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DISCRETE APPROXIMATIONS OF REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH RANDOM TERMINAL TIME

BY

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Abstract. We study convergence of discrete approximations of reflected backward stochastic differential equations with random terminal time in a general convex domain. Applications to investigation of the viability property for backward stochastic differential equations and to obstacle problem for partial differential equations are given.

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1. INTRODUCTION

In this paper we consider approximations of solutions (Y, Z, K) of the following reflected backward stochastic differential equation (RBSDE for short) with almost surely finite random terminal time τ in a given convex domain $D \subset \mathbb{R}^d$:

$$(1.1)\qquad Y_{t\wedge\tau} = g(X_\tau) + \int_{t\wedge\tau}^{\tau} f(s, X_s, Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s + K_\tau - K_{t\wedge\tau}$$

for $t \in \mathbb{R}^+$, where X is a given diffusion process, $g\colon \mathbb{R}^m \to \bar{D} = D \cup \partial D$ is a continuous function and $f\colon \mathbb{R}^+ \times \mathbb{R}^m \times \bar{D} \times \mathbb{R}^{d\times m} \to \mathbb{R}^d$ is a continuous function satisfying the monotonicity condition with respect to y and is Lipschitz with respect to z (precise definitions are given in Section 3).

Existence and uniqueness of RBSDE (1.1) was shown by Pardoux and Răşcanu [16]. They proved that the solution of (1.1) may be approximated by a sequence of non-reflected BSDEs with a penalization term. Moreover, they pointed out connections between RBSDE (1.1) and variational inequalities.

There are many papers about approximations of BSDEs and RBSDEs (see e.g. [1]–[3], [13], [14], [21]). However, it is worth pointing out that, up to now, the discrete approximation of RBSDEs was investigated only in one-dimensional case (see [1] and [14]).

The aim of this paper is to prove convergence of discrete approximations of a solution of RBSDE (1.1) in a general convex d-dimensional domain D. We present both weak and strong convergence of the discrete scheme to the solution of (1.1). The important point to note here is that our approximating sequence can be computed by simple recurrent formulas, and therefore is easy to implement. Our approximation methods correspond to the so-called projection scheme, which is a well-known method of approximation of classical stochastic differential equations (see e.g. [18]). Additionally, we give applications of the numerical scheme in solving partial differential equations (PDEs) and the viability property of BSDE.

The paper is organized as follows. First, in Section 2 we give an approximation scheme for RBSDE with fixed terminal time and formulate its properties. This is used in Section 3, which contains an approximation scheme for RBSDE with random terminal time and in which we formulate the main theorem of this paper. Section 4 is devoted to applications of the numerical scheme in solving PDE and the viability property of BSDE. Finally, Section 5 contains proofs of theorems from Sections 2 and 3.

Throughout this paper we will use the following notation. $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all mappings $x : \mathbb{R}^+ \to \mathbb{R}^d$ which are right continuous and admit left-hand limits endowed with the Skorokhod topology J_1 (see [10]). By $|x|$ we mean the Euclidean norm in $\mathbb{R}^d$, $x \in \mathbb{R}^d$, $\|x\|$ stands for $(\mathrm{trace}(x^\star x))^{1/2}$, $x \in \mathbb{R}^{d\times m}$. If $K = (K^1, \ldots, K^d)$ is a process with locally finite variation, then $|K|_t = \sum_{i=1}^d |K^i|_t$, where $|K^i|_t$ is a total variation of K^i on $[0,t]$. If $Y = (Y^1, \ldots, Y^d)$ is a semimartingale, then $[Y]_t = \sum_{i=1}^d [Y^i]_t$, where $[Y^i]$ is a quadratic variation process of Y^i. For a stopping time τ, by Y^τ we mean the process stopped at τ, i.e. $Y_t^\tau = Y_{t\wedge\tau}$. Finally, $\xrightarrow{\mathcal{P}}$ and $\xrightarrow{\mathcal{D}}$ denote convergence in probability and in law, respectively.

2. APPROXIMATIONS OF RBSDEs WITH FIXED TERMINAL TIME

Let $(\Omega, \mathcal{G}, \mathcal{P})$ be a complete probability space carrying a standard m-dimensional Wiener process $W = \{W_t\}_{t\in\mathbb{R}^+}$ and let $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ be the usual augmentation of the filtration generated by W. In this section we assume that $\tau = T = \mathrm{const}$ and RBSDE has the form

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t \tag{2.1}$$

for $t \in [0,T]$, where X is the solution of the following stochastic differential equation (SDE for short)

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in \mathbb{R}^+, \tag{2.2}$$

where $b:\mathbb{R}^+\times\mathbb{R}^m\to\mathbb{R}^m$ and $\sigma:\mathbb{R}^+\times\mathbb{R}^m\to\mathbb{R}^{m\times m}$ are continuous functions such that

$$|b(t,x)-b(t,x')|+\|\sigma(t,x)-\sigma(t,x')\|\leqslant L|x-x'|,\quad t\in\mathbb{R}^+,\ x,x'\in\mathbb{R}^m,$$

for some $L>0$.

By a *solution* of (2.1) we mean a triple (Y,Z,K) of $\mathcal{F}$ progressively measurable processes in $\bar{D}\times\mathbb{R}^{d\times m}\times\mathbb{R}^d$ satisfying (2.1) and such that

(a) $E\big(\sup_{t\leqslant T}|Y_t|^2+\int_0^T\|Z_t\|^2dt\big)<\infty$;

(b) K is a continuous process of bounded variation, such that $K_0=0$, and $\int_0^T(Y_t-A_t)dK_t\leqslant 0$ for every $\mathcal{F}$ progressively measurable process A with values in $\bar{D}$.

Let $g:\mathbb{R}^m\to\bar{D},\ f:\mathbb{R}^+\times\mathbb{R}^m\times\bar{D}\times\mathbb{R}^{d\times m}\to\mathbb{R}^d$ be continuous functions which satisfy the following conditions:

(i) there exist $q,\kappa\geqslant 0$ such that for any $x\in\mathbb{R}^m$

$$|g(x)|\leqslant\kappa(1+|x|^q);$$

(ii) there exists $L>0$ such that for any $t\in\mathbb{R}^+,\ x\in\mathbb{R}^m,\ y,y'\in\bar{D}$ and $z,z'\in\mathbb{R}^{d\times m}$

$$|f(t,x,y,z)-f(t,x,y',z')|\leqslant L(|y-y'|+\|z-z'\|);$$

(iii) $f(\cdot,\cdot,0,0)$ is bounded.

It is known that under assumptions (i)–(iii) RBSDE (2.1) has a unique strong solution (see [8]).

The discrete scheme we propose in this paper is based on approximation of a Wiener process W by a scaled random walk. Set $W_t^n=(\sqrt{n})^{-1}\sum_{j=1}^{[nt]}\varepsilon_j^n$, $t\in\mathbb{R}^+$, where for each $n\in N$, $\{\varepsilon_j^n\}_{j\in\mathbb{N}}$ is a sequence of independent symmetric Bernoulli random variables, and by $\mathcal{F}^n=\{\mathcal{F}_t^n\}_{t\in\mathbb{R}^+}$ denote the natural filtration of W^n.

Let us first consider the approximation scheme for SDE (2.2). Set $x_0^n=x$ and for $j=0,\dots,[nT]-1$ define $x_{(j+1)/n}^n$ by

$$x_{(j+1)/n}^n=x_{j/n}^n+\frac{1}{n}b(j/n,x_{j/n}^n)+\frac{1}{\sqrt{n}}\sigma(j/n,x_{j/n}^n)\varepsilon_{j+1}^n. \tag{2.3}$$

Notice that, for each j, $x_{j/n}^n$ is $\mathcal{F}_{j/n}^n$-measurable. Moreover, if we put $\varrho_t^n=[nt]/n$ and define $X_t^n=x_{[nt]/n}^n$, $t\in[0,T]$, then X^n is a solution of the discrete SDE

$$X_t^n=x+\int_0^t b(\varrho_{s-}^n,X_{s-}^n)d\varrho_s^n+\int_0^t\sigma(\varrho_{s-}^n,X_{s-}^n)dW_s^n,\quad t\in[0,T].$$

By Donsker's theorem, we have $W^n \xrightarrow{\mathcal{D}} W$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^m)$. Consequently, since $\sup_{t \leqslant T}(t - \varrho_t^n) \to 0$, $T \in \mathbb{R}^+$, it follows that

$$(X^n, W^n) \xrightarrow{\mathcal{D}} (X, W) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2m}).$$

Moreover, if the processes W^n, W are defined on the same probability space and $\sup_{t \leqslant T} |W_t^n - W_t| \xrightarrow{\mathcal{P}} 0$, then $\sup_{t \leqslant T} |X_t^n - X_t| \xrightarrow{\mathcal{P}} 0$, $T \in \mathbb{R}^+$. In fact, the convergence in probability can be strengthened to convergence in $\mathbb{L}^p$ norm for every $p \in \mathbb{N}$ (see e.g. [17]).

Now, consider the discrete version of RBSDE (2.1). In our scheme we combine some ideas from [2], where BSDE without reflection is considered, and from [18], where SDE with reflection is considered. For $j = [nT]$ put $y_{j/n}^n = g(x_{j/n}^n)$, $z_{j/n}^n = 0$, $\Delta k_{(j+1)/n}^n = 0$ and solve the equation

$$y_{j/n}^n = y_{(j+1)/n}^n + \frac{1}{n} f(j/n, x_{j/n}^n, y_{j/n}^n, z_{j/n}^n) - \frac{1}{\sqrt{n}} z_{j/n}^n \varepsilon_{j+1}^n + \Delta k_{(j+1)/n}^n \tag{2.4}$$

for $j = [nT] - 1, \ldots, 0$.

By a *solution* of (2.4) we mean a triple $(Y^n, Z^n, K^n) = (Y_t^n, Z_t^n, K_t^n)_{t \in [0,T]}$ of $\mathcal{F}^n$ adapted processes in $\bar{D} \times \mathbb{R}^{d \times m} \times \mathbb{R}^d$ such that $|K^n|_T < \infty$, $K_0^n = 0$, and $\int_0^T (Y_{t-}^n - A_{t-}^n) dK_t^n \leqslant 0$ for every $\mathcal{F}^n$ adapted process A^n with values in $\bar{D}$, where $Y_t^n = y_{[nt]/n}^n$, $Z_t^n = z_{[nt]/n}^n$, $K_t^n = \sum_{j=1}^{[nt]} \Delta k_{j/n}^n$.

Since (2.4) can be written in the equivalent form

$$Y_t^n = g(X_T^n) + \int_t^T f(\varrho_{s-}^n, X_{s-}^n, Y_{s-}^n, Z_{s-}^n) d\varrho_s^n - \int_t^T Z_{s-}^n dW_s^n + K_T^n - K_t^n \tag{2.5}$$

for $t \in [0, T]$, one can deduce from Lemma 5.2 below that (2.4) has a unique solution. Therefore, solving (2.4) is equivalent to finding the solution to the following iteration problem. The first step is to choose

$$z_{j/n}^n = \sqrt{n} E(y_{(j+1)/n}^n \varepsilon_{j+1}^n | \mathcal{F}_{j/n}^n),$$

then to find

$$h_{j/n}^n = y_{(j+1)/n}^n + \frac{1}{n} f\big(j/n, x_{j/n}^n, \pi(h_{j/n}^n), z_{j/n}^n\big) - \frac{1}{\sqrt{n}} z_{j/n}^n \varepsilon_{j+1}^n,$$

where $\pi(h) = \pi_D(h)$ means the projection of $h \in \mathbb{R}^d$ on $\bar{D}$. Notice that for n large enough ($n > L$), $h_{j/n}^n$ is well defined since f and π are Lipschitz. Now we put $y_{j/n}^n = \pi(h_{j/n}^n)$. Observe that $h_{j/n}^n$ and $y_{j/n}^n$ are $\mathcal{F}_{j/n}^n$-measurable. Indeed, by the representation theorem (see [22], the Lemma on page 154),

$$y_{(j+1)/n}^n - \frac{1}{\sqrt{n}} z_{j/n}^n \varepsilon_{j+1}^n = E\big(y_{(j+1)/n}^n | \mathcal{F}_{j/n}^n\big).$$

Finally, we take $\Delta k^n_{(j+1)/n} = y^n_{j/n} - h^n_{j/n}$, which is also $\mathcal{F}^n_{j/n}$-measurable. Therefore, Y^n and Z^n are $\mathcal{F}^n$ adapted, and K^n is an $\mathcal{F}^n$ predictable process. Since D is a convex set,

$$\langle \pi(h) - x', \pi(h) - h\rangle \leqslant 0, \quad h \in \mathbb{R}^d,\ x' \in \bar{D}. \tag{2.6}$$

Hence, in particular,

$$\langle y^n_{(j-1)/n} - x', \Delta k^n_{j/n}\rangle \leqslant 0 \tag{2.7}$$

for any $x' \in \bar{D}$ and $j = 1, \ldots, [nT]$. Therefore, for any $\mathcal{F}^n$ adapted process A^n with values in $\bar{D}$,

$$\int_0^T (Y^n_{t-} - A^n_{t-})dK^n_t = \sum_{j=1}^{[nT]} \langle y^n_{(j-1)/n} - A^n_{(j-1)/n}, \Delta k^n_{j/n}\rangle \leqslant 0. \tag{2.8}$$

THEOREM 2.1. *Assume that* (i)–(iii) *hold. Then*:

(a) *We have*

$$\Big(X^n, Y^n, \int_0^{\cdot} Z^n_{s-}dW^n_s, K^n, W^n\Big) \xrightarrow{\mathcal{D}} \Big(X, Y, \int_0^{\cdot} Z_s dW_s, K, W\Big)$$

in $\mathbb{D}([0,T], \mathbb{R}^m \times \bar{D} \times \mathbb{R}^{2d} \times \mathbb{R}^m)$.

(b) *If* $\sup_{t\leqslant T}|W^n_t - W_t| \xrightarrow{\mathcal{P}} 0$, *then*

$$E\Big(\sup_{t\leqslant T}|Y^n_t - Y_t|^2 + \int_0^T \|Z^n_{t-} - Z_t\|^2 dt + \sup_{t\leqslant T}|K^n_t - K_t|^2\Big) \to 0.$$

The proof of Theorem 2.1 is deferred to Section 5.

