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STOCHASTIC VOLATILITY: APPROXIMATION AND GOODNESS-OF-FIT TEST

BY

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Abstract. Let X be the unique solution started from x_0 of the stochastic differential equation $dX_t = \theta(t, X_t)dB_t + b(t, X_t)dt$ with B a standard Brownian motion. We consider an approximation of the volatility $\theta(t, X_t)$, the drift being considered as a nuisance parameter. The approximation is based on a discrete time observation of X and we study its rate of convergence as a process. A goodness-of-fit test is also constructed.

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1. INTRODUCTION

Let X be a one-dimensional diffusion process defined by

$$X_t = x_0 + \int_0^t \theta(s, X_s)dB_s + \int_0^t b(s, X_s)ds, \qquad t \in [0, 1], \tag{1.1}$$

where $x_0 \in \mathbb{R}$ and B is a standard one-dimensional Brownian motion. Assume that b and θ are unknown functions and that one observes X at discrete moments $\{i/2^n : i = 0, \ldots, 2^n\}$ of the finite time interval $[0, 1]$. One can ask the following two natural questions:

1. Is it possible to construct an estimator for the diffusion coefficient?
2. Given a known function ϑ, is it possible to decide if $\vartheta = \theta$ or not?

These two questions are classical and, in general, to answer the second question one uses the result of the first one.

Florens-Zmirou [7] was the first to answer the first question. In the case when $\theta = \theta(x)$ is a smooth function, a pointwise estimator for $\theta(x)^2$ (when the trajectory of the diffusion visits x) based on a discrete approximation of the local time is proposed. The rate of convergence which was obtained is $2^{\alpha n}$, $\alpha < \frac{1}{3}$. When θ has only Besov smoothness, asymptotically minimax estimators can be constructed using wavelet basis (see Hoffmann [11]) and their rate of convergence are $2^{ns/(1+2s)}$ (if $\theta \in B_{spq}$).

In the simpler case, when the diffusion coefficient does not depend on X_t but only on t, Genon-Catalot et al. [8] constructed a non-parametric estimator for $\theta = \theta(t)$. More precisely, one observes

$$X_t = x_0 + \int_0^t \theta(s) dB_s + \int_0^t b(s, X_s) ds, \quad t \in [0,1],$$

and one proceeds in two steps: firstly, one constructs the estimator of the quantity $\int_0^1 h(s)\theta(s)^2 ds$ (h being any smooth function) given by

$$\sum_{i=0}^{2^n-1} h\left(\frac{i}{2^n}\right)(X_{(i+1)/2^n} - X_{i/2^n})^2, \quad n \in \mathbb{N}, \tag{1.2}$$

and, secondly, one recovers the function θ^2 by using a wavelets basis. The rate of convergence which was obtained is $2^{n/2}$ (see also Hoffmann [10] for a study in Besov spaces).

What kind of result can be obtained by using the estimator (1.2) for the setting (1.1)? The present paper is an attempt to answer this question and we emphasize that our main interest is to construct a goodness-of-fit test. Precisely, a known function ϑ being given, we want to decide if $\vartheta = \theta$ or not. For that, we need some results on the convergence in law associated with a correct renormalization of (1.2). There are several works related to this topic: see, for instance, Jacod [12] (unpublished work), Delattre and Jacod [6], Becker [3], Barndorff-Nielsen and Shephard [2]. However, for the sake of completeness of this paper, we preferred to prove here all the results that we stated.

Let us then consider (1.1) with unknown functions θ and b. Based on the observations of X at discrete moments, we infer on $\theta(t, X_t)^2$ through the estimation of the primitive process

$$I(t) := \int_0^t \theta(s, X_s)^2 ds, \quad t \in [0,1]. \tag{1.3}$$

We consider the sequence of estimators given by (1.2) with $h = \mathbf{1}_{[0,t]}$. Precisely, we set

$$\widehat{I_n}(t) := \sum_{i=0}^{[2^n t]-1} (X_{(i+1)/2^n} - X_{i/2^n})^2, \quad t \in [0,1],\, n \in \mathbb{N}. \tag{1.4}$$

It is classical that, for each $t \geqslant 0$, $\widehat{I_n}(t)$ converges in probability towards $I(t)$ (see, for instance, Berman [4]) and one can prove that $\lim_{n\to\infty} \widehat{I_n} = I$ almost surely uniformly on $[0,1]$. Here, we study the convergence in distribution of $\widehat{I_n} - I$ as a process. In a certain sense, the rate of convergence which we obtain is $2^{n/2}$ (see Corollary 2.1 and Proposition 2.2 below) and, at least from a theoretical point of view, it suffices to differentiate $I(t)$ to recover the square of the volatility $\theta(t, X_t)$.

Let us briefly explain our approach. By means of Girsanov's theorem, we reduce the estimation of the diffusion coefficient of the semimartingale

$$X_t = x_0 + \int_0^t \theta(s, X_s) dB_s + \text{(a drift term)}$$

to the estimation of the martingale coefficient of the semimartingale

$$Y_t = x_0 + \int_0^t \sigma(s, B_s) dB_s + \text{(a finite variation term)},$$

with unknown σ linked to θ (the observations are $Y_{i/2^n}$, $i = 0, 1, \ldots, 2^n - 1$). The semimartingale model has, from our point of view, an interest by itself and our analysis is performed using stochastic calculus. We used similar ideas in Gradinaru and Nourdin [9] when studying the convergence at first and second order of approximations

$$\int_0^t \left(\frac{X_{s+\varepsilon} - X_s}{\sqrt{\varepsilon}} \right)^m ds$$

of m-order stochastic integrals (the main difference is that, in the present case, we are working with sum approximations instead of integral approximations and with the Skorokhod topology instead of the uniform topology). Although some results of convergence in law could be obtained by using some of the results in references which we have already quoted, the proofs given in the present paper maybe bring out a simple self-contained approach.

We turn now to the second question, which is somehow the novelty of the present work, that is the attempt to construct goodness-of-fit tests for this problem. Consider the diffusion X given by (1.1) and let ϑ be a known function. We introduce a test statistic T_n for testing the hypothesis (H): $\vartheta^2 = \theta^2$ against (A): $\vartheta^2 \neq \theta^2$. We evaluate the type I error probability and we prove that, under (A), $T_n \to \infty$ almost surely, as $n \to \infty$ (see Proposition 3.2 below).

As for the case of the diffusion X, we study the goodness-of-fit test problem linked to Y: if ψ is a known function, we observe the semimartingale $\varphi(B_t)$ with φ unknown, and we test $(\widetilde{H})$: $\psi(|\cdot|) = \varphi(|\cdot|)$ against $(\widetilde{A})$: $\psi(|\cdot|) \neq \varphi(|\cdot|)$ (see Proposition 3.1 below). Our method also applies for some cases when φ is not a bijection.

