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SPECTRAL GAP ESTIMATE FOR STABLE PROCESSES ON ARBITRARY BOUNDED OPEN SETS

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Abstract. In the paper we prove the lower bound estimate $\lambda_2^D - \lambda_1^D \geqslant c(\lambda_1^D)^{-d/\alpha}(\operatorname{diam} D)^{-d-\alpha}$ for the spectral gap of the Dirichlet fractional Laplacian $(-(-\Delta)^{\alpha/2})$ on an arbitrary bounded open set $D \subseteq \mathbf{R}^d$.

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1. INTRODUCTION AND MAIN RESULT

Let $\{X(t)\colon\ t \geqslant 0\}$ denote the d-dimensional isotropic stable process of index $\alpha \in (0,2)$, i.e. the rotation-invariant Lévy process with Fourier transform $\mathbf{E}_0 \exp\big(-i\langle X(t), z\rangle\big) = \exp(-t|z|^\alpha)$. For $\alpha = 2$ this reduces to the Brownian motion process $B(t)$, which, however, will not be considered below. Throughout this article it is assumed that $d \geqslant 1$ and $D \subset \mathbf{R}^d$ is an arbitrary bounded open set.

Let P_t^D denote the transition semigroup of $X(t)$ killed upon exiting D. It is known (see e.g. [10]) that there exists a complete orthonormal sequence $\varphi_n^D \in L^2(D)$ of eigenfunctions of P_t^D and a nondecreasing sequence λ_n^D of corresponding eigenvalues, i.e. $P_t^D \varphi_n^D = \exp(-\lambda_n^D t)\varphi_n^D$. The *ground state eigenvalue* λ_1^D is positive and simple. The corresponding *ground state eigenfunction* φ_1^D is positive on D. Various estimates of the *spectral gap* $\lambda_2^D - \lambda_1^D$ have been proved for convex D, some of which are discussed below. The result of this paper concerns arbitrary bounded open sets and follows from a standard estimate of the supremum norm of φ_1^D and a variational formula for $\lambda_2^D - \lambda_1^D$.

THEOREM 1.1. *There is a constant $c = c(d,\alpha)$ such that*

$$\lambda_2^D - \lambda_1^D \geqslant \frac{c}{(\lambda_1^D)^{d/\alpha}(\operatorname{diam} D)^{d+\alpha}}\,. \tag{1.1}$$

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The constant c is given by the explicit formula

$$c = \frac{2^{\alpha+1}(d/\alpha)^{d/\alpha}\Gamma\big((d+2)/2\big)\,\Gamma\big((d+\alpha)/2\big)}{\pi^{d-1}\,e^{d/\alpha}\,|\Gamma(-\alpha/2)|\,\Gamma\big((d+\alpha)/\alpha\big)}\,.$$

It is worth noting that without further assumptions on the domain D the exponent $d+\alpha$ in the estimate (1.1) cannot be improved. This remark will later be verified in Example 2.1.

Observe that no result of this type holds for the Brownian motion $B(t)$, even when only connected D are considered. Indeed, consider $D_\varepsilon = B(x_0, 1+\varepsilon) \cup B(-x_0, 1+\varepsilon)$, where $|x_0| = 1$. One can prove that two least eigenvalues of the semigroup of $B(t)$ killed upon exiting D_ε both tend to the ground state eigenvalue of a unit ball for $B(t)$ as $\varepsilon \to 0$.

The following result is a standard one and may be used together with Theorem 1.1 to obtain numerical estimates of the spectral gap.

PROPOSITION 1.1 (cf. [3], Corollary 2.2). *Suppose that $B(x,r) \subseteq D$. Then*

$$\lambda_1^D \leqslant \frac{\alpha\,(\alpha+d/2)\sqrt{\pi}\,\Gamma(\alpha/2)\,\Gamma(\alpha+d/2)}{(d+\alpha)\,\Gamma\big((1+\alpha)/2\big)\,\Gamma(d/2)}\,r^{-\alpha}\,. \tag{1.2}$$

For example, for $\alpha = 1$ we obtain $\lambda_2^D - \lambda_1^D \geqslant 0.00246$ for the unit disk in $\mathbf{R}^2$, and $\lambda_2^D - \lambda_1^D \geqslant 0.00174$ for a unit square.