Let us consider now another numerical scheme for RBSDE (2.1), which is simpler to simulate then the scheme described above. In this method in each step we take only once projection on the set $\bar{D}$ and do not look for a fixed point of a function. For $j = [nT]$ put $\hat{y}^n_{j/n} = g(x^n_{j/n})$, $\hat{z}^n_{j/n} = 0$, $\Delta\hat{k}^n_{(j+1)/n} = 0$. For $j = [nT]-1, \ldots, 0$ first choose $\hat{z}^n_{j/n} = \sqrt{n}E(\hat{y}^n_{(j+1)/n}\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n})$, then find

$$\hat{h}^n_{j/n} = \hat{y}^n_{(j+1)/n} + \frac{1}{n}f(j/n, x^n_{j/n}, \bar{y}^n_{j/n}, \hat{z}^n_{j/n}) - \frac{1}{\sqrt{n}}\hat{z}^n_{j/n}\varepsilon^n_{j+1},$$

where $\bar{y}^n_{j/n} = E(\hat{y}^n_{(j+1)/n}|\mathcal{F}^n_{j/n})$. Finally, take $\hat{y}^n_{j/n} = \pi(\hat{h}^n_{j/n})$ and $\Delta\hat{k}^n_{(j+1)/n} = \hat{y}^n_{j/n} - \hat{h}^n_{j/n}$. In a similar manner as before define processes $\hat{Y}^n_t = \hat{y}^n_{[nt]/n}$, $\hat{Z}^n_t = \hat{z}^n_{[nt]/n}$, $\hat{K}^n_t = \sum_{j=1}^{[nt]}\Delta\hat{k}^n_{j/n}$, $\bar{Y}^n_t = \bar{y}^n_{[nt]/n}$, which satisfy

$$\hat{Y}^n_t = g(X^n_T) + \int_t^T f(\varrho^n_{s-}, X^n_{s-}, \bar{Y}^n_{s-}, \hat{Z}^n_{s-})d\varrho^n_s - \int_t^T \hat{Z}^n_{s-}dW^n_s + \hat{K}^n_T - \hat{K}^n_t$$

for $t \in [0,T]$.

PROPOSITION 2.1. *Assume that* (i)–(iii) *hold. Then*

$$E\big(\sup_{t\leqslant T}|Y^n_t-\hat{Y}^n_t|^2+\int_0^T\|Z^n_{t-}-\hat{Z}^n_{t-}\|^2d\varrho^n_t+\sup_{t\leqslant T}|K^n_t-\hat{K}^n_t|^2\big)\to 0.$$

The proof of Proposition 2.1 is deferred to Section 5.

3. APPROXIMATIONS OF RBSDEs WITH RANDOM TERMINAL TIME

In this section we will consider the case where the terminal value of RBSDE is given by an $\mathcal{F}$ stopping time τ such that $P(\tau<\infty)=1$. We will assume that (i)–(iii) hold and, additionally,

(iv) there exists a constant $\mu\in\mathbb{R}$ such that for any $t\in\mathbb{R}^+$, $x\in\mathbb{R}^m$, $y,y'\in\bar{D}$, $z\in\mathbb{R}^{d\times m}$

$$\langle y-y', f(t,x,y,z)-f(t,x,y',z)\rangle\leqslant\mu|y-y'|^2;$$

(v) there exists $\lambda>2\mu+L^2$ such that

$$Ee^{\lambda\tau}(1+|X_\tau|^{2q})<\infty.$$

By a *solution* of RBSDE with random terminal time we mean a triple (Y,Z,K) of $\mathcal{F}$ progressively measurable processes in $\bar{D}\times\mathbb{R}^{d\times m}\times\mathbb{R}^d$ satisfying (1.1) and such that

(a) $E\big(\sup_{t\leqslant\tau}\exp(\lambda t)|Y_t|^2+\int_0^\tau\exp(\lambda t)\|Z_t\|^2dt\big)<\infty$;

(b) K is a continuous process with locally bounded variation, such that $K_0=0$, and $\int_0^\tau(Y_t-A_t)dK_t\leqslant 0$ for every $\mathcal{F}$ progressively measurable process A with values in $\bar{D}$. Moreover, $Y_t=\xi,\ Z_t=0,\ K_t=K_\tau$ on the set $\{t\geqslant\tau\}$.

In [16] it is proved that under the assumptions (i)–(v) there exists a unique strong solution of (1.1).

As in the previous section we shall approximate a Wiener process W by a scaled random walk W^n. We follow [21] and start with approximation of the stopping time τ by a sequence of bounded stopping times $\{\tau^n\}_n$. Since, for every $n\in\mathbb{N}$, τ^n is bounded, we can find $T_n\in\mathbb{N}$ such that $\tau^n\leqslant T_n$.

First let us introduce the approximation scheme for the forward equation. Write $\tau^n_j=(j/n)\wedge([n\tau^n]/n)$, $j\in\mathbb{N}$, and note that τ^n_j is an $\mathcal{F}^n$ stopping time. Now put $x^n_0=x$ and for $t\in[0,T_n]$ set $X^n_t=x^n_{\tau^n_{[nt]}}$, where x^n is given by (2.3).

In order to define the discrete RBSDE with random terminal time we take $j=nT_n,\dots,0$ and on the set $\{\tau^n\leqslant j/n\}$ we put $y^n_{\tau^n_j}=y^n_{\tau^n}=g(x^n_{\tau^n_j})$, $z^n_{\tau^n_j}=z^n_{\tau^n}=0$, $\Delta k^n_{\tau^n_{j+1}}=0$. Next, on the set $\{[n\tau^n]>j\}=\{[n\tau^n]/n>j/n\}\in\mathcal{F}^n_{j/n}$ we consider

$$y^n_{\tau^n_j}=y^n_{\tau^n_{j+1}}+\frac{1}{n}f(j/n,x^n_{\tau^n_j},y^n_{\tau^n_j},z^n_{\tau^n_j})\mathbf{1}_{\{[n\tau^n]>j\}}-\frac{1}{\sqrt{n}}z^n_{\tau^n_j}\varepsilon^n_{j+1}+\Delta k^n_{\tau^n_{j+1}}. \tag{3.1}$$

By a *solution* of (3.1) we mean a triple $(Y^n, Z^n, K^n) = (Y^n_t, Z^n_t, K^n_t)_{t\in[0,T_n]}$ of $\mathcal{F}^n$ adapted processes in $\bar{D}\times\mathbb{R}^{d\times m}\times\mathbb{R}^d$ such that $|K^n|_{\tau^n} < \infty$, $K^n_0 = 0$, and $\int_0^{\tau^n}(Y^n_{t-} - A^n_{t-})dK^n_t \leqslant 0$ for every $\mathcal{F}^n$ adapted process A^n with values in $\bar{D}$, where $Y^n_t = y^n_{\tau^n_{[nt]}}$, $Z^n_t = z^n_{\tau^n_{[nt]}}$, $K^n_t = \sum_{j=1}^{[nt]}\Delta k^n_{\tau^n_j}$. Moreover, on the set $\{t \geqslant \tau^n\}$, we have $Y^n_t = Y^n_{\tau^n}$, $Z^n_t = 0$, $K^n_t = K^n_{\tau^n}$.

Observe that (3.1) can be written in the equivalent form

$$
\begin{aligned}
(3.2)\qquad Y^n_{t\wedge\tau^n} = g(X^n_{\tau^n}) &+ \int_{t\wedge\tau^n}^{\tau^n} f(\varrho^n_{s-}, X^n_{s-}, Y^n_{s-}, Z^n_{s-})d\varrho^n_s \\
&- \int_{t\wedge\tau^n}^{\tau^n} Z^n_{s-}dW^n_s + K^n_{\tau^n} - K^n_{t\wedge\tau^n}, \quad t\in\mathbb{R}^+,
\end{aligned}
$$

and that (Y^n, Z^n, K^n) is a unique solution of equation (3.2) (see Lemma 5.4 (b)).

To solve (3.1) we first set

$$
z^n_{\tau^n_j} = z^n_{\tau^n_j}\mathbf{1}_{\{[n\tau^n]>j\}} = \sqrt{n}E(y^n_{\tau^n_{j+1}}\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n})\mathbf{1}_{\{[n\tau^n]>j\}},
$$

and next we find a solution $h^n_{\tau^n_j}$ of the equation

$$
h^n_{\tau^n_j} = y^n_{\tau^n_{j+1}} + \frac{1}{n}f\big(j/n, x^n_{\tau^n_j}, \pi(h^n_{\tau^n_j}), z^n_{\tau^n_j}\big)\mathbf{1}_{\{[n\tau^n]>j\}} - \frac{1}{\sqrt{n}}z^n_{\tau^n_j}\varepsilon^n_{j+1}.
$$

Note that since f and π are Lipschitz, $h^n_{\tau^n_j}$ is well defined for $n > L$. Now put $y^n_{\tau^n_j} = \pi(h^n_{\tau^n_j})$ and $\Delta k^n_{\tau^n_{j+1}} = y^n_{\tau^n_j} - h^n_{\tau^n_j}$. Observe that $h^n_{\tau^n_j}, y^n_{\tau^n_j}$ and $\Delta k^n_{\tau^n_{j+1}}$ are $\mathcal{F}^n_{j/n}$-measurable. Similarly to Section 2 it can be shown that, for any $\mathcal{F}^n$ adapted process A^n with values in $\bar{D}$,

$$
\int_0^{\tau^n}(Y^n_{t-} - A^n_{t-})dK^n_t = \sum_{j=1}^{[n\tau^n]}\langle y^n_{\tau^n_{j-1}} - A^n_{\tau^n_{j-1}}, \Delta k^n_{\tau^n_j}\rangle \leqslant 0.
$$

THEOREM 3.1. *Assume that* (i)–(v) *hold. Let* $\{\tau^n\}$ *be a sequence of* $\mathcal{F}^n$ *stopping times such that* $\sup_n E\exp(\lambda\tau^n)(1 + |X^n_{\tau^n}|^{2q}) < \infty$.

(a) *If* $(W^n, \tau^n)\xrightarrow{\mathcal{D}}(W,\tau)$, *then*

$$
\Big(X^n, Y^n, \int_0^{\cdot} Z^n_{s-}dW^n_s, K^n, W^n\Big)\xrightarrow{\mathcal{D}}\Big(X, Y, \int_0^{\cdot} Z_s dW_s, K, W\Big)
$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^m\times\bar{D}\times\mathbb{R}^{2d}\times\mathbb{R}^m)$.

(b) *If* $\sup_{t\leqslant T}|W^n_t - W_t|\xrightarrow{\mathcal{P}}0$, $T\in\mathbb{R}^+$ *and* $\tau^n\xrightarrow{\mathcal{P}}\tau$, *then*

$$
\Big(\sup_{t\leqslant T}|Y^{n,\tau^n}_t - Y^\tau_t| + \int_0^T\|Z^{n,\tau^n}_{t-} - Z^\tau_t\|^2dt + \sup_{t\leqslant T}|K^{n,\tau^n}_t - K^\tau_t|\Big)\xrightarrow{\mathcal{P}} 0.
$$

The proof will be given in Section 5.

Similarly to Section 2, we may consider another numerical scheme for RBSDE (1.1), which is easier to simulate. For $j = nT_n, \dots, 0$, on the set $\{\tau^n \leqslant j/n\}$ we put $\hat{y}^n_{\tau^n_j} = g(x^n_{\tau^n_j})$, $\hat{z}^n_{\tau^n_j} = 0$, $\Delta\hat{k}^n_{\tau^n_{j+1}} = 0$. On the set $\{[n\tau^n] > j\}$ we first take $\hat{z}^n_{\tau^n_j} = \sqrt{n}E(\hat{y}^n_{\tau^n_{j+1}}\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n})$, and then find a solution $\hat{h}^n_{\tau^n_j}$ of the equation

$$\hat{h}^n_{\tau^n_j} = \hat{y}^n_{\tau^n_{j+1}} + \frac{1}{n}f(j/n, x^n_{\tau^n_j}, \bar{y}^n_{\tau^n_j}, \hat{z}^n_{\tau^n_j})\mathbf{1}_{\{[n\tau^n]>j\}} - \frac{1}{\sqrt{n}}\hat{z}^n_{\tau^n_j}\varepsilon^n_{j+1},$$

where $\bar{y}^n_{\tau^n_j} = E(\hat{y}^n_{\tau^n_{j+1}}|\mathcal{F}^n_{j/n})$. Next, put $\hat{y}^n_{\tau^n_j} = \pi(\hat{h}^n_{\tau^n_j})$ and $\Delta\hat{k}^n_{\tau^n_{j+1}} = \hat{y}^n_{\tau^n_j} - \hat{h}^n_{\tau^n_j}$. Finally, let us define processes on $\mathbb{R}^+$ by setting $\hat{Y}^n_t = \hat{y}^n_{\tau^n_{[nt]}}$, $\hat{Z}^n_t = \hat{z}^n_{\tau^n_{[nt]}}$, $\hat{K}^n_t = \sum_{j=1}^{[nt]}\Delta\hat{k}^n_{\tau^n_j}$, $\bar{Y}^n_t = \bar{y}^n_{\tau^n_{[nt]}}$, so that

$$\hat{Y}^n_{t\wedge\tau^n} = g(X^n_{\tau^n}) + \int_{t\wedge\tau^n}^{\tau^n} f(\varrho^n_{s-}, X^n_{s-}, \bar{Y}^n_{s-}, \hat{Z}^n_{s-})d\varrho^n_s - \int_{t\wedge\tau^n}^{\tau^n} \hat{Z}^n_{s-}dW^n_s + \hat{K}^n_{\tau^n} - \hat{K}^n_{t\wedge\tau^n}$$

for $t \in \mathbb{R}^+$.

PROPOSITION 3.1. *Assume that* (i)–(v) *hold. Let* $\{\tau^n\}$ *be a sequence of* $\mathcal{F}^n$ *stopping times such that* $\sup_n E\exp(\lambda\tau^n)(1 + |X^n_{\tau^n}|^{2q}) < \infty$ *and assume that* (Y^n, Z^n, K^n), $(\hat{Y}^n, \hat{Z}^n, \hat{K}^n)$ *are given as above. Then, for every* $T \in \mathbb{R}^+$,

$$E\big(\sup_{t\leqslant T}|Y^{n,\tau^n}_t - \hat{Y}^{n,\tau^n}_t|^2 + \int_0^T \|Z^{n,\tau^n}_{t-} - \hat{Z}^{n,\tau^n}_{t-}\|^2 d\varrho^n_t + \sup_{t\leqslant T}|K^{n,\tau^n}_t - \hat{K}^{n,\tau^n}_t|^2\big) \to 0.$$

The proof of Proposition 3.1 is deferred to Section 5.

We end this section with a simple example of a sequence of stopping times $\{\tau^n\}$ satisfying the assumptions of Theorem 3.1.

EXAMPLE 3.1. Let $a \in \mathbb{R}^+$. Define

$$\tau = \inf\{t \geqslant 0;\ |W_t| > a\}, \qquad \tau^n = \inf\{t \geqslant 0;\ |W^n_t| > a\} \wedge n.$$

By [19] it is known that $(W^n, \tau^n) \xrightarrow{\mathcal{D}} (W, \tau)$. One can show that there exists a constant $C(a)$ such that, for every $\lambda < C(a)$, $\sup_n E\exp(\lambda\tau^n) < \infty$ (see Section 5 for details). On the other hand, it is not true that $\sup_n E\exp(\lambda\tau^n) < \infty$ for every $\lambda > 0$. It is a consequence of the fact that $E\exp(\lambda\tau) < \infty$ only for $\lambda < C(a)$ (see e.g. [12], Lemma 1.3).