The paper is organised as follows. In the next section, we state the results concerning the rate of convergence. Section 3 is devoted to the construction of goodness-of-fit tests. The proofs are given in Section 4.

2. CONVERGENCE IN DISTRIBUTION OF THE APPROXIMATIONS

2.1. Semimartingale model. Let Y be a semimartingale given by

$$Y_t = x_0 + \int_0^t \sigma(s, B_s)dB_s + \int_0^t M_s ds, \quad t \in [0,1], \tag{2.1}$$

where $\sigma \in \mathrm{C}^{1,2}([0,1] \times \mathbb{R}; \mathbb{R})$ has bounded derivatives with respect to the second variable, and M is a continuous adapted process.

We denote by $\widehat{J_n}$ and J the processes on $[0,1]$ given by

$$\widehat{J_n}(t) := \sum_{i=0}^{[2^n t]-1} (Y_{(i+1)/2^n} - Y_{i/2^n})^2, \; n \in \mathbb{N}, \text{ and } J(t) := \int_0^t \sigma(s, B_s)^2 ds, \tag{2.2}$$

respectively, and one can prove that, as $n \to \infty$, $\widehat{J_n}$ converges towards J, almost surely uniformly on $[0,1]$.

All processes will be considered as random elements of the space $\mathbb{D}([0,1]; \mathbb{R})$ of càdlàg real functions on $[0,1]$ endowed with the Skorokhod topology (see [5], p. 128).

We can state the following (see also [2], [3] and the unpublished work [12] for some similar questions):

THEOREM 2.1. *As $n \to \infty$,*

$$2^{n/2}(\widehat{J_n} - J) \xrightarrow{law} \sqrt{2} \int_0^{\bullet} \sigma(s, \beta_s^{(1)})^2 d\beta_s^{(2)}, \tag{2.3}$$

in the Skorokhod topology, where $\beta^{(1)}$ and $\beta^{(2)}$ are two independent standard Brownian motions.

Let us remark that the unknown function σ appears in the limit (2.3). To avoid this, we point out the following:

PROPOSITION 2.1. *Assume that $\sigma \not\equiv 0$. Set*

$$\widehat{V_n}(t) := 2^n \sum_{i=0}^{[2^n t]-1} (Y_{(i+1)/2^n} - Y_{i/2^n})^4, \quad t \in [0,1], \, n \in \mathbb{N}. \tag{2.4}$$

Then we have, for fixed $t \in (0,1]$, as $n \to \infty$,

$$2^{n/2}\big(\widehat{V_n}(t)\big)^{-1/2}\big(\widehat{J_n}(t) - J(t)\big) \xrightarrow{law} \sqrt{\frac{2}{3}}\mathcal{N}(0,1). \tag{2.5}$$

2.2. Diffusion model. Let us turn to the study of the strong solution of the stochastic differential equation (1.1). Here and elsewhere we denote by $(\Omega, \mathcal{F}, \mathrm{P})$ the probability space. We assume that $\theta \in \mathrm{C}^{1,2}([0,1] \times \mathbb{R}; \mathbb{R})$ has bounded derivatives with respect to the second variable and that $b \in \mathrm{C}^{1,1}([0,1] \times \mathbb{R}; \mathbb{R})$ is bounded with a bounded derivative with respect to the second variable.

To study the convergence in distribution, we shall assume henceforth that

$$\theta \text{ is elliptic: } \inf_{(t,x)\in[0,1]\times\mathbb{R}} |\theta(t,x)| > 0 \tag{2.6}$$

and

$$\sup_{(t,x)\in[0,1]\times\mathbb{R}} \left| \frac{\partial}{\partial t} \int\limits_{x_0}^{x} \frac{dy}{\theta(t,y)} \right| < \infty. \tag{2.7}$$

Clearly, (2.7) is trivially fulfilled if θ does not depend on t. We shall put

$$\begin{aligned} g(t,x) &:= \int\limits_{x_0}^{x} \frac{dy}{\theta(t,y)}, \quad G(t,x) := \big(t, g(t,x)\big), \\ F(t,x) &:= G^{-1}(t,x) =: \big(t, f(t,x)\big). \end{aligned} \tag{2.8}$$

The existence of F is a consequence of the Hadamard–Lévy theorem (see, for instance, [1], p. 130). We will set $\widetilde{B}_t := g(t, X_t)$ or, equivalently, $X_t = f(t, \widetilde{B}_t)$. Thanks to Itô's formula, by (1.1), we can write

$$\widetilde{B}_t = B_t - \int\limits_0^t C_s ds$$

with $C_s := -\{\frac{b}{\theta} - (\frac{\partial\theta}{\partial x})/2 + (\frac{\partial g}{\partial s})\}(s, X_s)$. By (2.7) we deduce that $\partial g/\partial s$ is bounded on $[0,1] \times \mathbb{R}$ and, using (2.6), that b/θ is bounded on $[0,1] \times \mathbb{R}$. Consequently, the Novikov criterion, that is, $\mathrm{E}\big(\exp(\frac{1}{2}\int_0^1 C_s^2 ds)\big) < \infty$, is easily verified. By applying Girsanov's theorem, we deduce that $\widetilde{B}$ is a Brownian motion under the probability Q given by

$$d\mathrm{Q} = \exp\Bigg(\int\limits_0^1 C_s\, dB_s - \frac{1}{2} \int\limits_0^1 C_s^2 ds \Bigg) d\mathrm{P} =: \exp(Z) d\mathrm{P}. \tag{2.9}$$

Hence, we can write

$$dX_t = \theta(t, X_t) dB_t + b(t, X_t) dt = \theta\big(t, f(t, \widetilde{B}_t)\big) d\widetilde{B}_t + M_t dt, \tag{2.10}$$

where $M_t = \{[(\frac{\partial\theta}{\partial x})/2 - (\frac{\partial g}{\partial s})]\theta\}\big(t, f(t, \widetilde{B}_t)\big)$. Thus X is related, by change of probability, to Y given by (2.1). Therefore, using Theorem 2.1 and Proposition 2.1, we obtain (see also [6] for some results concerning the convergence in law):

COROLLARY 2.1. 1. *As* $n \to \infty$,

$$2^{n/2}(\widehat{I_n} - I) \xrightarrow{law} \sqrt{2} \int_0^{\bullet} \theta\big(s, f(s, \beta_s^{(1)})\big)^2 \, d\beta_s^{(2)} \text{ under } \mathrm{Q} \tag{2.11}$$

in the Skorokhod topology. Here, $\beta^{(i)}$ $(i = 1, 2)$ *are two independent standard Brownian motions under* Q, *and* f *is given by the last inequality in* (2.8).