Let us briefly recall some known estimates of the spectral gap. If D is a convex planar domain symmetric about both coordinate axes, then the spectral gap is greater than a constant multiple of $(\operatorname{diam} D)^{-2}$. Precise asymptotics of $\lambda_2^D - \lambda_1^D$ are also known for rectangles; see [11], Theorems 1.2 and 1.4.

From Example 5.1 in [9] it follows that the spectral gap for a convex and bounded domain D is not less than $\frac{1}{2}\,(\widetilde{\lambda}_2)^{\alpha/2} - (\widetilde{\lambda}_1)^{\alpha/2}$, where $(-\widetilde{\lambda}_n)$ is the nth eigenvalue of the Dirichlet Laplacian on D. However, this may provide a nonnegative lower bound only for small d and large α. Indeed, by [14] and [15], we have $\widetilde{\lambda}_2 \leqslant (1+4/d)\widetilde{\lambda}_1$ (a better estimate, the so-called *Payne–Pólya–Weinberger conjecture*, is proved in [1]), so if $d \geqslant 4$ or $\alpha < 1$, the obtained lower bound is negative.

Theorem 1.1 provides an estimate of the spectral gap for arbitrary open and bounded sets, extending earlier results and demonstrating the difference between classical and "fractional" potential theory. Its proof avoids most of technical difficulties of [6] and [11].

The interested reader is referred to [4], [5], [6] and [11] for a more detailed introduction to the topic, and for many open problems concerning the shape of the ground state eigenfunction and the spectral gap for symmetric stable processes. More information on isotropic stable process can be found e.g. in [7], [8] and [13].

2. PROOF OF THE RESULTS

Denote by p_t^D the transition density of $X(t)$ killed upon exiting D. Let p_t be the (translation invariant) transition density of $X(t)$.

PROPOSITION 2.1. *For all open and bounded sets* D,

$$\sup \varphi_1^D \leqslant \sqrt{\frac{\pi^{d/2-1} e^{d/\alpha} \Gamma\big((d+\alpha)/\alpha\big)}{2(d/\alpha)^{d/\alpha} \Gamma\big((d+2)/2\big)}} \, (\lambda_1^D)^{d/(2\alpha)} \,. \tag{2.1}$$

Proof. Let $t = d/(2\alpha\,\lambda_1^D)$. By the Cauchy–Schwarz inequality and Plancherel's theorem, and using $\|\varphi_1^D\|_2 = 1$, $p_t^D(x,y) \leqslant p_t(x-y)$, we obtain

$$\varphi_1^D(x) = e^{d/(2\alpha)} \int p_t^D(x,y)\, \varphi_1^D(y)\, dy \leqslant e^{d/(2\alpha)} \sqrt{\int \big(p_t(x-y)\big)^2\, dy}$$
$$= e^{d/(2\alpha)} \sqrt{\frac{1}{2\pi} \int \exp(-2t|z|^\alpha)\, dz} = \sqrt{\frac{\pi^{d/2-1}\, e^{d/\alpha}\, \Gamma\big((d+\alpha)/\alpha\big)}{2\,\Gamma\big((d+2)/2\big)\, (2t)^{d/\alpha}}} \,. \quad \blacksquare$$

A similar argument was sketched to the author by Bañuelos (cf. [2]).

Proof of Theorem 1.1. The spectral gap is given by the variational formula (see [11], Proposition 1.1)

$$\lambda_2^D - \lambda_1^D = C \inf_{f \in \mathcal{F}} \iint \frac{\big(f(x) - f(y)\big)^2}{|x-y|^{d+\alpha}}\, \varphi_1^D(x)\, \varphi_1^D(y)\, dx\, dy \,,$$
$$\mathcal{F} = \Big\{ f\colon \int \big(f(x) \varphi_1^D(x)\big)^2 dx = 1 \,,\ \int f(x) \big(\varphi_1^D(x)\big)^2 dx = 0 \Big\} \,,$$

where $C = 2^{\alpha-1} \pi^{-d/2} \Gamma\big((d+\alpha)/2\big) |\Gamma(-\alpha/2)|^{-1}$. In addition, the infimum is attained for $f = \varphi_2^D / \varphi_1^D$.