4. APPLICATIONS

4.1. Discrete RBSDE and the obstacle problem for parabolic PDE. In this section we will consider the case where $D = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$ and

$\tau = T \in \mathbb{R}^+$. For each $(s,x) \in [0,T] \times \mathbb{R}^m$ let $X^{s,x}$ denote the solution of the SDE

$$(4.1) \qquad X_t^{s,x} = x + \int_s^t b(\theta, X_\theta^{s,x})d\theta + \int_s^t \sigma(\theta, X_\theta^{s,x})dW_\theta, \quad t \in [s,T],$$

and let $(Y^{s,x}, Z^{s,x}, K^{s,x})$ denote the solution of the equation

$$(4.2) \quad Y_t^{s,x} = (X_T^{s,x}) + \int_t^T f(\theta, X_\theta^{s,x}, Y_\theta^{s,x}, Z_\theta^{s,x})d\theta - \int_t^T Z_\theta^{s,x}dW_\theta + K_T^{s,x} - K_t^{s,x}$$

for $t \in [s,T]$. Under the assumptions (i)–(iii), $u(s,x) := Y_s^{s,x}$ is a continuous function of (s,x), which is a viscosity solution (see [7], p. 35, for a definition) of the following obstacle problem:

$$(4.3) \quad \begin{cases} \min\Big(u_i(t,x) - a_i, \max\big(u_i(t,x) - b_i, -F_u^i(t,x)\big)\Big) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [s,T),\ x \in \mathbb{R}^m, \\ u(T,x) = g(x), \qquad\qquad\qquad\quad x \in \mathbb{R}^m, \end{cases}$$

for $i = 1, \dots, d$, where

$$F_u^i(t,x) = \frac{\partial u_i}{\partial t}(t,x) + \frac{1}{2} \sum_{1 \leqslant j,k \leqslant d} \frac{\partial^2 u_i}{\partial x_j \partial x_k}(t,x)(\sigma\sigma^T)_{jk}(t,x)$$
$$+ \sum_{1 \leqslant j \leqslant d} \frac{\partial u_i}{\partial x_j}(t,x) b_j(t,x) + f_i\big(t, x, u(t,x), (\nabla u \sigma)(t,x)\big)$$

(see e.g. [23]). The approximation of the solution of parabolic PDE using the discrete scheme for BSDE (without reflection) was considered in [2]. Here, we propose a numerical scheme for the obstacle problem (4.3) which uses the discrete approximation of RBSDE (4.2). Fix $x \in \mathbb{R}^m$ and set $x_{j/n}^n = x, j = 0, \dots, [ns]$. Next, define $X_t^{s,x,n} = x_{[nt]/n}^n$, where $x_{(j+1)/n}^n$ is given by (2.3), $j = [ns], \dots, [nT] - 1$. Observe that $X^{s,x,n}$ is a strong solution of the SDE

$$X_t^{s,x,n} = x + \int_s^t b(\varrho_\theta^n, X_{\theta-}^{s,x,n})d\varrho_\theta^n + \int_s^t \sigma(\varrho_\theta^n, X_{\theta-}^{s,x,n})dW_\theta^n, \quad t \in [s,T].$$

Furthermore, $(X^{s,x,n}, W^n) \xrightarrow{\mathcal{D}} (X^{s,x}, W)$, where $X^{s,x}$ is given by (4.1). The next step is to solve the discrete RBSDE. To do this we put $y_{[nT]/n}^n = g(x_{[nT]/n}^n)$ and solve (2.4) for $j = [nT]-1, \dots, [ns]$. Let us put

$$\begin{aligned} D_+^n u(j,x) &= \frac{1}{2}\, u\left(j,\ x + \frac{1}{n}\, b\,(j/n,x) + \frac{1}{\sqrt{n}}\,\sigma\,(j/n,x)\right) \\ &\quad + \frac{1}{2}\, u\left(j,\ x + \frac{1}{n}\, b\,(j/n,x) - \frac{1}{\sqrt{n}}\,\sigma\,(j/n,x)\right), \\ D_-^n u(j,x) &= \frac{1}{2}\, u\left(j,\ x + \frac{1}{n}\, b\,(j/n,x) + \frac{1}{\sqrt{n}}\,\sigma\,(j/n,x)\right) \\ &\quad - \frac{1}{2}\, u\left(j,\ x + \frac{1}{n}\, b\,(j/n,x) - \frac{1}{\sqrt{n}}\,\sigma\,(j/n,x)\right). \end{aligned}$$

LEMMA 4.1. *Assume that* $v^n\colon \mathbb{N}\times\mathbb{R}^m \to D$, $n > L$, *is a function such that* $v^n([nT],x) = g(x)$ *and, for* $j = [ns],\dots,[nT]-1$, $v^n(j,x)$ *is defined as a unique solution of the equation*

$$v^n(j,x) = \pi\left(D_+^n v^n(j+1,x) + \frac{1}{n} f\big(j/n, x, v^n(j,x), \sqrt{n}\, D_-^n v^n(j+1,x)\big)\right). \tag{4.4}$$

Then $y^n_{j/n} = v^n(j, x^n_{j/n})$, $z^n_{j/n} = \sqrt{n} D^n_-\, v^n(j+1, x^n_{j/n})$.

P r o o f. We proceed by induction. First note that since f and π are Lipschitz functions, the solution of (4.4) is unique for $n > L$. By definition we have $v^n([nT], x^n_{[nT]/n}) = g(x^n_{[nT]/n}) = y^n_{[nT]/n}$. Suppose that

$$y^n_{(j+1)/n} = v^n(j+1, x^n_{(j+1)/n}).$$

As

$$D_+^n v^n(j+1, x^n_{j/n}) = E\big(v^n(j+1, x^n_{(j+1)/n}) | \mathcal{F}^n_{j/n}\big)$$

and

$$D_-^n v^n(j+1, x^n_{j/n}) = E\big(v^n(j+1, x^n_{(j+1)/n})\varepsilon^n_{j+1} | \mathcal{F}^n_{j/n}\big)$$

we have $z^n_{j/n} = \sqrt{n} D^n_- v^n(j+1, x^n_{j/n})$ and

$$h^n_{j/n} = D_+^n v^n(j+1, x^n_{j/n}) + \frac{1}{n} f\big(j/n, x^n_{j/n}, \pi(h^n_{j/n}), \sqrt{n} D^n_- v^n(j+1, x^n_{j/n})\big).$$

Since $y^n_{j/n} = \pi(h^n_{j/n})$ and the solution of (4.4) is unique, the lemma follows. ■

PROPOSITION 4.1. *Let* $u^n(t,x) = v^n(j,x)$, $t \in \big[j/n, (j+1)/n\big)$, $x \in \mathbb{R}^m$, $n > L$. *For each fixed* $s \in [0,T]$, *the sequence* $u^n(s,\cdot)$ *converges uniformly on compact sets in* $\mathbb{R}^m$ *to* $u(s,\cdot)$, *where* u *is a solution of the system* (4.3).

The proof of this proposition runs analogously to the proof of Theorem 5.2 in [2], so we omit it.

REMARK 4.1. Suppose that $\hat{v}^n$ satisfies $\hat{v}^n([nT], x) = g(x)$,

$$\hat{v}^n(j,x) = \pi\Big(D_+^n\hat{v}^n(j+1,x) + \frac{1}{n}\, f\big(j/n, x, D_+^n\hat{v}^n(j+1,x), \sqrt{n}\, D_-^n\hat{v}^n(j+1,x)\big)\Big)$$

for $j = [ns], \ldots, [nT]-1$. Then we have

$$\hat{y}^n_{j/n} = \hat{v}^n(j, x^n_{j/n})$$

and

$$\hat{z}^n_{j/n} = \sqrt{n} D_-^n \hat{v}^n(j+1, x^n_{j/n}),$$

where $\hat{y}^n_{j/n}$ satisfies

$$\hat{y}^n_{j/n} = \hat{y}^n_{(j+1)/n} + \frac{1}{n} f(j/n, x^n_{j/n}, \bar{y}^n_{j/n}, \hat{z}^n_{j/n}) - \frac{1}{\sqrt{n}}\hat{z}^n_{j/n}\varepsilon^n_{j+1} + \Delta\hat{k}^n_{(j+1)/n}$$

for $j = [nT]-1, \ldots, [ns]$. Moreover, if we define $\hat{u}^n(t,x) = \hat{v}^n(j,x)$, $t \in \big[j/n, (j+1)/n\big)$, $x \in \mathbb{R}^m$, then $\hat{u}^n(s,\cdot) \to u(s,\cdot)$, where u is a solution of the system (4.3).

Observe that the equality $\hat{y}^n_{(j+1)/n} = \hat{v}^n(j+1, x^n_{(j+1)/n})$ implies that $\bar{y}^n_{j/n} = (\hat{y}^n_{(j+1)/n} | \mathcal{F}^n_{j/n}) = D_+^n \hat{v}^n(j+1, x^n_{j/n})$. Therefore the result follows by Propositions 2.1 and 4.1.

4.2. Discrete RBSDE and the obstacle problem for elliptic PDE. As in the previous section let $D = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$ and let, moreover, assume that the functions b, σ, f do not depend on time. For each $x \in \mathbb{R}^m$ let X^x denote the solution of the SDE

$$X^x_t = x + \int_0^t b(X^x_s)ds + \int_0^t \sigma(X^x_s)dW_s, \quad t \in \mathbb{R}^+. \tag{4.5}$$

Let G be an open bounded set in $\mathbb{R}^m$. Define $\tau^x = \inf\{t \geqslant 0; X^x_t \notin G\}$ and assume that $\sup_{x\in\bar{G}} E\exp(\lambda\tau^x)(1 + |X_{\tau^x}|^{2q}) < \infty$. It can be shown that the mapping $x \mapsto \tau^x$ is a.s. continuous ([15], Proposition 4.1). Consider now for each $x \in \bar{G}$ the following RBSDE:

$$Y^x_{t\wedge\tau^x} = g(X^x_{\tau^x}) + \int_{t\wedge\tau^x}^{\tau^x} f(X^x_s, Y^x_s, Z^x_s)ds - \int_{t\wedge\tau^x}^{\tau^x} Z^x_s dW_s + K^x_{\tau^x} - K^x_{t\wedge\tau^x} \tag{4.6}$$

for $t \in \mathbb{R}^+$. Under the assumptions (i)–(iii), $u(x) = Y^x_0$, $x \in \bar{G}$, is a continuous function, which is a viscosity solution of the following obstacle problem:

$$\begin{cases} \min\Big(u_i(x) - a_i, \max\big(u_i(x) - b_i, -F^i_u(x)\big)\Big) = 0, & x \in G, \\ u(x) = g(x), & x \in \partial G, \end{cases} \tag{4.7}$$

for $i = 1, \ldots, d$, where

$$F_u^i(x) = f_i\big(x, u(x), \nabla u\sigma(x)\big) + Lu_i(x) = f_i\big(x, u(x), \nabla u\sigma(x)\big) + \frac{1}{2}\sum_{1\leqslant j,l\leqslant m} \frac{\partial^2 u_i}{\partial x_j \partial x_l}(x)(\sigma\sigma^T)_{jl}(x) + \sum_{1\leqslant j\leqslant m} \frac{\partial u_i}{\partial x_j}(x) b_j(x)$$

(see [16]). In what follows we propose a numerical scheme for (4.7) which uses the discrete approximation of RBSDE (4.6).

Let us fix $x \in \bar{G}$. As in Theorem 3.1, we need to approximate the stopping time τ^x by a sequence of bounded stopping times $\{\tau^{x,n}\}$. We also assume that $(W^n, \tau^{x,n}) \xrightarrow{\mathcal{D}} (W, \tau^x)$. Let $T_{x,n} \in \mathbb{N}$ be such that $\tau^{x,n} \leqslant T_{x,n}$ and let $\tau_j^{x,n} = (j/n) \wedge ([n\tau^{x,n}]/n)$, $j \in \mathbb{N}$. We set $x_0^n = x$ and

$$x^n_{\tau^{x,n}_{j+1}} = x^n_{\tau^{x,n}_j} + \frac{1}{n} b(x^n_{\tau^{x,n}_j}) + \frac{1}{\sqrt{n}} \sigma(x^n_{\tau^{x,n}_j}) \varepsilon^n_{j+1}$$

for $j = 0, 1, \ldots, nT_{x,n} - 1$. Observe that $X^{x,n}$ defined as $X_t^{x,n} = x^n_{\tau^{x,n}_{[nt]}}$ satisfies

$$X_t^{x,n} = x + \int_0^{t\wedge\tau^{x,n}} b(X_{s-}^{x,n}) d\varrho_s^n + \int_0^{t\wedge\tau^{x,n}} \sigma(X_{s-}^{x,n}) dW_s^n, \quad t \in \mathbb{R}^+.$$

Furthermore, $(X^{x,n}, W^n, \tau^{x,n}) \xrightarrow{\mathcal{D}} (X^x, W, \tau^x)$, where X^x is given by (4.5).

The next step is to solve the discrete RBSDE. Assume that

$$\sup_n E \exp(\lambda \tau^{x,n})(1 + |X^n_{\tau^{x,n}}|^{2q}) < \infty.$$

Put $y^n_{\tau^{x,n}_j} = g(x^n_{\tau^{x,n}_j})$, $z^n_{\tau^{x,n}_j} = 0$, $\Delta k^n_{\tau^{x,n}_{j+1}} = 0$ on the set $\{\tau^{x,n} \leqslant j/n\}$ and as in Section 3 solve the equation

$$y^n_{\tau^{x,n}_j} = y^n_{\tau^{x,n}_{j+1}} + \frac{1}{n} f(x^n_{\tau^{x,n}_j}, y^n_{\tau^{x,n}_j}, z^n_{\tau^{x,n}_j}) \mathbf{1}_{\{[n\tau^{x,n}]>j\}} - \frac{1}{\sqrt{n}} z^n_{\tau^{x,n}_j} \varepsilon^n_{j+1} + \Delta k^n_{\tau^{x,n}_{j+1}}$$

on the set $\{[n\tau^{x,n}] > j\}$ for $j = nT_{x,n}, \ldots, 0$. Let us put

$$D_+^n u(x) = \frac{1}{2}\, u\left(x + \frac{1}{n} b\,(x) + \frac{1}{\sqrt{n}}\,\sigma\,(x)\right) + \frac{1}{2}\, u\left(x + \frac{1}{n} b\,(x) - \frac{1}{\sqrt{n}}\,\sigma\,(x)\right),$$
$$D_-^n u(x) = \frac{1}{2}\, u\left(x + \frac{1}{n} b\,(x) + \frac{1}{\sqrt{n}}\,\sigma\,(x)\right) - \frac{1}{2}\, u\left(x + \frac{1}{n} b\,(x) - \frac{1}{\sqrt{n}}\,\sigma\,(x)\right).$$

LEMMA 4.2. *Assume that $u^n \colon \bar{G} \to D$, $n > L$, satisfies $u^n(x) = g(x)$ for $x \in \partial G$ and, for $x \in G$, $u^n(x)$ is defined as a unique solution of the equation*

$$u^n(x) = \pi\left(D_+^n u^n(x) + \frac{1}{n} f\big(x, u^n(x), \sqrt{n}\, D_-^n u^n(x)\big)\right). \tag{4.8}$$

Then $y^n_{\tau^{x,n}_j} = u^n(x^n_{\tau^{x,n}_j})$, $z^n_{\tau^{x,n}_j} = \sqrt{n} D_-^n u^n(x^n_{\tau^{x,n}_j}) \mathbf{1}_{\{[n\tau^{x,n}]>j\}}$, $j = 0, \ldots, nT_{x,n}$.