2. *Set*

$$\widehat{U_n}(t) := 2^n \sum_{i=0}^{[2^n t]-1} (X_{(i+1)/2^n} - X_{i/2^n})^4, \quad t \in [0,1],\ n \in \mathbb{N}. \tag{2.12}$$

Then we have, for fixed $t \in (0, 1]$, *as* $n \to \infty$,

$$2^{n/2}\big(\widehat{U_n}(t)\big)^{-1/2}\big(\widehat{I_n}(t) - I(t)\big) \xrightarrow{law} \sqrt{\frac{2}{3}}\,\mathcal{N}(0,1) \text{ under } \mathrm{Q}. \tag{2.13}$$

If we want to use the initial probability P, the following proposition explains the rate $2^{n/2}$ of convergence and allows to construct an asymptotic confidence interval for $I(t)$, both under P:

PROPOSITION 2.2. 1. *Let* $\gamma > \frac{1}{2}$. *We have, for fixed* $t \in (0, 1]$,

$$\forall R > 0 \ \lim_{n\to\infty} \mathrm{P}\Big(2^{n\gamma}\big(\widehat{U_n}(t)\big)^{-1/2}|\widehat{I_n}(t) - I(t)| \geqslant R\Big) = 1. \tag{2.14}$$

2. *Assume that* $\kappa > 0$ *is such that* $\mathrm{E_P}(Z^2) \leqslant \kappa^2$ *with* Z *given by* (2.9). *Let* $\phi_\kappa \colon [0,1] \to [0, e^{-\kappa}]$ *be continuous bijection given by* $\phi_\kappa(x) = x\exp(-\kappa/\sqrt{x})$. *We have, for fixed* $t \in (0, 1]$ *and for all* $\eta > 0$,

$$\limsup_{n\to\infty} \mathrm{P}\Big(2^{n/2}\big(\widehat{U_n}(t)\big)^{-1/2}|\widehat{I_n}(t) - I(t)| \geqslant \eta\Big) \leqslant \phi_\kappa^{-1}\bigg(\frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant \sqrt{3/2}\eta} \exp\Big(-\frac{x^2}{2}\Big)dx\bigg). \tag{2.15}$$

3. GOODNESS-OF-FIT TEST

3.1. Semimartingale model. Let us denote by $\mathcal{C}_{b,0}$ the set of non-constant analytic functions $\psi : \mathbb{R} \to \mathbb{R}$, with bounded first and second derivatives such that $\psi(0) = 0$.

We will consider a semimartingale of the form $Y_t = \varphi(B_t)$, $t \in [0, 1]$, with $\varphi \in \mathcal{C}_{b,0}$ unknown: $\{Y_{i/2^n} \colon i = 0, 1, \ldots, [2^n t] - 1\}$. If $\psi \in \mathcal{C}_{b,0}$ is known, we are interested by testing the hypothesis $(\widetilde{H})$: $\varphi(|\cdot|) = \psi(|\cdot|)$ against the alternative $(\widetilde{A})$: $\varphi(|\cdot|) \neq \psi(|\cdot|)$. More precisely, we study the following two situations:

ψ is a strictly monotone bijection or ψ verifies $\psi'^2 = F(\psi)$, with F a real C^1-function.

Let us remark that if ψ is a strictly monotone bijection, then it is classical to test if $\psi^{-1}(Y_t)$ is a standard Brownian motion. Indeed, it can be performed as follows: we need to investigate if

$$\Big(2^{n/2}\psi^{-1}(Y_{1/2^n}), 2^{n/2}\big(\psi^{-1}(Y_{2/2^n}) - \psi^{-1}(Y_{1/2^n})\big), \dots, \\ 2^{n/2}\big(\psi^{-1}(Y_1) - \psi^{-1}(Y_{1-1/2^n})\big)\Big)$$

is a sample of the standard Gaussian distribution. Here we propose an alternative procedure which can be applied even if ψ is not a bijection.

If at least one of the observed values $\{Y_{i/2^n}, i = 0, 1, \dots, [2^n t] - 1\}$ lies outside of the range of ψ, $(\widetilde{H})$ is rejected. Otherwise, we set, for $t \in [0,1]$ and $n \in \mathbb{N}$,

(3.1)
$$\widehat{\mathrm{Jint}_n}(t) := \begin{cases} \dfrac{1}{2^n} \displaystyle\sum_{i=0}^{[2^n t]-1} (\psi' \circ \psi^{-1})(Y_{i/2^n})^2 \text{ if } \psi \text{ is a strictly monotone bijection,} \\ \\ \dfrac{1}{2^n} \displaystyle\sum_{i=0}^{[2^n t]-1} F(Y_{i/2^n}) \text{ if } \psi \text{ verifies } \psi'^2 = F(\psi) \text{ with } F \text{ a } C^1(\mathbb{R};\mathbb{R}). \end{cases}$$

Let us note that if ψ is a strictly monotone bijection verifying at the same time $\psi'^2 = F(\psi)$, then $(\psi' \circ \psi^{-1})^2 = F$ and $\widehat{\mathrm{Jint}_n}$ is well defined. An example of a strictly monotone bijection (respectively, a function verifying $\psi'^2 = F(\psi)$) is $\arctan x$ (respectively, $\sin x$).

Recall that $\widehat{J_n}$ is given by (2.2) and $\widehat{V_n}$ by (2.4).

PROPOSITION 3.1. *Introduce the decision statistic*:

$$\widetilde{T}_n(t) = \sqrt{\frac{3}{2}}\, 2^{n/2}\big(\widehat{V_n}(t)\big)^{-1/2} |\widehat{J_n}(t) - \widehat{\mathrm{Jint}_n}(t)|, \quad t \in [0,1],\ n \in \mathbb{N}. \tag{3.2}$$

1. *Assume that* $(\widetilde{H})$ *holds. Then for all* $t \in (0,1]$:

$$\widetilde{T}_n(t) \xrightarrow{law} |N| \quad \textit{as } n \to \infty, \tag{3.3}$$

where N *is a standard Gaussian random variable.*

2. *Assume that* $(\widetilde{A})$ *holds. Then, for all* $t \in (0,1]$:

$$\widetilde{T}_n(t) \xrightarrow{a.s.} \infty \quad \textit{as } n \to \infty. \tag{3.4}$$

