}$ in terms of d_t and of an independent 2-dimensional process, which consists of a meander and a Brownian motion, as we now state.

THEOREM. *There is the identity in law between* $(B_{ud_t}/\sqrt{d_t}, u \leqslant 1)$ *and*

$$(2)\qquad \left(u\sqrt{\left(\frac{d_t}{t}-1\right)}\left(\hat{m}\left\{\left(\frac{1}{u}-1\right)\frac{1}{d_t/t-1}\wedge 1\right\}+\beta\left\{\left(\frac{1/u-1}{d_t/t-1}-1\right)^+\right\}\right); u \leqslant 1\right),$$

where d_t, $\hat{m}$ *and* β *are independent, with* $\hat{m}(u) \equiv \varepsilon m(u)$, m *a meander independent of* ε, *a symmetric Bernoulli variable, and* β *a Brownian motion.*

COROLLARY. $B^{[0,d_t]}$ *and* d_t *are not independent.*

Assuming the truth of the Theorem for a moment, we prove the Corollary by showing that the quantity for a generic test function $f: \boldsymbol{R}_+ \to \boldsymbol{R}_+$:

$$E\left(f\left(\frac{1}{d_t}B^2_{ud_t}\right)\middle|\frac{d_t}{t}-1=\delta\right),$$

for fixed u, depends on δ. We distinguish two cases:

Case 1. $\left(\frac{1}{u}-1\right)\frac{1}{\delta} < 1.$

Then, by (2), we get

$$(3)\qquad E\left(f\left(\frac{1}{d_t}B^2_{ud_t}\right)\middle|\frac{d_t}{t}-1=\delta\right) = E\left(f\left(u^2\delta m^2_{\left(\frac{1}{u}-1\right)\frac{1}{\delta}}\right)\right).$$

But, from (1) we deduce $m_t^2 \sim (1-t)tN^2+2et$, where $N \sim \mathcal{N}(0, 1)$ and e is an exponential variable with parameter 1, independent of N. Thus, the quantity in (3) is

$$E\left(f\left(u^2\delta\left\{\left(1-\left(\frac{1}{u}-1\right)\frac{1}{\delta}\right)\left(\frac{1}{u}-1\right)\frac{N^2}{\delta}+2e\left(\frac{1}{u}-1\right)\frac{1}{\delta}\right\}\right)\right)$$

$$= E\left(f\left(\left\{u^2\left(\frac{1}{u}-1\right)\left(1-\left(\frac{1}{u}-1\right)\frac{1}{\delta}\right)N^2+2e(u-u^2)\right\}\right)\right).$$

Clearly, the latter quantity depends on δ.

Case 2. $\left(\frac{1}{u}-1\right)\frac{1}{\delta} > 1.$

Then, by (2), we get

$$E\left(f\left(\frac{1}{d_t}B_{ud_t}^2\right)\middle|\frac{d_t}{t}-1=\delta\right)=E\left(f\left(u^2\delta\left\{\hat{m}(1)+\beta\left(\frac{1/u-1}{\delta}-1\right)\right\}^2\right)\right)$$

and, as in Case 1, we can show that this quantity depends on δ.

It now remains to prove the Theorem.

Proof of the Theorem. (i) We use time inversion, i.e. $B_v = vB'_{1/v}$ for B' being another Brownian motion. Then, with obvious notation, we have

$$\frac{1}{d_t} = g'_{1/t}, \tag{4}$$

and

$$\frac{B_{ud_t}}{\sqrt{d_t}} = \frac{u}{\sqrt{g'_{1/t}}}B'_{\frac{1}{u}g'_{1/t}}, \qquad u \leqslant 1.$$

(ii) Thus, with the help of this time inversion trick, we have now transformed the problem into a study of the quantity

$$\frac{1}{\sqrt{g'_T}}B'_{g'_T(1+h)}, \qquad h \geqslant 0,$$

where we have put $T = 1/t$. For h's such that $g'_T(1+h) \leqslant T$, we may write $B'_{T'_T(1+h)}$ in terms of the meander

$$m'_u \equiv \frac{1}{\sqrt{T-g'_T}}B'_{g'_T+u(T-g'_T)}.$$

Thus, we obtain

$$\frac{B_{ud_t}}{\sqrt{d_t}} = \frac{u}{\sqrt{g'_T}}B'_{\frac{1}{u}g'_T} = \frac{u}{\sqrt{\frac{g'_T}{T-g'_T}}}m'_{\left(\frac{1}{u}-1\right)\frac{g'_T}{T-g'_T}}.$$

Now, using the relationship (4) between d_t and g'_T, we get

$$\frac{B_{ud_t}}{\sqrt{d_t}} = u\sqrt{\left(\frac{d_t}{t}-1\right)}\,m'_{\frac{1/u-1}{d_t/t-1}}.$$

However, our argument could only be developed when $(1+h)g'_T \leqslant T$.

In the other case, that is when $(1+h)g'_T > T$, we can write

$$B'_{g'_T(1+h)} \equiv B'_{T+[(h+1)g'_T-T]} \equiv B'_T+\beta\left((h+1)g'_T-T\right),$$

where $(\beta(s),\, s \geqslant 0)$ is a Brownian motion independent of $\mathscr{F}'_T$, hence of g'_T. Now, we may write the full decomposition:

$$\frac{1}{\sqrt{g'_T}} B'_{g'_T(1+h)}$$

$$\equiv \frac{1}{\sqrt{\frac{g'_T}{T-g'_T}}} m'_{h\frac{g'_T}{T-g'_T}} 1_{((1+h)g'_T \leqslant T)} + \frac{1}{\sqrt{g'_T}} \left(B'_T + \beta\left((h+1)\, g'_T - T\right)\right)_{((1+h)g'_T \geqslant T)}$$

$$= \frac{1}{\sqrt{\frac{g'_T}{T-g'_T}}} m'_{h\frac{g'_T}{T-g'_T} \wedge 1} + \frac{1}{\sqrt{g'_T}} \beta\left((h+1)\, g'_T - T\right) 1_{\left(\frac{hg'_T}{T-g'_T} \geqslant 1\right)}.$$

By Brownian scaling β after g'_T, we may still write

$$\frac{1}{\sqrt{g'_T}} B'_{g'_T(1+h)} = \frac{1}{\sqrt{\frac{g'_T}{T-g'_T}}} \left\{ m'_{h\frac{g'_T}{T-g'_T} \wedge 1} + \hat{\beta}\left(\frac{\left((1+h)\, g'_T - T\right)^+}{T-g'_T}\right)\right\}. \tag{5}$$

Coming back to $B_{ud_t}/\sqrt{d_t}$, we then obtain the identity asserted in the Theorem, which can be found in the statement of this theorem after making some elementary substitutions in (5). Finally, we leave to the interested reader some possible extensions of this study to the case of a Bessel process with dimension $0 < \delta < 2$, possibly using results about corresponding Bessel meanders as discussed in [3].

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