P r o o f. We proceed by induction. Since $x^n_{\tau^{x,n}}\in\partial G$, we have $y^n_{\tau^{x,n}}=g(x^n_{\tau^{x,n}}) = u^n(x^n_{\tau^{x,n}})$. Note that it is enough to consider the case $\{[n\tau^{x,n}] > j\}$. Suppose that $y^n_{\tau^{x,n}_{j+1}} = u^n(x^n_{\tau^{x,n}_{j+1}})$. Then

$$z^n_{\tau^{x,n}_j} = \sqrt{n}E\big(u^n(x^n_{\tau^{x,n}_{j+1}})\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n}\big)\mathbf{1}_{\{[n\tau^{x,n}]>j\}} = \sqrt{n}D^n_- u^n(x^n_{\tau^{x,n}_j})\mathbf{1}_{\{[n\tau^{x,n}]>j\}}.$$

On the set $\{[n\tau^{x,n}] > j\}$ we have $E\big(u^n(x^n_{\tau^{x,n}_{j+1}})|\mathcal{F}^n_{j/n}\big) = D^n_+ u^n(x^n_{\tau^{x,n}_j})$ and

$$\begin{aligned} h^n_{\tau^{x,n}_j} &= E\big(u^n(x^n_{\tau^{x,n}_{j+1}})|\mathcal{F}^n_{j/n}\big) + \frac{1}{n}f\big(x^n_{\tau^{x,n}_j}, \pi(h^n_{\tau^{x,n}_j}), z^n_{\tau^{x,n}_j}\big)\\ &= D^n_+ u^n(x^n_{\tau^{x,n}_j}) + \frac{1}{n}f\big(x^n_{\tau^{x,n}_j}, \pi(h^n_{\tau^{x,n}_j}), \sqrt{n}D^n_- u^n(x^n_{\tau^{x,n}_j})\big). \end{aligned}$$

Since $y^n_{\tau^{x,n}_j} = \pi(h^n_{\tau^{x,n}_j})$ and the solution of (4.8) is unique (functions f and π are Lipschitz), the proof is complete. ■

PROPOSITION 4.2. *Let $u^n(x)$, $x \in \bar{G}$, $n > L$, be as defined above. Then $u^n(x) \to u(x)$, where u is a solution of* (4.7).

P r o o f. Observe that if $x \in \partial G$, then $u^n(x) = g(x) = u(x)$. Take $x \in G$. We have $u^n(x) = u^n(x^n_0) = y^n_0 = Y^{x,n}_0$ and $u(x) = Y^x_0$. Since $Y^{x,n}_0, Y^x_0$ are constant, by Theorem 3.1 we get the result. ■

REMARK 4.2. Assume that $\hat{u}^n(x) = g(x)$, $x \in \partial G$, and

$$\hat{u}^n(x) = \pi\bigg(D^n_+\hat{u}^n(x) + \frac{1}{n}f\big(x, D^n_+\hat{u}^n(x), \sqrt{n}D^n_-\hat{u}^n(x)\big)\bigg),$$

$x\in G$. Then we have $\hat{y}^n_{\tau^{x,n}_j} = \hat{u}^n(x^n_{\tau^{x,n}_j})$, $\hat{z}^n_{\tau^{x,n}_j} = \sqrt{n}D^n_-\hat{u}^n(x^n_{\tau^{x,n}_j})\mathbf{1}_{\{[\tau^{x,n}]>j\}}$, $j = 0, \ldots, T_{x,n}$, where $\hat{y}^n, \hat{z}^n$ are as in Section 3.

Moreover, for every $x \in \bar{G}$, $\hat{u}^n(x) \to u(x)$, where u is a solution of (4.7).

4.3. Backward stochastic viability property. In this section we investigate a viability property in a general convex domain $\bar{D} \subset \mathbb{R}^d$ for solutions of non-reflected BSDEs with random terminal time

$$Y_{t\wedge\tau} = g(X_\tau) + \int_{t\wedge\tau}^{\tau} f(s, X_s, Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s, \quad t \in \mathbb{R}^+. \tag{4.9}$$

We recall that a stochastic process $\{Y_t\}_{t\in\mathbb{R}^+}$ is *viable* in $\bar{D}$ if and only if for each $t \in \mathbb{R}^+$

$$Y_t(\omega) \in \bar{D} \quad \mathcal{P}\text{-a.s.}$$

Following [4], we assume that the generator f satisfies

$$\big\langle h - \pi(h), f\big(t, x, \pi(h), z\big)\big\rangle \leqslant Cd^2(h) \quad \mathcal{P}\text{-a.s.} \tag{4.10}$$

for every $(t, x, z) \in \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^{d\times m}$ and for all $h \in \mathbb{R}^d$ such that $d^2(\cdot)$ is twice differentiable at h, where $d(\cdot)$ denotes the distance function from the set $\bar{D}$.

COROLLARY 4.1. *Assume that* (i)–(v) *and* (4.10) *hold and that* τ *is an* $\mathcal{F}$ *stopping time of the form* $\tau = \inf\{t \geqslant 0; |W_t| > a\}$. *Then the process* $\{Y_t\}_{t\in\mathbb{R}^+}$ *being the first component of the solution* (4.9) *is viable in* $\bar{D}$.

Proof. Consider the sequence of stopping times τ^n as in Example 3.1, i.e. $\tau^n = \inf\{t \geqslant 0; |W_t^n| > a\}\wedge n$. Put $\tau^{(M)} = \tau\wedge M$ and $\tau^{n,(M)} = \tau^n\wedge M$, $M \in \mathbb{N}$. Define X^n, Y^n, Z^n, K^n as in Section 3 but with τ^n replaced by $\tau^{n,(M)}$. By (2.6) and (2.7), for every $j = 0, \ldots, [nT_n] - 1$ we have

$$\begin{aligned}
|\Delta k^n_{\tau^n_{j+1}}|^2 &= \langle h^n_{\tau^n_j} - \pi(h^n_{\tau^n_j}), h^n_{\tau^n_j} - \pi(h^n_{\tau^n_j})\rangle \\
&= \Big\langle y^n_{\tau^n_{j+1}} + \frac{1}{n} f\big(j/n, \pi(h^n_{\tau^n_j}), z^n_{\tau^n_j}\big) - \frac{1}{\sqrt{n}} z^n_{\tau^n_j}\varepsilon^n_{j+1} - y^n_{\tau^n_j}, h^n_{\tau^n_j} - \pi(h^n_{\tau^n_j})\Big\rangle \\
&\leqslant \Big\langle y^n_{\tau^n_{j+1}} - y^n_{\tau^n_j} - \frac{1}{\sqrt{n}} z^n_{\tau^n_j}\varepsilon^n_{j+1}, h^n_{\tau^n_j} - \pi(h^n_{\tau^n_j})\Big\rangle + \frac{C}{n}|h^n_{\tau^n_j} - \pi(h^n_{\tau^n_j})|^2 \\
&\leqslant \Big\langle -\frac{1}{\sqrt{n}} z^n_{\tau^n_j}\varepsilon^n_{j+1}, h^n_{\tau^n_j} - \pi(h^n_{\tau^n_j})\Big\rangle + \frac{C}{n}|\Delta k^n_{\tau^n_{j+1}}|^2,
\end{aligned}$$

where the first inequality follows by (4.10) and the second one by (2.7). Thus, for sufficiently large n, $E|\Delta k^n_{\tau^n_{j+1}}|^2 = 0$, and hence $K^n = 0$. Since $\tau^{n,(M)}$ is bounded uniformly in n, $\sup_n E\exp(\lambda\tau^{n,(M)})(1 + |X^n_{\tau^{n,(M)}}|^{2q}) < \infty$. Moreover, $(W^n, \tau^{n,(M)}) \xrightarrow{\mathcal{D}} (W, \tau^{(M)})$. Therefore, by Theorem 3.1,

$$\Big(X^n, Y^n, \int_0^{\cdot} Z^n_{s-}dW^n_s, K^n, W^n\Big) \xrightarrow{\mathcal{D}} \Big(X^{(M)}, Y^{(M)}, \int_0^{\cdot} Z^{(M)}_s dW_s, 0, W\Big)$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^m\times\bar{D}\times\mathbb{R}^{2d}\times\mathbb{R}^m)$, where $(Y^{(M)}, Z^{(M)})$ is a solution of the equation

$$Y^{(M)}_{t\wedge\tau^{(M)}} = g(X_{\tau^{(M)}}) + \int_{t\wedge\tau^{(M)}}^{\tau^{(M)}} f(s, X_s, Y^{(M)}_s, Z^{(M)}_s)ds - \int_{t\wedge\tau^{(M)}}^{\tau^{(M)}} Z^{(M)}_s dW_s, \quad t \in \mathbb{R}^+,$$

such that, for every $M \in \mathbb{N}$ and $t \in \mathbb{R}^+$, $Y^{(M)}_t \in \bar{D}$ $\mathcal{P}$-a.s. Since $\tau^{(M)} \to \tau$ a.s. and $g(X_{\tau^{(M)}}) \to g(X_\tau)$ a.s., $Y^{\tau^{(M)}}_t \to Y^\tau_t$ a.s., where Y_t is the first component of the solution (4.9). Hence $Y_t \in \bar{D}$ $\mathcal{P}$-a.s., i.e., Y is viable in $\bar{D}$. ■

5. PROOFS

Before proving main results from Sections 2 and 3 we will give four technical lemmas.

LEMMA 5.1. *Assume that* (i)–(iii) *hold. For every* $T>0$ *and* $p\geqslant 1$ *there exists a constant* $C>0$ *such that for every* $n\in\mathbb{N}$ *and* $a\in D$

$$E\Big(\sup_{t\leqslant T}|Y^n_t-a|^{2p}+\big(\int_0^T\|Z^n_{t-}\|^2d\varrho^n_t\big)^p+|K^n|^{2p}_T\Big)$$
$$\leqslant CE\big(|g(X^n_T)-a|^{2p}+\int_0^T|f(\varrho^n_{t-},X^n_{t-},a,0)|^{2p}d\varrho^n_t\big).$$

P r o o f. Let us first show that

$$E(\sup_{t\leqslant T}|Y^n_t|^{2p}+\|Z^n_t\|^{2p})<\infty. \tag{5.1}$$

To prove it, we proceed by induction. For $j=[nT]-1$,

$$E|y^n_{(j+1)/n}|^{2p}=E|g(x^n_{(j+1)/n})|^{2p}\leqslant\kappa^{2p}E(1+|X^n_T|^q)^{2p}<\infty.$$

Therefore, $\|z^n_{j/n}\|^{2p}\leqslant n^p\big\|E(\|y^n_{(j+1)/n}\varepsilon_{j+1}\|^{2p}|\mathcal{F}^n_{j/n})\big\|$ is integrable. Since

$$\begin{aligned}|y^n_{j/n}|=|\pi(h^n_{j/n})|\leqslant|h^n_{j/n}|&=\Big|E(y^n_{(j+1)/n}|\mathcal{F}^n_{j/n})+\frac{1}{n}f(j/n,x^n_{j/n},y^n_{j/n},z^n_{j/n})\Big|\\&\leqslant E(|y^n_{(j+1)/n}||\mathcal{F}^n_{j/n})+\frac{1}{n}|f(j/n,x^n_{j/n},y^n_{j/n},z^n_{j/n})|\\&\leqslant E(|y^n_{(j+1)/n}||\mathcal{F}^n_{j/n})+\frac{L}{n}(|y^n_{j/n}|+\|z^n_{j/n}\|)+\frac{1}{n}|f(j/n,x^n_{j/n},0,0)|,\end{aligned}$$

$E|y^n_{j/n}|^{2p}<\infty$ for n such that $n>L$. It implies also that $E|\Delta k^n_{(j+1)/n}|^{2p}<\infty$. Inductively, $E(|Y^n_t|^{2p}+|K^n_t|^{2p})<\infty,\ t\in[0,T]$, and hence

$$E\sup_{t\leqslant T}|Y^n_t|^{2p}\leqslant E\sum_{j=1}^{[nT]}|y^n_{(j-1)/n}|^{2p}<\infty.$$

Moreover, observe that $z^n_{j/n}=\sqrt{n}E\big((y^n_{(j+1)/n}-y^n_{j/n})\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n}\big)$ implies that for any $p\in\mathbb{N}$

$$E\big(\int_t^T\|Z^n_{s-}\|^2d\varrho^n_s\big)^p=E\Big(\sum_{j=[nt]}^{[nT]-1}\|z^n_{j/n}\|^2\frac{1}{n}\Big)^p\leqslant p^pE([Y^n]^T_t)^p. \tag{5.2}$$

Fix $a \in D$. By Itô's formula,

$$\begin{aligned}(5.3)\quad |Y_t^n - a|^{2p} &+ p(2p-1)\int_t^T |Y_{s-}^n - a|^{2p-2} d[Y^n]_s \\ &\leqslant |g(X_T^n) - a|^{2p} \\ &\quad + 2p\int_t^T |Y_{s-}^n - a|^{2p-2}(Y_{s-}^n - a) f(\varrho_{s-}^n, X_{s-}^n, Y_{s-}^n, Z_{s-}^n) d\varrho_s^n \\ &\quad + 2p\int_t^T |Y_{s-}^n - a|^{2p-2}(Y_{s-}^n - a) dK_s^n \\ &\quad - 2p\int_t^T |Y_{s-}^n - a|^{2p-2}(Y_{s-}^n - a) Z_{s-}^n dW_s^n.\end{aligned}$$