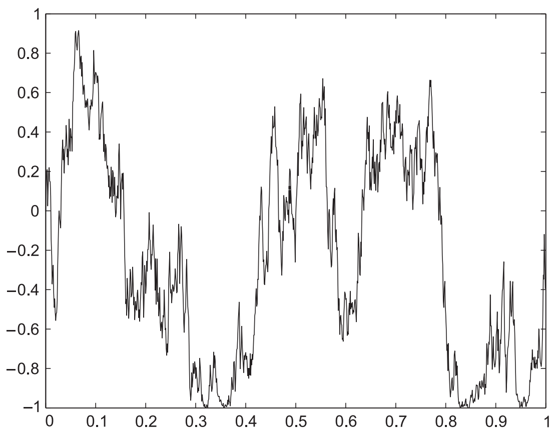

FIGURE 1. Observed semimartingale

EXAMPLE 3.1. Let us describe an example of application of this test. All computations below can be made using, for instance, a Matlab procedure. We observe $n = \log_2(10\,000)$ values of a path Y given by Figure 1. We note that $Y_t \in [-1,1]$ when $t \in [0,1]$. Hence, if we suspect that $Y_t = \varphi(B_t)$, we are looking for a function φ the range of which is contained in $[-1,1]$. For instance, we shall test $(\widetilde{H})$ with the monotone bijection $\psi(x) = \frac{2}{\pi}\arctan x$. In this case, we obtain

$$\widetilde{T}_n(1)(\omega) = \sqrt{\frac{3}{2}}2^{n/2}\big(\widehat{V_n}(1)(\omega)\big)^{-1/2}|\widehat{J_n}(1)(\omega) - \widehat{\mathrm{Jint}_n}(1)(\omega)| = 36.9889.$$

Since $\mathrm{P}(|N| > 36.9889) < 10^{-2}$, $N \sim \mathcal{N}(0,1)$, we can reject $(\widetilde{H})$ by using (3.3).

Let us now test $(\widetilde{H})$ with $\psi(x) = \sin x$ which verifies $\psi'^2 = F(\psi)$ with $F(x) = 1 - x^2$. We obtain

$$\widetilde{T}_n(1)(\omega) = \sqrt{\frac{3}{2}}2^{n/2}\big(\widehat{V_n}(1)(\omega)\big)^{-1/2}|\widehat{J_n}(1)(\omega) - \widehat{\mathrm{Jint}_n}(1)(\omega)| = 0.6759.$$

Since $\mathrm{P}(|N| > 0.6759) = 0.5$, we cannot reject $(\widetilde{H})$.

3.2. Diffusion model. Let us denote by $\mathcal{C}_{b,\pm}$ the set of the non-constant analytic functions $\vartheta\colon \mathbb{R} \to \mathbb{R}$, with bounded first and second derivatives, which do not vanish.

Assume, to simplify, that we observe a diffusion X of the form

$$X_t = x_0 + \int_0^t \theta(X_s)dB_s + \int_0^t b(s, X_s)ds, \quad t \in [0,1], \tag{3.5}$$

with $\theta \in \mathcal{C}_{b,\pm}$ unknown and $b \in \mathrm{C}^1(\mathbb{R})$ either known or unknown: $\{X_{i/2^n}\colon\ i = 0, 1, \ldots, [2^n t] - 1\}$.

If $\vartheta \in \mathcal{C}_{b,\pm}$ is known, we are interested in testing the hypothesis $(H)\colon\ \theta^2 = \vartheta^2$ against the alternative $(A)\colon\ \theta^2 \neq \vartheta^2$.

We set

$$\widehat{\mathrm{Iint}_n}(t) := 2^{-n} \sum_{i=0}^{[2^n t]-1} \vartheta(X_{i/2^n})^2, \quad t \in [0,1],\ n \in \mathbb{N}, \tag{3.6}$$

and we recall that $\widehat{I_n}$ and $\widehat{U_n}$ are given by (1.4) and (2.12), respectively.

PROPOSITION 3.2. *Introduce the decision statistic T_n given by*

$$T_n(t) = \sqrt{\frac{3}{2}} 2^{n/2} \big(\widehat{U_n}(t)\big)^{-1/2} |\widehat{I_n}(t) - \widehat{\mathrm{Iint}_n}(t)|, \quad t \in [0,1],\ n \in \mathbb{N}. \tag{3.7}$$

1. *Assume that (H) holds. Then, for all $\eta > 0$ and for fixed $t \in (0,1]$,*

$$\limsup_{n\to\infty} \mathrm{P}\big(T_n(t) \geqslant \eta\big) \leqslant \phi_\kappa^{-1}\bigg(\frac{1}{\sqrt{2\pi}} \int_{|x|\geqslant\eta} \exp\Big(-\frac{x^2}{2}\Big) dx\bigg). \tag{3.8}$$

2. *Assume that (A) holds. Then, for all $t \in (0,1]$,*

$$T_n(t) \xrightarrow{a.s.} \infty \quad \text{as } n \to \infty. \tag{3.9}$$

EXAMPLE 3.2. Let us describe an example of application of this test. All computations below can be made using again a Matlab procedure. We observe $n = \log_2(10\,000)$ values of a path X (see Figure 2). Considering the diffusion

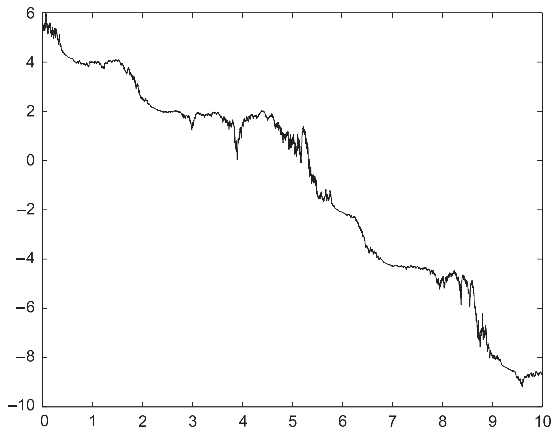

FIGURE 2. Observed diffusion

on the time interval $[0,10]$ instead of $[0,1]$ is not a constraint. We test (H) with $\vartheta(x) = 2 + \cos x$, that is, let us inspect that X verifies

$$X_0 = 5 \quad \text{and} \quad dX_t = (2 + \cos X_t) dB_t - dt,\ t \in [0,10].$$

In this case, we obtain

$$T_n(10)(\omega)=\sqrt{\frac{3}{2}}2^{n/2}\big(\widehat{U_n}(10)(\omega)\big)^{-1/2}|\widehat{I_n}(10)(\omega)-\widehat{\mathrm{Iint}_n}(10)(\omega)|=115.2572$$

and we have (see (2.9)):

$$Z=\int_0^{10}C_s dB_s-\frac{1}{2}\int_0^{10}C_s^2 ds\qquad\text{with } |C_s|=\left|\frac{1}{2+\cos X_s}-\frac{\sin X_s}{2}\right|\leqslant\frac{3}{2}.$$

This implies that $\mathrm{E_P}(Z^2)\leqslant\frac{82440}{32}$, and hence we can choose $\kappa=51$. Since

$$\phi_\kappa^{-1}\bigg(\frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant 115.2572}\exp\Big(-\frac{x^2}{2}\Big)dx\bigg)<10^{-2},$$

we can reject (H).