For simplicity, let us put $R = \operatorname{diam} D$. Let $f = \varphi_2^D / \varphi_1^D$ in the variational formula. Since $|x-y| \leqslant R$ and $\varphi_1^D(z) \leqslant M = \sup \varphi_1^D$, we have

$$\lambda_2^D - \lambda_1^D \geqslant \frac{C}{R^{d+\alpha} M^2} \iint \left(\frac{\varphi_2^D(x)}{\varphi_1^D(x)} - \frac{\varphi_2^D(y)}{\varphi_1^D(y)} \right)^2 \big(\varphi_1^D(x) \varphi_1^D(y)\big)^2\, dx\, dy.$$

By the properties $\|\varphi_1^D\|_2 = \|\varphi_2^D\|_2 = 1$ and $\langle \varphi_1^D, \varphi_2^D \rangle = 0$, the double integral equals 2, and hence

$$\lambda_2^D - \lambda_1^D \geqslant \frac{2C}{R^{d+\alpha} M^2}.$$

An application of Proposition 2.1 completes the proof. $\blacksquare$

EXAMPLE 2.1. Let us consider the set $D = B(-a, 1) \cup B(a, 1)$, where $a = (r, 0, \dots, 0)$ and $r > 2$. Since D is invariant under reflection in the origin, so is φ_1^D. By the variational formula with $f = \mathbf{1}_{B(a,1)} - \mathbf{1}_{B(-a,1)}$ and by Proposition 2.1,

$$\lambda_2^D - \lambda_1^D \leqslant \int\limits_{B(-a,1)} \int\limits_{B(a,1)} \frac{8\,C}{r^{d+\alpha}}\, \varphi_1^D(x)\, \varphi_1^D(y)\, dx\, dy \leqslant C'\, (\lambda_1^D)^{d/\alpha}\, r^{-d-\alpha}.$$

The constant C' depends only on d and α, and λ_1^D does not exceed the ground state eigenvalue of the unit ball (recall that λ_1^D is a decreasing function of D). We conclude that in this case we have

$$\frac{c'(\alpha, d)}{(\lambda_1^D)^{d/\alpha}(\operatorname{diam} D)^{d+\alpha}} \leqslant \lambda_2^D - \lambda_1^D \leqslant \frac{c''(\alpha, d)}{(\lambda_1^D)^{d/\alpha}(\operatorname{diam} D)^{d+\alpha}}\,.$$

This implies that the degree of the estimate (1.1) cannot be improved, justifying the note following Theorem 1.1.

Let $\big(-(-\Delta)^{\alpha/2}\big)$ denote the generator of $X(t)$ and $s_{B(x,r)}(y) = \mathbf{E}_y\, \tau_{B(x,r)}$, where τ_D is the first exit time from D. Then (cf. [12])

$$s_{B(x,r)}(y) = \frac{2^{1-\alpha}\,\Gamma(d/2)}{\alpha\,\Gamma\big((d+\alpha)/2\big)\Gamma(\alpha/2)}\,(r^2 - |x-y|^2)^{\alpha/2}, \quad y \in B(x, r).$$

It is known that $(-\Delta)^{\alpha/2} s_{B(x,r)}(y) = 1$ for all $y \in B(x, r)$.

Proof of Proposition 1.1. Observe that $\lambda_1^D \|f\|_2^2 \leqslant \langle (-\Delta)^{\alpha/2} f, f\rangle$ for any $f \in L^2(D)$. Let $f = s_{B(x,r)}$. The explicit formula for $s_{B(x,r)}$ and integration in spherical coordinates yield

$$\lambda_1^D \int\limits_0^r t^{d-1}\,(r^2 - t^2)^{\alpha}\, dt \leqslant \frac{2^{\alpha-1}\,\alpha\,\Gamma\big((d+\alpha)/2\big)\,\Gamma(\alpha/2)}{\Gamma(d/2)} \int\limits_0^r t^{d-1}\,(r^2 - t^2)^{\alpha/2}\, dt\,.$$

By letting $s = t^2/r^2$ we obtain beta integrals. Thus the estimate (1.2) holds true. ■

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