In the rest of the proof C will denote a constant the values of which may change from line to line but do not depend on n. By (2.7) and the fact that $|Y_{s-}^n - a|^{2p-2}$ is nonnegative, the third component on the right-hand side of the above inequality is less than or equal to zero. By the Burkholder–Davis–Gundy inequality and by Hölder's inequality,

$$\begin{aligned}(5.4)\quad & E\sup_{t\leqslant T}\Big|\int_t^T |Y_{s-}^n - a|^{2p-2}(Y_{s-}^n - a) Z_{s-}^n dW_s^n\Big| \\ &\leqslant C\big(E\sup_{t\leqslant T}|Y_t^n - a|^{2p}\big)^{1/2}\Big(E\int_0^T |Y_{s-}^n - a|^{2p-2}\|Z_{s-}^n\|^2 d\varrho_s^n\Big)^{1/2} \\ &\leqslant \frac{1}{4p}E\sup_{t\leqslant T}|Y_t^n - a|^{2p} + C^2 p\, E\int_0^T |Y_{s-}^n - a|^{2p-2}\|Z_{s-}^n\|^2 d\varrho_s^n < \infty.\end{aligned}$$

(It follows by (5.1) and the fact that the last component of (5.4) is bounded by $C\big(E\int_0^T |Y_{s-}^n - a|^{2p} d\varrho_s^n\big)^{(2p-2)/2p}\big(E\int_0^T \|Z_{s-}^n\|^{2p} d\varrho_s^n\big)^{1/p} < \infty$.) Therefore, taking the expectation and using Young's inequality from (5.3) we obtain

$$\begin{aligned}E\Big(|Y_t^n - a|^{2p} &+ p(2p-1)\int_t^T |Y_{s-}^n - a|^{2p-2}\|Z_{s-}^n\|^2 d\varrho_s^n\Big) \\ &\leqslant E|g(X_T^n) - a|^{2p} \\ &\quad + CE\int_t^T |f(\varrho_{s-}^n, X_{s-}^n, a, 0)|^{2p} d\varrho_s^n + CE\int_t^T |Y_{s-}^n - a|^{2p} d\varrho_s^n \\ &\quad + \frac{p(2p-1)}{2}E\int_t^T |Y_{s-}^n - a|^{2p-2}\|Z_{s-}^n\|^2 d\varrho_s^n.\end{aligned}$$

Hence, by Gronwall's lemma,

$$E\big(|Y_t^n - a|^{2p} + \int_t^T |Y_{s-}^n - a|^{2p-2}\|Z_{s-}^n\|^2 d\varrho_s^n\big) \leqslant CE\big(|g(X_T^n) - a|^{2p} + \int_t^T |f(\varrho_{s-}^n, X_{s-}^n, a, 0)|^{2p} d\varrho_s^n\big). \tag{5.5}$$

Combining (5.3) with (5.4) and (5.5) we get

$$E\sup_{t\leqslant T}|Y_t^n - a|^{2p} \leqslant CE\big(|g(X_T^n) - a|^{2p} + \int_0^T |f(\varrho_{s-}^n, X_{s-}^n, a, 0)|^{2p} d\varrho_s^n\big).$$

Now, observe that by (5.3)

$$\begin{aligned}[Y^n]_T \leqslant |g(X_T^n) - a|^2 &+ 2\int_0^T (Y_{s-}^n - a) f(\varrho_{s-}^n, X_{s-}^n, Y_{s-}^n, Z_{s-}^n) d\varrho_s^n \\ &- 2\int_0^T (Y_{s-}^n - a) Z_{s-}^n dW_s^n.\end{aligned}$$

Therefore, by (5.2) and arguments used previously, we obtain

$$E\big(\int_0^T \|Z_{s-}^n\|^2 d\varrho_s^n\big)^p \leqslant CE\big(|g(X_T^n) - a|^{2p} + \int_0^T |f(\varrho_{s-}^n, X_{s-}^n, a, 0)|^{2p} d\varrho_s^n\big).$$

To complete the proof note that since $E(y_{(j+1)/n}^n|\mathcal{F}_{j/n}^n) \in \bar{D}$ and

$$h_{j/n}^n = E(y_{(j+1)/n}^n|\mathcal{F}_{j/n}^n) + \frac{1}{n} f(j/n, x_{j/n}^n, y_{j/n}^n, z_{j/n}^n),$$

we have

$$\begin{aligned}|K^n|_T = \sum_{j=1}^{[nT]} |\Delta k_{j/n}^n| &= \sum_{j=1}^{[nT]} |\mathrm{dist}(h_{(j-1)/n}^n, \bar{D})| \\ &\leqslant \sum_{j=1}^{[nT]} \frac{1}{n}\big|f\big((j-1)/n, x_{(j-1)/n}^n, y_{(j-1)/n}^n, z_{(j-1)/n}^n\big)\big| \\ &= \int_0^T |f(\varrho_{s-}^n, X_{s-}^n, Y_{s-}^n, Z_{s-}^n)| d\varrho_s^n \\ &\leqslant \int_0^T \big(L(|Y_{s-}^n - a| + \|Z_{s-}^n\|) + |f(\varrho_{s-}^n, X_{s-}^n, a, 0)|\big) d\varrho_s^n.\end{aligned}$$

Therefore, by previous estimates we obtain

$$E|K^n|_T^{2p} \leqslant CE\big(|g(X_T^n) - a|^{2p} + \int_0^T |f(\varrho_{s-}^n, X_{s-}^n, a, 0)|^{2p} d\varrho_s^n\big). \quad \blacksquare$$

Before proving the second lemma we introduce some notation. We denote by $\mathcal{M}_d^2(0,T;\gamma)$ the set of $\mathcal{F}$ progressively measurable d-dimensional processes X such that $E\int_0^T e^{\gamma t}|X_t|^2dt < \infty$, and by $\mathcal{M}_d^{n,2}(0,T;\gamma)$ the set of $\mathcal{F}^n$ adapted d-dimensional processes X^n such that $E\int_0^T \exp(\gamma\varrho_t^n)|X_{t-}^n|^2 d\varrho_t^n < \infty$. Moreover, we set

$$\mathcal{B}^2 = \mathcal{M}_d^2(0,T;\gamma) \times \mathcal{M}_{d\times m}^2(0,T;\gamma)$$

and

$$\mathcal{B}_n^2 = \mathcal{M}_d^{n,2}(0,T;\gamma) \times \mathcal{M}_{d\times m}^{n,2}(0,T;\gamma).$$

Clearly, if we set

$$\|(Y,Z)\|_\gamma^2 = E\int_0^T \exp(\gamma t)(|Y_t|^2 + \|Z_t\|^2)dt$$

and

$$\|(Y^n,Z^n)\|_{n,\gamma}^2 = E\int_0^T \exp(\gamma\varrho_t^n)(|Y_{t-}^n|^2 + \|Z_{t-}^n\|^2)d\varrho_t^n,$$

then $(\mathcal{B}^2, \|\cdot\|_\gamma)$ and $(\mathcal{B}_n^2, \|\cdot\|_{n,\gamma})$ are Banach spaces.

LEMMA 5.2. (a) *Define* $\Phi: \mathcal{B}^2 \to \mathcal{B}^2$ *by putting* $\Phi(U,V) = (Y,Z)$, *where* Y *and* Z *are the first two components of the solution* (Y,Z,K) *of the following RBSDE*:

$$Y_t = g(X_T) + \int_t^T f(s,X_s,U_s,V_s)ds - \int_t^T Z_s dW_s + K_T - K_t.$$

Then, for $\gamma \geqslant 4L^2+1$, Φ *is a contraction in* $(\mathcal{B}^2, \|\cdot\|_\gamma)$.

(b) *Define* $\Phi_n: \mathcal{B}_n^2 \to \mathcal{B}_n^2$ *by putting* $\Phi(U^n,V^n) = (Y^n,Z^n)$, *where* Y^n *and* Z^n *are the first two components of the solution* (Y^n,Z^n,K^n) *of the following RBSDE*:

$$Y_t^n = g(X_T^n) + \int_t^T f(\varrho_{s-}^n, X_{s-}^n, U_{s-}^n, V_{s-}^n)d\varrho_s^n - \int_t^T Z_{s-}^n dW_s^n + K_T^n - K_t^n.$$

Then, for $\gamma \geqslant 4L^2+1$, Φ_n *is a contraction in* $(\mathcal{B}_n^2, \|\cdot\|_{n,\gamma})$.

P r o o f. We will only give the proof of (b). The proof of (a) follows by the same method. Let (U^n,V^n), $(\bar U^n,\bar V^n) \in \mathcal{B}_n^2$ and set $(Y^n,Z^n) = \Phi_n(U^n,V^n)$, $(\bar Y^n,\bar Z^n) = \Phi_n(\bar U^n,\bar V^n)$. Using Itô's formula for any $\gamma \in \mathbb{R}$ we get

$$E\exp(\gamma\varrho_t^n)|Y_t^n - \bar Y_t^n|^2 + \gamma E\int_t^T \exp(\gamma\varrho_s^n)|Y_{s-}^n - \bar Y_{s-}^n|^2 ds + E\int_t^T \exp(\gamma\varrho_s^n)d[Y^n - \bar Y^n]_s$$

$$= 2E\int_t^T \exp(\gamma\varrho_s^n)(Y_{s-}^n - \bar{Y}_{s-}^n)\big(f(\varrho_{s-}^n, X_{s-}^n, U_{s-}^n, V_{s-}^n)$$
$$- f(\varrho_{s-}^n, X_{s-}^n, \bar{U}_{s-}^n, \bar{V}_{s-}^n)\big)d\varrho_s^n$$
$$+ 2E\int_t^T \exp(\gamma\varrho_s^n)(Y_{s-}^n - \bar{Y}_{s-}^n)d(K_s^n - \bar{K}_s^n)$$
$$\leqslant 4L^2 E\int_t^T \exp(\gamma\varrho_s^n)|Y_{s-}^n - \bar{Y}_{s-}^n|^2 d\varrho_s^n$$
$$+ \frac{1}{2}E\int_t^T \exp(\gamma\varrho_s^n)(|U_{s-}^n - \bar{U}_{s-}^n|^2 + \|V_{s-}^n - \bar{V}_{s-}^n\|^2)d\varrho_s^n.$$

Notice that $\exp(\gamma\varrho_s^n) \leqslant \exp(\gamma s)$ for $s \in [(j-1)/n, j/n)$. Since

$$E\int_0^T \|Z_{s-}^n - \bar{Z}_{s-}^n\|^2 d\varrho_s^n \leqslant E[Y^n - \bar{Y}^n]_T,$$

putting $\gamma = 4L^2 + 1$ we obtain

$$E\int_0^T \exp(\gamma\varrho_s^n)(|Y_{s-}^n - \bar{Y}_{s-}^n|^2 + \|Z_{s-}^n - \bar{Z}_{s-}^n\|^2)d\varrho_s^n$$
$$\leqslant \frac{1}{2}E\int_0^T \exp(\gamma\varrho_s^n)(|U_{s-}^n - \bar{U}_{s-}^n|^2 + \|V_{s-}^n - \bar{V}_{s-}^n\|^2)d\varrho_s^n,$$

which completes the proof. ∎

LEMMA 5.3. *Assume (Y, H, K) are continuous and (Y^n, H^n, K^n), $n \in \mathbb{N}$, are càdlàg processes satisfying $Y_t = H_t + K_T - K_t$ and $Y_t^n = H_t^n + K_T^n - K_t^n$, where Y, $\bar{Y}$ take values in $\bar{D}$, K, K^n are processes of bounded variation such that $K_0 = K_0^n = 0$ and $\int_0^T (Y_t - A_t)dK_t \leqslant 0$, $\int_0^T (Y_{t-}^n - A_t)dK_t^n \leqslant 0$ for every process A with values in $\bar{D}$. Then*

$$|Y_t^n - Y_t|^2 + \sum_{t<s\leqslant T} |\Delta K_s^n|^2$$
$$\leqslant |H_t^n - H_t|^2 + 2\int_t^T \big(H_t^n - H_t - (H_{s-}^n - H_s)\big)d(K_s^n - K_s).$$

P r o o f. By Itô's formula and assumptions,

$$|K_T^n - K_t^n - (K_T - K_t)|^2$$
$$= 2\int_t^T \big(K_T^n - K_{s-}^n - (K_T - K_s)\big)d(K_s^n - K_s) - \sum_{t<s\leqslant T} |\Delta K_s^n - \Delta K_s|^2$$

$$= 2\int_t^T \big(Y^n_{s-} - H^n_{s-} - (Y_s - H_s)\big)d(K^n_s - K_s) - \sum_{t<s\leqslant T} |\Delta K^n_s|^2$$
$$\leqslant -2\int_t^T (H^n_{s-} - H_s)d(K^n_s - K_s) - \sum_{t<s\leqslant T} |\Delta K^n_s|^2.$$

Since

$$|Y^n_t - Y_t|^2 = |H^n_t - H_t|^2 + |K^n_T - K^n_t - (K_T - K_t)|^2 + 2\langle H^n_t - H_t, K^n_T - K^n_t - (K_T - K_t)\rangle,$$

by the above inequality, we have

$$|Y^n_t - Y_t|^2 + \sum_{t<s\leqslant T} |\Delta K^n_s|^2$$
$$\leqslant |H^n_t - H_t|^2 - 2\int_t^T (H^n_{s-} - H_s)d(K^n_s - K_s) + 2\int_t^T (H^n_t - H_t)d(K^n_s - K_s),$$

which completes the proof. ■

LEMMA 5.4. *Assume that* (i)–(v) *hold and*

$$\sup_n E\exp(\lambda\tau^n)(1 + |X^n_{\tau^n}|^{2q}) < \infty.$$

(a) *There exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $a \in D$*

$$E\Big(\sup_{t\leqslant\tau^n} \exp(\lambda t)|Y^n_t - a|^2 + \int_0^{\tau^n} \exp(\lambda\varrho^n_t)\|Z^n_{t-}\|^2 d\varrho^n_t + |K^n|^2_{\tau^n}\Big)$$
$$\leqslant CE\Big(\exp(\lambda\tau^n)|g(X^n_{\tau^n}) - a|^2 + \int_0^{\tau^n} \exp(\lambda\varrho^n_t)|f(\varrho^n_{t-}, X^n_{t-}, a, 0)|^2 d\varrho^n_t\Big).$$

(b) *The solution of* (3.2) *is unique.*

P r o o f. (a) Similarly to the proof of Lemma 5.1 one can show that

$$E(\sup_{t\leqslant\tau^n} \exp(\lambda t)|Y^n_t|^2 + \|Z^n_t\|^2 + |K^n_t|^2) < \infty.$$

By Itô's formula,

$$\exp\big(\lambda(t\wedge\tau^n)\big)|Y^n_{t\wedge\tau^n} - a|^2 + \lambda\int_{t\wedge\tau^n}^{\tau^n} \exp(\lambda s)|Y^n_{s-} - a|^2 d\varrho^n_s + \int_{t\wedge\tau^n}^{\tau^n} \exp(\lambda\varrho^n_s)d[Y^n]_s$$
$$\leqslant \exp(\lambda\tau^n)|g(X^n_{\tau^n}) - a|^2 + 2\int_{t\wedge\tau^n}^{\tau^n} \exp(\lambda\varrho^n_s)(Y^n_{s-} - a)f(\varrho^n_{s-}, X^n_{s-}, Y^n_{s-}, Z^n_{s-})d\varrho^n_s$$
$$- 2\int_{t\wedge\tau^n}^{\tau^n} \exp(\lambda\varrho^n_s)(Y^n_{s-} - a)Z^n_{s-}dW^n_s$$

$$\leqslant \exp(\lambda\tau^n)|g(X^n_{\tau^n})-a|^2+(2\mu+L^2/\varepsilon+\eta)\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)|Y^n_{s-}-a|^2d\varrho^n_s$$
$$+\varepsilon\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)\|Z^n_{s-}\|^2d\varrho^n_s+\frac{1}{\eta}\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)|f(\varrho^n_{s-},X^n_{s-},a,0)|^2d\varrho^n_s$$
$$-2\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)(Y^n_{s-}-a)Z^n_{s-}dW^n_s.$$

Choosing $\varepsilon<1$ and $\eta>0$ such that $2\mu+L^2/\varepsilon+\eta<\lambda$, by (5.2) we get

$$E\Big(\exp\big(\lambda(t\wedge\tau^n)\big)|Y^n_{t\wedge\tau^n}-a|^2+\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)\|Z^n_{s-}\|^2d\varrho^n_s\Big)$$
$$\leqslant CE\big(\exp(\lambda\tau^n)|g(X^n_{\tau^n})-a|^2+\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)|f(\varrho^n_{s-},X^n_{s-},a,0)|^2d\varrho^n_s\big).$$

Using similar arguments to those in the proof of Lemma 5.1 we complete the proof of part (a).