4. PROOFS

Proof of Theorem 2.1. First, by using a localization argument, it is not very difficult to prove that the finite variation part of Y does not have any contribution to the limit. Consequently, in the following, we shall suppose that $M\equiv 0$ in (2.1).

We can write

$$\widehat{J_n}(t)-J(t)=j_n(t)+r_n(t)$$

with

$$j_n(t):=2\sum_{i=0}^{[2^nt]-1}\sigma\Big(\frac{i}{2^n},B_{i/2^n}\Big)^2\int_{i/2^n}^{(i+1)/2^n}dB_s\int_{i/2^n}^{s}dB_u$$

and

$$r_n(t):=2\sum_{i=0}^{[2^nt]-1}\Bigg(\int_{i/2^n}^{(i+1)/2^n}\bigg(\sigma(s,B_s)-\sigma\Big(\frac{i}{2^n},B_{i/2^n}\Big)\bigg)dB_s\int_{i/2^n}^{s}\sigma\Big(\frac{i}{2^n},B_{i/2^n}\Big)dB_u$$

$$+\int_{i/2^n}^{(i+1)/2^n}\sigma(s,B_s)dB_s\int_{i/2^n}^{s}\bigg(\sigma(u,B_u)-\sigma\Big(\frac{i}{2^n},B_{i/2^n}\Big)\bigg)dB_u\Bigg)-\int_{[2^nt]/2^n}^{t}\sigma(s,B_s)^2ds.$$

By the classical Burkholder–Davis–Gundy inequality and using the fact that σ has bounded derivatives with respect to the second variable, we can prove that

$$\lim_{n\to\infty}\mathrm{E}\{\sup_{t\in[0,1]}|2^{n/2}r_n(t)|^2\}=0.$$

Consequently, to obtain the convergence (2.3), it suffices to show that $2^{n/2}j_n$ converges in distribution to $\sqrt{2}\int_0^\bullet\sigma(s,\beta_s^{(1)})^2\,d\beta_s^{(2)}$, as we can see by the following classical lemma which is a consequence of Theorem 4.1, p. 25, in [5]:

LEMMA 4.1. *Consider two sequences* $\{\mathcal{X}_n\colon n\in\mathbb{N}\}$ *and* $\{\mathcal{Y}_n\colon n\in\mathbb{N}\}$ *of random elements with values in* $\mathrm{D}([0,1];\mathbb{R})$ (*or càdlàg real stochastic processes starting from* 0). *Assume that, as* $n\to\infty$, $\mathcal{X}_n \xrightarrow{law} \mathcal{X}$ *and* $\mathrm{E}\{\sup_{t\in[0,1]}|\mathcal{Y}_n(t)|^2\}\to 0$. *Then, as* $n\to\infty$, $\mathcal{X}_n+\mathcal{Y}_n \xrightarrow{law} \mathcal{X}$.

Let us introduce, for $n\in\mathbb{N}$, the process Z_n given by

$$
\begin{aligned}
Z_n(t) &= 2^{1+n/2}\sum_{i=0}^{[2^n t]-1}\int_{i/2^n}^{(i+1)/2^n} dB_s \int_{i/2^n}^{s} dB_u \\
&= 2^{n/2}\sum_{i=0}^{[2^n t]-1}\left[(B_{(i+1)/2^n}-B_{i/2^n})^2-\frac{1}{2^n}\right], \quad t\in[0,1].
\end{aligned}
\tag{4.1}
$$

We split the proof of the convergence of $2^{n/2}\jmath_n=\int_0^{\bullet}\sigma(s,B_s)dZ_n(s)$ into several steps.

(a) Convergence in law in the particular case where $\sigma\equiv 1$.

Let $\{N_i\}_{i\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables. We have, for all $n\in\mathbb{N}$:

$$
2^{n/2}\sum_{i=0}^{[2^n t]-1}\left[(B_{(i+1)/2^n}-B_{i/2^n})^2-\frac{1}{2^n}\right] \overset{\text{(law)}}{=} 2^{-n/2}\sum_{i=0}^{[2^n t]-1}(N_i^2-1).
$$

Then the convergence in law when $\sigma\equiv 1$ is an immediate consequence of the functional central limit theorem.

(b) Convergence in law for any function σ.

(i) We can write $Z_n(t)=\tilde{Z}_n(t)-R_n(t)$, where the martingale $\tilde{Z}_n$ is given by

$$
\tilde{Z}_n(t) := 2\int_0^t dB_s\int_0^s dB_u f_n(s,u)
$$

with

$$
f_n(s,u)=2^{n/2}\sum_{i=0}^{2^n-1}\mathbf{1}_{[i/2^n,(i+1)/2^n)}(s)\mathbf{1}_{[i/2^n,1]}(u),
$$

and the remainder is defined by

$$
R_n(t) := 2^{1+n/2}\int_{[2^n t]/2^n}^{t} dB_s\int_{[2^n t]/2^n}^{s} dB_u.
$$

It is not difficult to show that

$$
\lim_{n\to\infty}\mathrm{E}\{\sup_{t\in[0,1]}|R_n(t)|^2\}=0.
$$

Again, by Lemma 4.1, it suffices to study the convergence in distribution of $\tilde{Z}_n$. We fix $t \in [0,1]$. By applying successively Itô's formula and the stochastic version of the Fubini theorem (see, for instance, [15], p. 175), the quadratic variation of $\tilde{Z}_n$ satisfies

$$[\tilde{Z}_n, \tilde{Z}_n](t) = 8 \int_0^t dB_u \int_0^u dB_v \int_u^t ds\, f_n(s,u)\, f_n(s,v) + 4 \int_0^t ds \int_0^s du\, f_n(s,u)^2.$$

On the one hand, note that

$$4 \int_0^t ds \int_0^s du f_n(s,u)^2 = 2\, \frac{[2^n t]}{2^n} + 2^{1+n} \left(t - \frac{[2^n t]}{2^n} \right)^2 \to 2t \qquad \text{as } n \to \infty.$$

By the isometry formula, we also have

$$\begin{aligned} &\mathrm{E}\left[\int_0^t dB_u \int_0^u dB_v \int_u^t ds\, f_n(s,u)\, f_n(s,v) \right]^2 \\ &\quad = 2^{2n+1} \sum_{i=0}^{[2^n t]-1} \int_{i/2^n}^{(i+1)/2^n} du \int_{i/2^n}^{u} dv \int_u^{(i+1)/2^n} ds \int_u^s dw \\ &\quad\quad + 2^{2n+1} \int_{[2^n t]/2^n}^{t} du \int_{[2^n t]/2^n}^{u} dv \int_u^t ds \int_u^s dw = O\left(\frac{1}{2^n} \right) \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, we deduce that in L^2 we have $\lim_{n\to\infty} [\tilde{Z}_n, \tilde{Z}_n](t) = 2\,t$.

(ii) By using the stochastic version of the Fubini theorem, the covariation between $\tilde{Z}_n$ and B satisfies

$$[\tilde{Z}_n, B](t) = 2^{1+n/2} \Big(\sum_{i=0}^{[2^n t]-1} \int_{i/2^n}^{(i+1)/2^n} dB_u \int_u^{(i+1)/2^n} ds + \int_{[2^n t]/2^n}^{t} dB_u \int_u^t ds \Big).$$

We deduce that

$$\begin{aligned} \mathrm{E}\{[\tilde{Z}_n, B](t)^2\} &= 2^{n+2} \Big[\sum_{i=0}^{[2^n t]-1} \int_{i/2^n}^{(i+1)/2^n} du \Big(\int_u^{(i+1)/2^n} ds \Big)^2 + \int_{[2^n t]/2^n}^{t} (t-u)^2 du \Big] \\ &= O\left(\frac{1}{2^n} \right) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

(iii) With a similar reasoning, we can prove that in L^2 we have

$$\lim_{n\to\infty} [\tilde{Z}_n, B]\big([\tilde{Z}_n, \tilde{Z}_n]^{-1}(t) \big) = 0.$$

(iv) Let us denote by β_n the Dubins–Schwarz Brownian motion associated with $\tilde{Z}_n$. By steps (i)–(iii) and the asymptotic version of Knight's theorem (see, for instance, [15], p. 524), we deduce that, as $n \to \infty$, $(B, \beta_n) \xrightarrow{\text{law}} (\beta^{(1)}, \beta^{(2)})$, where $\beta^{(1)}$ and $\beta^{(2)}$ are two independent standard Brownian motions. Since in L^2 the limit of $[\tilde{Z}_n, \tilde{Z}_n](t)$ is $2t$, we see that, as $n \to \infty$, $(B, \tilde{Z}_n) \xrightarrow{\text{law}} (\beta^{(1)}, \sqrt{2}\,\beta^{(2)})$. Set $\iota(t) = t$, $t \in [0,1]$. Then $(\iota, B, \tilde{Z}_n) \xrightarrow{\text{law}} (\iota, \beta^{(1)}, \sqrt{2}\,\beta^{(2)})$ as $n \to \infty$ and, by Lemma 4.1,

$$(\iota, B, Z_n) \xrightarrow{\text{law}} (\iota, \beta^{(1)}, \sqrt{2}\,\beta^{(2)}) \quad \text{as } n \to \infty. \tag{4.2}$$

Since σ is a continuous function, we get $\big(\sigma(\iota, B)^2, Z_n\big) \xrightarrow{\text{law}} \big(\sigma(\iota, \beta^{(1)})^2, \sqrt{2}\,\beta^{(2)}\big)$ as $n \to \infty$. Using the result concerning the convergence in distribution of stochastic integrals (see Jakubowski et al. [13]), we obtain, as $n \to \infty$,

$$2^{n/2} j_n = \int_0^{\bullet} \sigma(s, B_s)^2\, dZ_n(s) \xrightarrow{\text{law}} \sqrt{2} \int_0^{\bullet} \sigma(s, \beta_s^{(1)})^2\, d\beta_s^{(2)}.$$

Indeed, it suffices to verify the uniform tightness hypothesis of the result in [13], p. 125: for all $t \geqslant 0$, for every predictable process A bounded by 1, and for all $n \in \mathbb{N}$, we have for some constant a

$$\mathrm{P}\Big(\Big|\int_0^t A_s\, dZ_n(s)\Big| > R\Big) \leqslant \frac{1}{R^2} \mathrm{E}\Big(2^{1+n/2} \sum_{i=0}^{[2^n t]-1} A_{i/2^n} \int_{i/2^n}^{(i+1)/2^n} dB_s \int_{i/2^n}^{s} dB_u\Big)^2 \leqslant \frac{a}{R^2}.$$

Consequently, the proof of (2.3) is complete. ■

P r o o f o f P r o p o s i t i o n 2.1. We denote by S_n the process given by

$$S_n(t) = 2^n \sum_{i=0}^{[2^n t]-1} (B_{(i+1)/2^n} - B_{i/2^n})^4, \quad t \in [0,1],\ n \in \mathbb{N}.$$

Let us note that, for any $t \in [0,1]$, $S_n(t)$ converges towards $3t$, a deterministic limit, in probability, as $n \to \infty$. Indeed, for fixed $t \geqslant 0$, $S_n(t)$ has the same law as $2^{-n} \sum_{i=0}^{[2^n t]-1} N_i^4$ with $\{N_i\}_{i \in \mathbb{N}}$ a sequence of independent standard Gaussian random variables. The convergence is then obtained by the law of large numbers. One can also prove that S_n converges towards 3ι, in probability uniformly on $[0,1]$. We deduce, by using (4.2), that, as $n \to \infty$,

$$(\iota, B, Z_n, S_n) \xrightarrow{\text{law}} (\iota, \beta^{(1)}, \sqrt{2}\,\beta^{(2)}, 3\,\iota),$$

with $\beta^{(1)}$ and $\beta^{(2)}$ again two independent standard Brownian motions and $\iota(t)=t$, $t\in[0,1]$. Moreover, by a similar reasoning to that in step (iv) of the proof of Theorem 2.1, for fixed $t\in[0,1]$, we obtain firstly, as $n\to\infty$,

$$\Big(\int_0^t \sigma(s,B_s)^2\,dZ_n(s),\ \int_0^t \sigma(s,B_s)^4\,dS_n(s)\Big)$$
$$\xrightarrow{\text{law}} \Big(\sqrt{2}\int_0^t \sigma(s,\beta_s^{(1)})^2 d\beta_s^{(2)},\ 3\int_0^t \sigma(s,\beta_s^{(1)})^4 ds\Big)$$

and secondly, as $n\to\infty$,

$$\Big(2^{n/2}\big(\widehat{J_n}(t)-J(t)\big),\ \widehat{V_n}(t)\Big)\xrightarrow{\text{law}}\Big(\sqrt{2}\int_0^t \sigma(s,\beta_s^{(1)})^2 d\beta_s^{(2)},\ 3\int_0^t \sigma(s,\beta_s^{(1)})^4 ds\Big).$$

Finally, since the function $(x,y)\mapsto x/\sqrt{y}$ is continuous on $\mathbb{R}\times\mathbb{R}_+^*$, we deduce that, as $n\to\infty$,

$$2^{n/2}\big(\widehat{V_n}(t)\big)^{-1/2}\big(\widehat{J_n}(t)-J(t)\big)\xrightarrow{\text{law}}\sqrt{\frac{2}{3}}\,\frac{\int_0^t \sigma(s,\beta_s^{(1)})^2 d\beta_s^{(2)}}{\big(\int_0^t \sigma(s,\beta_s^{(1)})^4 ds\big)^{1/2}}.$$