(b) Suppose that (Y^n,Z^n,K^n) and $(\tilde Y^n,\tilde Z^n,\tilde K^n)$ are two solutions of (3.2). By Itô's formula,

$$\exp\big(\lambda(t\wedge\tau^n)\big)|Y^n_{t\wedge\tau^n}-\tilde Y^n_{t\wedge\tau^n}|^2+\lambda\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda s)|Y^n_{s-}-\tilde Y^n_{s-}|^2d\varrho^n_s$$
$$+\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)d[Y^n-\tilde Y^n]_s$$
$$\leqslant(2\mu+L^2/\varepsilon)\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)|Y^n_{s-}-\tilde Y^n_{s-}|^2d\varrho^n_s+\varepsilon\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)\|Z^n_{s-}-\tilde Z^n_{s-}\|^2d\varrho^n_s$$
$$-2\int\limits_{t\wedge\tau^n}^{\tau^n}\exp(\lambda\varrho^n_s)(Y^n_{s-}-\tilde Y^n_{s-})(Z^n_{s-}-\tilde Z^n_{s-})dW^n_s.$$

Now, we choose $\varepsilon<1$ such that $2\mu+L^2/\varepsilon<\lambda$. Integrating the above inequality, by (5.2) we prove the lemma. ∎

Proof of Theorem 2.1. (a) Note that (b) implies (a) easily. By the Skorokhod representation theorem there exists a probability space $(\tilde\Omega,\tilde{\mathcal G},\tilde{\mathcal P})$ with a Wiener process $\tilde W$ and Bernoulli symmetric sequences $\{\tilde\varepsilon^n_j\}_{j\in\mathbb N}$ such that

$$\sup_{t\leqslant T}|\tilde W^n_t-\tilde W_t|\xrightarrow{\mathcal P}0,$$

where $\tilde W^n_t=n^{-1/2}\sum_{j=1}^{[nt]}\tilde\varepsilon^n_j,\ t\in\mathbb R^+$. Then, by part (b), as $n\to\infty$, we get

$$\tilde E\big(\sup_{t\leqslant T}|\tilde Y^n_t-\tilde Y_t|^2+\int\limits_0^T\|\tilde Z^n_{t-}-\tilde Z_t\|^2dt+\sup_{t\leqslant T}|\tilde K^n_t-\tilde K_t|^2\big)\to 0.$$

Since $\mathcal{L}\big(\tilde{X}^n, \tilde{Y}^n, \int_0^\cdot \tilde{Z}^n_{s-} d\tilde{W}^n_s, \tilde{K}^n, \tilde{W}^n\big) = \mathcal{L}\big(X^n, Y^n, \int_0^\cdot Z^n_{s-} dW^n_s, K^n, W^n\big)$ and $\mathcal{L}\big(\tilde{X}, \tilde{Y}, \int_0^\cdot \tilde{Z}_s d\tilde{W}_s, \tilde{K}, \tilde{W}\big) = \mathcal{L}\big(X, Y, \int_0^\cdot Z_s dW_s, K, W\big)$, part (a) easily follows.

(b) We will follow the proof of Theorem 2.1 in [2]. Consider the decompositions:

$$
\begin{aligned}
Y^n - Y &= (Y^n - Y^{n,(q)}) + (Y^{n,(q)} - Y^{(q)}) + (Y^{(q)} - Y),\\
Z^n - Z &= (Z^n - Z^{n,(q)}) + (Z^{n,(q)} - Z^{(q)}) + (Z^{(q)} - Z),\\
K^n - K &= (K^n - K^{n,(q)}) + (K^{n,(q)} - K^{(q)}) + (K^{(q)} - K),
\end{aligned}
\tag{5.6}
$$

where the superscript (q) stands for the approximation of the solution to the RBSDE by the Picard method. More precisely, set $Y_t^{(0)} = E\big(g(X_T)|\mathcal{F}_t\big)$, $Y_t^{n,(0)} = E\big(g(X^n_T)|\mathcal{F}^n_t\big)$, $K_t^{(0)} = K_t^{n,(0)} = 0$,

$$
\begin{aligned}
\int_t^T Z_s^{(0)} dW_s &= g(X_T) - E\big(g(X_T)|\mathcal{F}_t\big),\\
\int_t^T Z_{s-}^{n,(0)} dW^n_s &= g(X^n_T) - E\big(g(X^n_T)|\mathcal{F}^n_t\big),
\end{aligned}
$$

and then define $(Y^{(q+1)}, Z^{(q+1)}, K^{(q+1)})$ and $(Y^{n,(q+1)}, Z^{n,(q+1)}, K^{n,(q+1)})$ as solutions of the equations

$$
\begin{aligned}
Y_t^{(q+1)} = g(X_T) &+ \int_t^T f(s, X_s, Y_s^{(q)}, Z_s^{(q)}) ds\\
&- \int_t^T Z_s^{(q+1)} dW_s + K_T^{(q+1)} - K_t^{(q+1)}
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
Y_t^{n,(q+1)} = g(X^n_T) &+ \int_t^T f(\varrho^n_{s-}, X^n_{s-}, Y^{n,(q)}_{s-}, Z^{n,(q)}_{s-}) d\varrho^n_s\\
&- \int_t^T Z_{s-}^{n,(q+1)} dW^n_s + K_T^{n,(q+1)} - K_t^{n,(q+1)},
\end{aligned}
\tag{5.8}
$$

respectively. Norms $\|\cdot\|_\gamma$ and $\|\cdot\|_0$ are equivalent, so by Lemma 5.2 it follows that

$$
E\int_0^T (|Y_t^{(q)} - Y_t|^2 + \|Z_t^{(q)} - Z_t\|^2) dt \to 0
$$

and

$$\sup_{n\in\mathbb{N}} E\int_0^T (|Y_{t-}^{n,(q)} - Y_{t-}^n|^2 + \|Z_{t-}^{n,(q)} - Z_{t-}^n\|^2)d\varrho_t^n \to 0$$

as $q \to \infty$. Since, by the Burkholder–Davis–Gundy inequality,

$$\begin{aligned} E\sup_{t\leqslant T}|Y_t^{(q)} - Y_t|^2 \leqslant CE\int_0^T (|Y_t^{(q)} - Y_t|^2 + \|Z_t^{(q)} - Z_t\|^2)dt \\ + E\int_0^T (|Y_t^{(q-1)} - Y_t|^2 + \|Z_t^{(q-1)} - Z_t\|^2)dt \end{aligned}$$

and

$$\begin{aligned} E\sup_{t\leqslant T}|Y_t^{n,(q)} - Y_t^n|^2 \leqslant CE\int_0^T (|Y_{t-}^{n,(q)} - Y_{t-}^n|^2 + \|Z_{t-}^{n,(q)} - Z_{t-}^n\|^2)d\varrho_t^n \\ + E\int_0^T (|Y_{t-}^{n,(q-1)} - Y_{t-}^n|^2 + \|Z_{t-}^{n,(q-1)} - Z_{t-}^n\|^2)d\varrho_t^n, \end{aligned}$$

we have

$$E\sup_{t\leqslant T}|Y_t^{(q)} - Y_t|^2 + \sup_{n\in\mathbb{N}} E\sup_{t\leqslant T}|Y_t^{n,(q)} - Y_t^n|^2 \to 0 \quad \text{as } q \to \infty,$$

which implies also that

$$E\sup_{t\leqslant T}|K_t^{(q)} - K_t|^2 \to 0 \quad \text{and} \quad \sup_{n\in\mathbb{N}} E\sup_{t\leqslant T}|K_t^{n,(q)} - K_t^n|^2 \to 0$$

as $q \to \infty$. Hence the first and the third components of the decompositions (5.6) converge to zero. The convergence of the second term will be shown by induction on q. Let $q=0$. Recall that $K^{n,(0)} = K^{(0)} = 0$. Since $\{W^n\}$ is a sequence of processes with independent increments such that $\sup_{t\leqslant T}|W_t^n - W_t| \xrightarrow{\mathcal{P}} 0$, we obtain

$$\sup_{t\leqslant T}|E(H|\mathcal{F}_t^n) - E(H|\mathcal{F}_t)| \xrightarrow{\mathcal{P}} 0 \tag{5.9}$$

for every $\mathcal{F}_T$-measurable integrable random variable H (see, e.g., [6], Proposition 2 and Remark 1). From (5.9) and the maximal Doob inequality it follows that if $E|H|^p < \infty$, $p \in \mathbb{N}$, then

$$E\sup_{t\leqslant T}|E(H|\mathcal{F}_t^n) - E(H|\mathcal{F}_t)|^p \to 0, \quad p \in \mathbb{N}. \tag{5.10}$$

Hence, in particular, $E\sup_{t\leqslant T}|Y_t^{n,(0)} - Y_t^{(0)}|^2$ tends to zero and

$$E\int_0^T \|Z_{s-}^{n,(0)} - Z_s^{(0)}\|^2 ds \to 0.$$

Now, assume that the convergence holds for fixed q. We will prove it for $q+1$. To simplify the notation we drop the superscript (q), so that equations (5.7) and (5.8) become

$$Y_t = g(X_T) + \int_t^T f(s, X_s, U_s, V_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \quad t \in [0,T],$$

and

$$Y_t^n = g(X_T^n) + \int_t^T f(\varrho_{s-}^n, X_{s-}^n, U_{s-}^n, V_{s-}^n)d\varrho_s^n - \int_t^T Z_{s-}^n dW_s^n + K_T^n - K_t^n,$$

respectively, where

$$(U,V) = (Y^{(q)}, Z^{(q)}), \quad (Y,Z) = (Y^{(q+1)}, Z^{(q+1)}),$$
$$(U^n, V^n) = (Y^{n,(q)}, Z^{n,(q)}), \quad (Y^n, Z^n) = (Y^{n,(q+1)}, Z^{n,(q+1)}).$$

By assumption, (U^n, V^n) converges to (U,V). We have to prove that (Y^n, Z^n, K^n) converges to (Y, Z, K). We begin by studying the convergence of K_T^n to K_T. By Itô's formula,

$$\begin{aligned}
|K_T - K_T^n|^2 + \sum_{t \leqslant T} |\Delta K_t^n|^2 &= 2\int_0^T \big(K_T - K_t - (K_T^n - K_{t-}^n)\big)d(K_t - K_t^n) \\
= 2\int_0^T (Y_t - Y_{t-}^n)d(K_t - K_t^n) &- 2\int_0^T \big(g(X_T) - g(X_T^n)\big)d(K_t - K_t^n) \\
- 2\int_0^T \Big(\int_t^T f(s, X_s, U_s, V_s)ds &- \int_{t-}^T f(\varrho_{s-}^n, X_{s-}^n, U_{s-}^n, V_{s-}^n)d\varrho_s^n\Big)d(K_t - K_t^n) \\
+ 2\int_0^T \Big(\int_t^T Z_s dW_s &- \int_{t-}^T Z_{s-}^n dW_s^n\Big)d(K_t - K_t^n).
\end{aligned}$$

Due to (2.7) the first term on the right-hand side of the above equality is less than or equal to zero. On the other hand, by Lemma 5.1, $\sup_n E|K^n|_T^p < \infty,\ p \in \mathbb{N}$, and by the arguments from the proof of Lemma 5.2 in [8] (pp. 122 and 126) $E|K|_T^p < \infty$. Therefore, the convergence of (X^n, U^n, V^n) to (X, U, V) implies that the second and the third terms converge to zero in probability. What is left is to show that the fourth component of the above equality converges to zero. To simplify the notation let us put $N_t = \int_0^t Z_s dW_s$ and $N_t^n = \int_0^t Z_{s-}^n dW_s^n$. Now the fourth component is equal to

$$\begin{aligned}
\int_0^T (N_T - N_t)dK_t + \int_0^T (N_T^n - N_{t-}^n)dK_t^n - \int_0^T (N_T - N_t)dK_t^n \\
- \int_0^T (N_T^n - N_{t-}^n)dK_t = I_1 + I_2 - I_3 - I_4.
\end{aligned}$$

Clearly, $E|N_T|^p+\sup_n E|N^n_T|^p<\infty, p\in\mathbb{N}$. Since N is a continuous martingale with respect to $\mathcal{F}$ and K is a continuous process of finite variation,

$$EI_1=EN_TK_T-E\int_0^T N_t\,dK_t=EN_TK_T-E\big(N_TK_T-\int_0^T K_t\,dN_t\big)=0.$$

Similarly, since N^n is a martingale with respect to $\mathcal{F}^n$ and K^n is a process of finite variation,

$$EI_2=EN^n_TK^n_T-E\big(N^n_TK^n_T-\int_0^T K^n_{t-}dN^n_t-\sum_{t\leqslant T}\Delta N^n_t\Delta K^n_t\big)=0.$$

To estimate EI_3 set $B^n_t=E(N_T|\mathcal{F}^n_t)$. Since B^n is an $\mathcal{F}^n$ martingale,

$$\begin{aligned}
E\int_0^T E(N_T|\mathcal{F}^n_t)dK^n_t&=E\int_0^T B^n_t dK^n_t=E\sum_{t\leqslant T}\Delta B^n_t\Delta K^n_t+E\int_0^T B^n_{t-}dK^n_t\\
&=E\sum_{t\leqslant T}\Delta B^n_t\Delta K^n_t+\,E\big(B^n_TK^n_T-\int_0^T K^n_{t-}dB^n_t-\sum_{t\leqslant T}\Delta B^n_t\Delta K^n_t\big)\\
&=EB^n_TK^n_T=EN_TK^n_T=E\int_0^T N_TdK^n_t.
\end{aligned}$$

By the above and (5.10),

$$\begin{aligned}
EI_3&=E\int_0^T(N_T-N_t)dK^n_t=E\int_0^T\big(E(N_T|\mathcal{F}^n_t)-N_t\big)dK^n_t\\
&=E\int_0^T\big(E(N_T|\mathcal{F}^n_t)-E(N_T|\mathcal{F}_t)\big)dK^n_t\\
&\leqslant\big(E\sup_{t\leqslant T}|E(N_T|\mathcal{F}^n_t)-E(N_T|\mathcal{F}_t)|^2\big)^{1/2}(E|K^n|^2_T)^{1/2}\to 0
\end{aligned}$$

as $n\to\infty$. It remains to prove that $EI_4\to 0$. Let us write $K^{\varrho^n}_t=K_{j/n}$ for $t\in\big[j/n,(j+1)/n\big)$ and $G^n_t=E(K^{\varrho^n}_t|\mathcal{F}^n_t)$. Since N^n_t is $\mathcal{F}^n_t$-measurable,

$$\begin{aligned}
EN^n_{(j-1)/n}K_{j/n}&=EN^n_{(j-1)/n}G^n_{j/n},\\
EN^n_{(j-1)/n}K_{(j-1)/n}&=EN^n_{(j-1)/n}G^n_{(j-1)/n},
\end{aligned}$$

and, as a consequence,

$$
\begin{aligned}
E\int_0^T N^n_{t-}dK_t &= E\sum_{j=1}^{[nT]} N^n_{(j-1)/n}(G^n_{j/n}-G^n_{(j-1)/n}) = E\int_0^T N^n_{t-}dG^n_t\\
&= E\big(N^n_T G^n_T - \int_0^T G^n_{t-}dN^n_t - \sum_{j=1}^{[nT]}\Delta G^n_{j/n}\Delta N^n_{j/n}\big)\\
&= EN^n_T G^n_T - E\sum_{j=1}^{[nT]}\Delta G^n_{j/n}\,\Delta N^n_{j/n}.
\end{aligned}
$$

Since $EN^n_T G^n_T = EN^n_T E(K_T|\mathcal{F}^n_T) = EN^n_T K_T$,

$$
\begin{aligned}
EI_4 &= E\sum_{j=1}^{[nT]}\Delta G^n_{j/n}\,\Delta N^n_{j/n}\\
&\leqslant \big(E\sum_{j=1}^{[nT]}|N^n_{j/n}-N^n_{(j-1)/n}|^2\big)^{1/2}\big(E\sum_{j=1}^{[nT]}|G^n_{j/n}-G^n_{(j-1)/n}|^2\big)^{1/2}\\
&\leqslant (\sup_n E[N^n]_T)^{1/2}(E[G^n]_T)^{1/2}.
\end{aligned}
$$

The first term on the right-hand side of the above inequality is bounded, so we need to prove that the second one converges to zero. Since K^{ϱ^n} is a process of finite variation, it can be decomposed into a difference of two increasing processes $K^{\varrho^n} = K^{\varrho^n+} - K^{\varrho^n-}$ such that $K^{\varrho^n+}_T \leqslant |K^{\varrho^n}|_T$ and $K^{\varrho^n-}_T \leqslant |K^{\varrho^n}|_T$. Moreover, the processes $G^{n+}_t = E(K^{\varrho^n+}_t|\mathcal{F}^n_t)$, $G^{n-}_t = E(K^{\varrho^n-}_t|\mathcal{F}^n_t), t\in[0,T]$, are submartingales, so from [6], p. 317, and by Lemma 5.1 in Section 5 it follows that for any $p\in\mathbb{N}$

$$
\begin{aligned}
\sup_n \|G^n\|_{H^p(S)} &\leqslant \sup_n \|G^{n+}\|_{H^p(S)} + \sup_n \|G^{n-}\|_{H^p(S)}\\
&\leqslant C_p\sup_n(E\sup_{t\leqslant T}|G^{n+}_t|^p)^{1/p} + C_p\sup_n(E\sup_{t\leqslant T}|G^{n-}_t|^p)^{1/p}\\
&\leqslant 2C_p\sup_n(E|K^{\varrho^n}|^p_T)^{1/p} \leqslant 2C_p(E|K|^p_T)^{1/p} < \infty,
\end{aligned}
$$

where $\|\cdot\|_{H^p(S)}$ means the norm of a special semimartingale X with a canonical decomposition $X = M + A$ which is defined by $\|X\|_{H^p(S)} = \|[M]^{1/2}_T\|_{\mathbb{L}^p} + \||A|_T\|_{\mathbb{L}^p}$. Therefore, G^n satisfies the (UT) condition considered in [11]. Moreover $\sup_{t\leqslant T}|G^n_t - K_t|\xrightarrow{\mathcal{P}}0$. To see this we first note that from Theorem 1 and the Comment on p. 319 in [6] we have

$$
\sup_{t\leqslant T}|E(K_t|\mathcal{F}^n_t) - K_t|\xrightarrow{\mathcal{P}}0.
$$

Next, by Doob's inequality,

$$E\sup_{t\leqslant T}|G^n_t-E(K_t|\mathcal{F}^n_t)|^2\leqslant E\sup_{t\leqslant T}\big|E(\sup_{t\leqslant T}|K^{\varrho^n}_t-K_t|\,|\mathcal{F}^n_t)\big|^2\leqslant 4E\sup_{t\leqslant T}|K^{\varrho^n}_t-K_t|^2.$$

Since K is a continuous process, $G^n\xrightarrow{\mathcal{P}}K$. Hence, by Theorem 1.4 in [9] (see also [11]) we deduce that also $\sum_{j=1}^{[nT]}|\Delta G^n_{j/n}|^2=[G^n]_T\xrightarrow{\mathcal{P}}[K]_T=0$. Moreover, since $E[G^n]^2_T\leqslant C\|G^n\|_{H^4(S)}\leqslant C(E|K^{\varrho^n}|^4_T)^{1/4}<\infty$, we obtain $E[G^n]_T\to 0$. This implies that $E|K^n_T-K_T|^2\to 0$.

In order to complete the proof notice that the process

$$M^n_t=Y^n_t+\int_0^t f(\varrho^n_{s-},X^n_{s-},U^n_{s-},V^n_{s-})d\varrho^n_s+K^n_t$$

is an $\mathcal{F}^n$ martingale, which satisfies

$$M^n_t=M^n_0+\int_0^t Z^n_{s-}dW^n_s. \tag{5.11}$$

Set $M_t=Y_t+\int_0^t f(s,X_s,U_s,V_s)ds+K_t$. Then we have

$$\begin{aligned}|M^n_T-M_T|&=\big|M^n_T-Y_T-\int_0^T f(s,X_s,U_s,V_s)ds-K_T\big|\\&\leqslant|g(X^n_T)-g(X_T)|+|K^n_T-K_T|\\&\quad+\big|\int_0^T f(\varrho^n_{s-},X^n_{s-},U^n_{s-},V^n_{s-})d\varrho^n_s-\int_0^T f(s,X_s,U_s,V_s)ds\big|.\end{aligned}$$

Since we have already known that $K^n_T\to K_T$ in $\mathbb{L}^2$, from our assumptions on X^n, U^n, V^n it follows that $M^n_T\to M_T$ in $\mathbb{L}^2$. Hence, by (5.10) we deduce that $E\sup_{t\leqslant T}|M^n_t-M_t|^2\to 0$ and, in particular, $Y^n_0\to Y_0$. By (5.11) and by Theorem 3.1 in [2],

$$E\int_0^T\|Z^n_{t-}-Z_t\|^2dt\to 0,$$

where $M_t=M_0+\int_0^t Z_s dW_s$. Thus, $E\sup_{t\leqslant T}|Y^n_t-Y_t+K^n_t-K_t|^2\to 0$. Now, set $H^n_t=g(X^n_T)+\int_t^T f(\varrho^n_{s-},X^n_{s-},U^n_{s-},V^n_{s-})d\varrho^n_s-\int_t^T Z^n_{s-}dW^n_s$ and $H_t=g(X_T)+\int_t^T f(s,X_s,U_s,V_s)ds-\int_t^T Z_s dW_s$. Since we have shown that

$$\sup_{t\leqslant T}|H^n_t-H_t|\xrightarrow{\mathcal{P}}0\quad\text{and}\quad E|K|_T+\sup_n E|K^n|_T<\infty,$$

by Lemma 5.3 it follows that $\sup_{t\leqslant T}|Y^n_t-Y_t|\xrightarrow{\mathcal{P}}0$. Hence $\sup_{t\leqslant T}|K^n_t-K_t|\xrightarrow{\mathcal{P}}0$.

Moreover, by Lemma 5.1, $E\sup_{t\leqslant T}|Y_t|^4+\sup_n E\sup_{t\leqslant T}|Y_t^n|^4<\infty$, which implies that

$$E\sup_{t\leqslant T}(|Y_t^n-Y_t|^2+|K_t^n-K_t|^2)\to 0$$

and the proof is complete. ∎

P r o o f o f P r o p o s i t i o n 2.1. Similarly to the proof of Lemma 5.1 one can show that

$$\sup_n E\big(\sup_{t\leqslant T}|\hat{Y}_t^n-a|^2+\int_0^T\|\hat{Z}_{t-}^n\|^2d\varrho_t^n+\sup_{t\leqslant T}|\hat{K}_t^n|^2\big)<\infty. \tag{5.12}$$

By Itô's formula and (2.8),

$$\begin{aligned}\exp(\gamma\varrho_t^n)|Y_t^n-\hat{Y}_t^n|^2&+\gamma\int_t^T\exp(\gamma s)|Y_{s-}^n-\hat{Y}_{s-}^n|^2ds\\&+\int_t^T\exp(\gamma\varrho_s^n)d[Y^n-\hat{Y}^n]_s\\&\leqslant 2\int_t^T\exp(\gamma\varrho_s^n)(Y_{s-}^n-\hat{Y}_{s-}^n)\big(f(\varrho_{s-}^n,X_{s-}^n,Y_{s-}^n,Z_{s-}^n)\\&\qquad\qquad-f(\varrho_{s-}^n,X_{s-}^n,\bar{Y}_{s-}^n,\hat{Z}_{s-}^n)\big)d\varrho_s^n\\&+2\int_t^T\exp(\gamma\varrho_s^n)(Y_{s-}^n-\hat{Y}_{s-}^n)d(K_s^n-\hat{K}_s^n)\\&-2\int_t^T\exp(\gamma\varrho_s^n)(Y_{s-}^n-\hat{Y}_{s-}^n)(Z_{s-}^n-\hat{Z}_{s-}^n)dW_s^n\\&\leqslant 4L^2\int_t^T\exp(\gamma\varrho_s^n)|Y_{s-}^n-\hat{Y}_{s-}^n|^2d\varrho_s^n\\&+\frac{1}{2}\int_t^T\exp(\gamma\varrho_s^n)(|Y_{s-}^n-\bar{Y}_{s-}^n|^2+\|Z_{s-}^n-\hat{Z}_{s-}^n\|^2)d\varrho_s^n\\&-2\int_t^T\exp(\gamma\varrho_s^n)(Y_{s-}^n-\hat{Y}_{s-}^n)(Z_{s-}^n-\hat{Z}_{s-}^n)dW_s^n.\end{aligned} \tag{5.13}$$

Note that for $s=j/n,\ j\in\mathbb{N}$,

$$\begin{aligned}E|Y_{s-}^n-\bar{Y}_{s-}^n|^2&\leqslant 2E|Y_{s-}^n-\hat{Y}_{s-}^n|^2+2E|\hat{Y}_{s-}^n-\bar{Y}_{s-}^n|^2\\&\leqslant 2E|Y_{s-}^n-\hat{Y}_{s-}^n|^2+\frac{2}{n^2}E|f(\varrho_{s-}^n,X_{s-}^n,\bar{Y}_{s-}^n,\hat{Z}_{s-}^n)|^2,\end{aligned}$$

where the last inequality follows from the fact that

$$\hat{Y}_{s-}^n=\pi\bigg(\bar{Y}_{s-}^n+\frac{1}{n}f(\varrho_{s-}^n,X_{s-}^n,\bar{Y}_{s-}^n,\hat{Z}_{s-}^n)\bigg).$$

Moreover, $E|\bar{Y}^n_{s-}-a|^2=E|E(\hat{Y}^n_s-a|\mathcal{F}^n_{s-})|^2\leqslant E|\hat{Y}^n_s-a|^2$. Therefore, setting $\gamma=4L^2+1$ and integrating (5.13) we have

$$\begin{aligned}
&E\exp(\gamma\varrho^n_t)|Y^n_t-\hat{Y}^n_t|^2+\frac{1}{2}E\int_t^T\exp(\gamma\varrho^n_s)\|Z^n_{s-}-\hat{Z}^n_{s-}\|^2d\varrho^n_s\\
&\quad\leqslant\frac{1}{n^2}E\int_t^T\exp(\gamma\varrho^n_s)|f(\varrho^n_{s-},X^n_{s-},\bar{Y}^n_{s-},\hat{Z}^n_{s-})|^2d\varrho^n_s\\
&\quad\leqslant\frac{C}{n^2}E\int_t^T\exp(\gamma\varrho^n_s)\big(|\bar{Y}^n_{s-}-a|^2+\|\hat{Z}^n_{s-}\|^2+|f(\varrho^n_{s-},X^n_{s-},a,0)|^2\big)d\varrho^n_s\\
&\quad\leqslant\frac{C}{n^2}E\int_t^T\exp(\gamma\varrho^n_s)\big(|\hat{Y}^n_s-a|^2+\|\hat{Z}^n_{s-}\|^2+|f(\varrho^n_{s-},X^n_{s-},a,0)|^2\big)d\varrho^n_s.
\end{aligned}$$

By (5.12) it is obvious that the right-hand side of the above inequality tends to zero. Moreover, since

$$\begin{aligned}
&2E\sup_{t\leqslant T}\Big|\int_t^T\exp(\gamma\varrho^n_s)(Y^n_{s-}-\hat{Y}^n_{s-})(Z^n_{s-}-\hat{Z}^n_{s-})dW^n_s\Big|\\
&\qquad\leqslant CE\Big(\int_0^T\exp(2\gamma\varrho^n_s)|Y^n_{s-}-\hat{Y}^n_{s-}|^2\|Z^n_{s-}-\hat{Z}^n_{s-}\|^2d\varrho^n_s\Big)^{1/2}\\
&\qquad\leqslant\frac{1}{2}E\sup_{t\leqslant T}\exp(\gamma t)|Y^n_t-\hat{Y}^n_t|^2+CE\int_0^T\exp(\gamma\varrho^n_t)\|Z^n_{t-}-\hat{Z}^n_{t-}\|^2d\varrho^n_t,
\end{aligned}$$

in the same manner as before one can prove that

$$E\sup_{t\leqslant T}\exp(\gamma\varrho^n_t)|Y^n_t-\hat{Y}^n_t|^2\to 0.$$

It implies also that $E\sup_{t\leqslant T}|K^n_t-\hat{K}^n_t|^2\to 0$. ∎

P r o o f o f T h e o r e m 3.1. (a) Note that (b) implies (a) similarly to the proof of Theorem 2.1. To see this it is sufficient to use the arguments from the proof of Theorem 2.1 and to observe that if $\mathcal{L}(\tilde{\tau},\tilde{W})=\mathcal{L}(\tau,W)$ (respectively, $\mathcal{L}(\tilde{\tau}^n,\ \tilde{W}^n)=\mathcal{L}(\tau^n,\ W^n)$), then $\tilde{\tau}$ is a stopping time with respect to the natural filtration generated by a Wiener process $\tilde{W}$ (respectively, $\tilde{\tau}^n$ is a stopping time with respect to the natural filtration generated by a Wiener process $\tilde{W}^n$) (see e.g. [20], lemme 1.1.21).