By the independence between $\beta^{(1)}$ and $\beta^{(2)}$, it is easy to see that, for any fixed $t\in(0,1]$,

$$\sqrt{\frac{2}{3}}\,\frac{\int_0^t \sigma(s,\beta_s^{(1)})^2 d\beta_s^{(2)}}{\big(\int_0^t \sigma(s,\beta_s^{(1)})^4 ds\big)^{1/2}}\overset{\text{law}}{=}\sqrt{\frac{2}{3}}\,\mathcal{N}(0,1).$$

Thus the proof of (2.5) is complete. ■

Proof of Proposition 2.2. We begin by stating the following:

LEMMA 4.2. *Let ϕ_κ be as in Proposition 2.2. We have, for any set $A\in\mathcal{F}$,*

$$\mathrm{Q}(A)\geqslant\phi_\kappa\big(\mathrm{P}(A)\big)\quad\textit{or, equivalently},\quad \mathrm{P}(A)\leqslant\phi_\kappa^{-1}\big(\mathrm{Q}(A)\big).\tag{4.3}$$

In particular, if (A_m) is a sequence of sets in $\mathcal{F}$ such that $\mathrm{Q}(A_m)\to 0$ (respectively, $\mathrm{Q}(A_m)\to 1$), then $\mathrm{P}(A_m)\to 0$ (respectively, $\mathrm{P}(A_m)\to 1$) as $m\to\infty$.

Let us complete the proof of Proposition 2.2. We fix $t\in(0,1]$ and let $R>0$. We can write, for all $n_0\in\mathbb{N}$ and $n\geqslant n_0$,

$$\mathrm{Q}\Big(2^{n\gamma}\big(\widehat{U_n}(t)\big)^{-1/2}|\widehat{I}_n(t)-I(t)|\geqslant R\Big)$$
$$\geqslant \mathrm{Q}\Big(2^{n/2}\big(\widehat{U_n}(t)\big)^{-1/2}|\widehat{I}_n(t)-I(t)|\geqslant\frac{R}{2^{n_0(\gamma-1/2)}}\Big)$$
$$\to \mathrm{Q}\Big(\sqrt{\frac{2}{3}}|N|\geqslant\frac{R}{2^{n_0(\gamma-1/2)}}\Big)\to 1\quad\text{as } n\to\infty.$$

Here and below we denote by N a standard Gaussian random variable under Q. Consequently, by Lemma 4.2, we obtain the first part of Proposition 2.2. For the second part, using successively Lemma 4.2 and (2.13), we see that

$$\limsup_{n\to\infty} \mathrm{P}\biggl(\bigl(\widehat{U_n}(t)\bigr)^{-1/2}|\widehat{I_n}(t) - I(t)| \geqslant \frac{\beta}{2^{n/2}}\biggr)$$
$$\leqslant \limsup_{n\to\infty} \phi_\kappa^{-1}\biggl(\mathrm{Q}\biggl(\bigl(\widehat{U_n}(t)\bigr)^{-1/2}|\widehat{I_n}(t) - I(t)| \geqslant \frac{\beta}{2^{n/2}}\biggr)\biggr)$$
$$= \phi_\kappa^{-1}\bigl(\mathrm{Q}(\sqrt{2/3}\,|N| \geqslant \beta)\bigr). \ \blacksquare$$

P r o o f o f L e m m a 4.2. We have, using the Jensen and Cauchy–Schwarz inequalities,

$$\mathrm{Q}(A) = \mathrm{E}_{\mathrm{P}}(e^Z \mathbf{1}_A) \geqslant \exp\left[\frac{1}{\mathrm{P}(A)} \int_A Z(\omega) d\mathrm{P}(\omega)\right] \mathrm{P}(A)$$
$$= \exp\left[\frac{\mathrm{E}_{\mathrm{P}}(Z\mathbf{1}_A)}{\mathrm{P}(A)}\right] \mathrm{P}(A) \geqslant \phi_\kappa\bigl(\mathrm{P}(A)\bigr). \ \blacksquare$$

P r o o f o f P r o p o s i t i o n 3.1. Let us first prove (3.3). Assume that $(\widetilde{H})$ holds and fix $t \in [0,1]$. We then have

$$\widehat{J_n}(t) = \sum_{i=0}^{[2^n t]-1} \bigl(\psi(B_{(i+1)/2^n}) - \psi(B_{i/2^n})\bigr)^2$$
$$= \sum_{i=0}^{[2^n t]-1} \psi'(B_{i/2^n})^2 (B_{(i+1)/2^n} - B_{i/2^n})^2$$
$$+ (\psi'\psi'')(B_{i/2^n})(B_{(i+1)/2^n} - B_{i/2^n})^3 + r_n(t)$$

with

$$\sup_{n\geqslant 1} \mathrm{E}\left\{|2^n r_n(t)|\right\} < +\infty.$$

Moreover, we also have

$$E\Bigl[\sum_{i=0}^{[2^n t]-1} (\psi'\psi'')(B_{i/2^n})(B_{(i+1)/2^n} - B_{i/2^n})^3\Bigr]^2$$
$$= \sum_{i,j=0}^{[2^n t]-1} \mathrm{E}[(\psi'\psi'')(B_{i/2^n})(\psi'\psi'')(B_{j/2^n})(B_{(i+1)/2^n} - B_{i/2^n})^3$$
$$\times (B_{(j+1)/2^n} - B_{j/2^n})^3]$$
$$= 15 \cdot 2^{-3n} \sum_{i=0}^{[2^n t]-1} \mathrm{E}[(\psi'\psi'')(B_{i/2^n})^2] \leqslant 15 \cdot 2^{-2n} \|\psi'\|_\infty^2 \|\psi''\|_\infty^2.$$

Consequently,

$$\widehat{J_n}(t) - \widehat{\mathrm{Jint}_n}(t) = \sum_{i=0}^{[2^n t]-1} \psi'(B_{i/2^n})^2 \left[(B_{(i+1)/2^n} - B_{i/2^n})^2 - \frac{1}{2^n} \right] + \tilde{r}_n(t)$$

with

$$\sup_{n \geqslant 1} \mathrm{E}\left\{ |2^n \tilde{r}_n(t)| \right\} < +\infty.$$

Finally, we can complete the proof of (3.3) as in the proof of Theorem 2.1.

Now, let us prove (3.4) by considering the two situations (i) and (ii).

(i) The case when ψ is a monotone bijection.