(b) S t e p 1. Let $M\in\mathbb{N}$ and let $\xi_M=E\big(g(X_\tau)|\mathcal{F}_M\big)$. Since ξ_M is $\mathcal{F}_M$-measurable and f satisfies (i) and (ii) it follows by results of [8] that there exists a unique solution $(Y^M_t,Z^M_t,K^M_t)_{t\in[0,M]}$ of the following RBSDE:

$$Y^M_t=\xi_M+\int_t^M\mathbf{1}_{[0,\tau]}(s)f(s,X_s,Y^M_s,Z^M_s)ds-\int_t^M Z^M_s dW_s+K^M_M-K^M_t \tag{5.14}$$

for $t \in [0, M]$. Notice that on the set $\{t \geqslant \tau\}$ we have $\xi_M = g(X_\tau) = Y_t^M$ and $Z_t^M = 0$, so (5.14) can be rewritten as

$$Y_t^M = \xi_M + \int_{t\wedge\tau}^{M\wedge\tau} f(s, X_s, Y_s^M, Z_s^M)ds - \int_{t\wedge\tau}^{M\wedge\tau} Z_s^M dW_s + K_{M\wedge\tau}^M - K_{t\wedge\tau}^M.$$

Moreover, note that

$$Y_t = Y_{M\wedge\tau} + \int_{t\wedge\tau}^{M\wedge\tau} f(s, X_s, Y_s, Z_s)ds - \int_{t\wedge\tau}^{M\wedge\tau} Z_s dW_s + K_{M\wedge\tau} - K_{t\wedge\tau}, \quad t \in [0, M].$$

Now, similarly to the proof of Theorem 3.1 in [15], using Itô's formula we have for $t \in [0, M]$

$$\begin{aligned}
&\exp\bigl(\lambda(t\wedge\tau)\bigr)|Y_{t\wedge\tau} - Y_{t\wedge\tau}^M|^2 + \int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)(\lambda|Y_s - Y_s^M|^2 + \|Z_s - Z_s^M\|^2)ds \\
&= \exp\bigl(\lambda(M\wedge\tau)\bigr)|Y_{M\wedge\tau} - Y_{M\wedge\tau}^M|^2 \\
&\quad + 2\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)(Y_s - Y_s^M)\bigl(f(s, X_s, Y_s, Z_s) - f(s, X_s, Y_s^M, Z_s^M)\bigr)ds \\
&\quad - 2\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)(Y_s - Y_s^M)(Z_s - Z_s^M)dW_s \\
&\quad + 2\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)(Y_s - Y_s^M)d(K_s - K_s^M) \\
&\leqslant \exp\bigl(\lambda(M\wedge\tau)\bigr)|Y_{M\wedge\tau} - \xi_M|^2 + (2\mu + L^2/\varepsilon)\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)|Y_s - Y_s^M|^2 ds \\
&\quad + \varepsilon\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)\|Z_s - Z_s^M\|^2 ds - 2\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)(Y_s - Y_s^M)(Z_s - Z_s^M)dW_s.
\end{aligned}$$

Choosing ε such that $\varepsilon < 1$ and $2\mu + L^2/\varepsilon < \lambda$ we get

$$\begin{aligned}
(5.15)\quad E\exp\bigl(\lambda(t\wedge\tau)\bigr)|Y_{t\wedge\tau} - Y_{t\wedge\tau}^M|^2 + E\int_{t\wedge\tau}^{M\wedge\tau} \exp(\lambda s)\|Z_s - Z_s^M\|^2 ds \\
\leqslant CE\exp\bigl(\lambda(M\wedge\tau)\bigr)|Y_{M\wedge\tau} - \xi_M|^2.
\end{aligned}$$

Analogously, choosing $\varepsilon < 1$ such that $2\mu + L^2/\varepsilon = \lambda' < \lambda$ we get (5.15) with λ' in place of λ. Since

$$\begin{aligned}
E\exp\bigl(\lambda(M\wedge\tau)\bigr)|Y_{M\wedge\tau} - \xi_M|^2 \\
\leqslant 2E\sup_{t\leqslant\tau}\exp(\lambda t)|Y_t|^2 + 2E\exp(\lambda\tau)|g(X_\tau)|^2 < \infty
\end{aligned}$$

and

$$E\exp\big(\lambda'(M\wedge\tau)\big)|Y_{M\wedge\tau}-\xi_M|^2 \\ \leqslant \exp\big((\lambda'-\lambda)M\big)E\exp\big(\lambda(M\wedge\tau)\big)|Y_{M\wedge\tau}-\xi_M|^2,$$

we have

$$E\exp\big(\lambda'(t\wedge\tau)\big)|Y_{t\wedge\tau}-Y^M_{t\wedge\tau}|^2+E\int_{t\wedge\tau}^{M\wedge\tau}\exp(\lambda's)\|Z_s-Z^M_s\|^2ds\to 0$$

as $M\to\infty$. Moreover, by the Burkholder–Davis–Gundy inequality,

$$E\sup_{t\leqslant M}\exp\big(\lambda'(t\wedge\tau)\big)|Y_{t\wedge\tau}-Y^M_{t\wedge\tau}|^2\to 0$$

and, as a consequence,

$$E\sup_{t\leqslant M}|K_{t\wedge\tau}-K^M_{t\wedge\tau}|^2\to 0 \quad \text{as } M\to\infty.$$

Step 2. Let $\xi^n_M=E\big(g(X^n_{\tau^n})|\mathcal{F}^n_M\big)$. For $j=nM$ let us put $y^{n,M}_{\tau^n_j}=\xi^n_M=E\big(g(X^n_{\tau^n})|\mathcal{F}^n_M\big)$, $z^{n,M}_{\tau^n_j}=0$, $\Delta k^{n,M}_{\tau^n_{j+1}}=0$ and, as in Section 3, solve

$$y^{n,M}_{\tau^n_j}=y^{n,M}_{\tau^n_{j+1}}+\frac{1}{n}f(j/n,x^n_{\tau^n_j},y^{n,M}_{\tau^n_j},z^{n,M}_{\tau^n_j})\mathbf{1}_{(\tau^n>j/n)}-\frac{1}{\sqrt{n}}z^{n,M}_{\tau^n_j}\varepsilon^n_{j+1}+\Delta k^{n,M}_{\tau^n_{j+1}}$$

for $j=nM-1,\dots,0$. Defining $Y^{n,M}_t=y^{n,M}_{\tau^n_{[nt]}}$, $Z^{n,M}_t=z^{n,M}_{\tau^n_{[nt]}}$, and $K^{n,M}_t=\sum_{j\leqslant[nt]}\Delta k^{n,M}_{\tau^n_j}$ we see that

$$Y^{n,M}_{t\wedge\tau^n}=\xi^n_M+\int_{t\wedge\tau^n}^{M\wedge\tau^n}f(\varrho^n_{s-},X^n_{s-},Y^{n,M}_{s-},Z^{n,M}_{s-})d\varrho^n_s \\ -\int_{t\wedge\tau^n}^{M}Z^{n,M}_{s-}dW^n_s+K^{n,M}_{M\wedge\tau^n}-K^{n,M}_{t\wedge\tau^n},\quad t\in[0,M]. \tag{5.16}$$

By (3.2) we have

$$Y^n_{t\wedge\tau^n}=Y^n_{M\wedge\tau^n}+\int_{t\wedge\tau^n}^{M\wedge\tau^n}f(\varrho^n_{s-},X^n_{s-},Y^n_{s-},Z^n_{s-})d\varrho^n_s \\ -\int_{t\wedge\tau^n}^{M\wedge\tau^n}Z^n_{s-}dW^n_s+K^n_{M\wedge\tau^n}-K^n_{t\wedge\tau^n},\quad t\in[0,M].$$

Therefore, using the fact that $E\int_0^{t\wedge\tau^n}\|Z^n_{s-}-Z^{n,M}_{s-}\|^2d\varrho^n_s \leqslant E[Y^n-Y^{n,M}]_{t\wedge\tau^n}$ and

$$\begin{aligned}&\sup_n\ E\exp\big(\lambda(M\wedge\tau^n)\big)|Y^n_{M\wedge\tau^n}-\xi^{n,M}|^2\\&\leqslant 2\sup_n E\Big(\exp\big(\lambda(M\wedge\tau^n)\big)|Y^n_{M\wedge\tau^n}|^2+\exp\big(\lambda(M\wedge\tau^n)\big)\big|E\big(g(X^n_{\tau^n})|\mathcal{F}^n_M\big)\big|^2\Big)\\&\leqslant 2\sup_n E\sup_{t\leqslant\tau^n}\exp(\lambda t)|Y^n_t|^2+2\sup_n E\exp(\lambda\tau^n)|g(X^n_{\tau^n})|^2<\infty\end{aligned}$$

(see Lemma 5.4 (a)) and arguing as in Step 1 we conclude that

$$\begin{aligned}\sup_n E\big(&\sup_{t\leqslant M}\exp(\lambda' t\wedge\tau^n)|Y^n_{t\wedge\tau^n}-Y^{n,M}_{t\wedge\tau^n}|^2\\&+\int_{t\wedge\tau^n}^{M\wedge\tau^n}\exp(\lambda' s)\|Z^n_{s-}-Z^{n,M}_{s-}\|^2d\varrho^n_s+\sup_{t\leqslant M}|K^n_{t\wedge\tau^n}-K^{n,M}_{t\wedge\tau^n}|^2\big)\to 0\end{aligned}$$

as $M\to\infty$.

S t e p 3. By the same method as in the proof of Theorem 2.1 (b) we show that for any $M\in\mathbb{N}$

$$\big(\sup_{t\leqslant M}|Y^{n,M}_{t\wedge\tau^n}-Y^M_{t\wedge\tau}|+\int_0^M\|Z^{n,M}_{t-\wedge\tau^n}-Z^M_{t\wedge\tau}\|^2dt+\sup_{t\leqslant M}|K^{n,M}_{t\wedge\tau^n}-K^M_{t\wedge\tau}|\big)\xrightarrow{\mathcal{P}}0.$$

Combing this with results from Steps 1 and 2 completes the proof. ∎

P r o o f o f P r o p o s i t i o n 3.1. The proof runs similarly to the proof of Proposition 2.1, so we omit it. ∎

P r o o f o f E x a m p l e 3.1. Obviously, it suffices to consider the case where $\lambda>0$. It is known (see [5], [20]) that $P(W_t\in[-A,A])\leqslant 2A/\sqrt{2\pi t}$ for any $A>0$ and $P^x(\tau^n>t)\leqslant P(|W^n_t+x|\leqslant a)=P(W^n_t\in[-a-x,a-x])$, where P^x is such that $P^x(W^n_0=x)=1$. Since

$$\sup_{x\in[-a,a]}|P(W^n_t\in[-a-x,a-x])-P(W_t\in[-a-x,a-x])|\to 0$$

as $n\to\infty$, there exists $n_0\in\mathbb{N}$ such that for any $n\geqslant n_0$

$$\sup_{x\in[-a,a]}P^x(\tau^n>t)\leqslant\frac{2a}{\sqrt{\pi t}}.$$

Moreover, for $n\geqslant n_0$ and $k\in\mathbb{N}$

$$\begin{aligned}P\big(\tau^n>(k+1)t\big)&=P(\tau^n>kt,\sup_{s\in(kt,(k+1)t]}|W^n_s|\leqslant a)\\&\leqslant E\mathbf{1}_{\{\tau^n>kt\}}P^{W^n_{kt}}(\sup_{s\in(0,t]}|W^n_s|\leqslant a)\\&\leqslant E\mathbf{1}_{\{\tau^n>kt\}}P^{W^n_{kt}}(\tau^n>t)\leqslant P(\tau^n>kt)\frac{2a}{\sqrt{\pi t}},\end{aligned}$$

so iterating the above inequality yields $P(\tau^n > kt) \leqslant \big(2a/\sqrt{\pi t}\big)^k$ for $n \geqslant n_0$. Therefore, for $u > 1$ we have

$$\begin{aligned} E\exp(\lambda\tau^n) &= \int_0^\infty P\big(\alpha \leqslant \exp(\lambda\tau^n)\big)d\alpha \\ &= \int_0^1 P\big(\alpha \leqslant \exp(\lambda\tau^n)\big)d\alpha + \sum_{k=0}^\infty \int_{u^k}^{u^{k+1}} P\big(\alpha \leqslant \exp(\lambda\tau^n)\big)d\alpha \\ &\leqslant 1 + \sum_{k=0}^\infty u^k(u-1)P\big(u^k \leqslant \exp(\lambda\tau^n)\big) \\ &\leqslant 1 + \sum_{k=0}^\infty u^k(u-1)P(\tau^n \geqslant k\ln u/\lambda) \\ &\leqslant 1 + (u-1)\sum_{k=0}^\infty \big(u\, 2a\sqrt{\lambda}/\sqrt{\pi\ln u}\big)^k, \end{aligned}$$

where the last inequality follows from the previous calculations for $t = \ln u/\lambda$. Hence $\sup_n E\exp(\lambda\tau^n) < \infty$ if $2ua\sqrt{\lambda}/\sqrt{\pi\ln u} < 1$. ■

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