We have, for fixed $t \in (0,1]$,

$$|\widehat{J_n}(t) - \widehat{\mathrm{Jint}_n}(t)| \xrightarrow{\text{a.s.}} \Big| \int_0^t \big(\varphi'(B_u)^2 - (\psi' \circ \psi^{-1} \circ \varphi)(B_u)^2 \big) du \Big| \quad \text{as } n \to \infty.$$

Assume that

$$\mathrm{P}\Big(\int_0^t \big(\varphi'(B_u)^2 - (\psi' \circ \psi^{-1} \circ \varphi)(B_u)^2 \big) du = 0 \Big) > 0.$$

It follows that the law of the random variable $\int_0^t \big(\varphi'(B_u)^2 - (\psi' \circ \psi^{-1} \circ \varphi)(B_u)^2 \big) du$ is not absolutely continuous with respect to the Lebesgue measure. At this level, we need the following:

LEMMA 4.3. *Let h be a real analytic function and let $T \in (0,1]$. The law of $\int_0^T h(B_u) du$ is absolutely continuous with respect to the Lebesgue measure if and only if h is not a constant funtion.*

Admit this result (the proof of which is postponed to the end of this section). We deduce that $\varphi'^2 - (\psi' \circ \psi^{-1} \circ \varphi)^2 = c$, $c \in \mathbb{R}$. Moreover, necessarily $c = 0$ because

$$0 < \mathrm{P}\Big(\int_0^t \big(\varphi'(B_u)^2 - (\psi' \circ \psi^{-1} \circ \varphi)(B_u)^2 \big) du = 0 \Big) = \mathrm{P}(c\, t = 0).$$

Consequently, $\varphi'^2 = (\psi' \circ \psi^{-1} \circ \varphi)^2$. By a connectedness argument, we obtain $\varphi' = \varepsilon\, \psi' \circ \psi^{-1} \circ \varphi$ with $\varepsilon \in \{\pm 1\}$. Then φ is one-to-one and we have $\varphi' \circ \varphi^{-1} = \varepsilon\, \psi' \circ \psi^{-1}$ or, equivalently, $(\varphi^{-1})' = \varepsilon (\psi^{-1})'$. We finally get $\varphi^{-1}(x) = \varepsilon\, \psi^{-1}(x)$ for all $x \in \mathbb{R}$ or, equivalently, $\varphi(x) = \psi(\varepsilon x)$ for all $x \in \mathbb{R}$. This is a contradiction to $(\widetilde{A})$. Consequently, almost surely $\int_0^t \big(\varphi'(B_u)^2 - (\psi' \circ \psi^{-1} \circ \varphi)(B_u)^2 \big) du$ does not vanish and (3.4) holds.

(ii) The case when $\psi'^2 = F(\psi)$.
In this case,

$$|\widehat{J_n}(t) - \widehat{\mathrm{Jint}_n}(t)| \xrightarrow{\text{a.s.}} \Big|\int\limits_0^t \big(\varphi'(B_u)^2 - F(\varphi)(B_u)\big)du\Big| \quad \text{as } n \to \infty.$$

Assume that

$$\mathrm{P}\Big(\int\limits_0^t \big(\varphi'(B_u)^2 - F(\varphi)(B_u)\big)du = 0\Big) > 0.$$

Again, it follows that the law of the random variable $\int_0^t \big(\varphi'(B_u)^2 - F(\varphi)(B_u)\big)du$ is not absolutely continuous with respect to the Lebesgue measure. Again, using Lemma 4.3, we obtain $\varphi'^2 = F(\varphi)$, and then $2\varphi'\varphi'' = \varphi' F'(\varphi)$. On the one hand, by real analyticity of φ, the set $\{x : \varphi'(x) \neq 0\}$ is dense in $\mathbb{R}$ and it allows us to simplify: $2\varphi'' = F'(\varphi)$. On the other hand, we have $\varphi'^2(0) = F(0) = \psi'^2(0)$ and $\varphi(0) = \psi(0)$. By uniqueness of the Cauchy problem, we deduce that $\varphi(|\cdot|) = \psi(|\cdot|)$, which is a contradiction to $(\widetilde{A})$. ■

P r o o f o f P r o p o s i t i o n 3.2. If we assume that (H) holds, then (3.8) is a consequence of the first point of Proposition 3.1 and Lemma 4.2. Assume that (A) holds. We have, for fixed $t \in (0,1]$,

$$|\widehat{I_n}(t) - \widehat{\mathrm{Iint}_n}(t)| \xrightarrow{\text{a.s.}} \Big|\int\limits_0^t \big(\theta(X_u)^2 - \vartheta(X_u)^2\big)du\Big| \quad \text{as } n \to \infty.$$

Assume that

$$\mathrm{P}\Big(\int\limits_0^t \big(\theta(X_u)^2 - \vartheta(X_u)^2\big)du = 0\Big) > 0.$$

Once more, it follows that the law of the random variable $\int_0^t \big(\theta(X_u)^2 - \vartheta(X_u)^2\big)du$ is not absolutely continuous with respect to the Lebesgue measure. We need a similar result to Lemma 4.3:

LEMMA 4.4. *Let h be a real analytic function and let $T \in (0,1]$. The law of $\int_0^T h(X_u)du$ is absolutely continuous with respect to the Lebesgue measure if and only if h is not a constant function.*

Using this result, we deduce that $\vartheta^2 - \theta^2 = c$, $c \in \mathbb{R}$. Moreover, necessarily $c = 0$ because

$$0 < \mathrm{P}\bigg(\int\limits_0^t \big(\theta(X_u)^2 - \vartheta(X_u)^2\big)du = 0\bigg) = \mathrm{P}(c\,t = 0).$$

This is a contradiction to (A). ■

P r o o f o f L e m m a s 4.3 a n d 4.4. We use the Malliavin calculus. By Theorem 2.1.3 in [14], p. 87, we know that if F lies in the Malliavin space $\mathbb{D}^{1,2}$ and if $\int_0^T (D_t F)^2 dt > 0$ almost surely, then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Firstly, if $F = \int_0^T h(B_u)du$, then $D_t F = \int_0^T D_t\big(h(B_u)\big)du = \int_t^T h'(B_u)du$. We have

$$\mathrm{P}(\forall t,\ D_t F = 0) = \mathrm{P}\big(\forall t,\ h'(B_t) = 0\big).$$

If we assume $\mathrm{P}\big(\forall t,\ h'(B_t) = 0\big) > 0$, then, in particular, $\mathrm{P}\big(h'(B_T) = 0\big) > 0$. Since h is analytic and the random variable B_T is absolutely continuous with respect to the Lebesgue measure, we deduce that $h' \equiv 0$. The assertion of Lemma 4.3 follows easily.

Secondly, if $F = \int_0^T h(X_u)du$ with X given by (3.5), then

$$\begin{aligned} D_t F &= \int_0^T D_t\big(h(X_u)\big)du \\ &= \theta(X_t)\int_t^T h'(X_u)\exp\left[\int_0^u \theta'(X_v)dB_v + \int_0^u \left(b' - \frac{1}{2}\theta'^2\right)(X_v)dv\right]du \end{aligned}$$

and the assertion of Lemma 4.4 follows as previously. ∎

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