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INTRINSIC ULTRACONTRACTIVITY FOR LÉVY PROCESSES*

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Abstract. We prove the intrinsic ultracontractivity for semigroups generated by a large class of symmetric Lévy processes killed on exiting a bounded and connected Lipschitz set under some conditions about the behavior of the Lévy measure in the neighborhood of the origin.

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1. INTRODUCTION

Intrinsic ultracontractivity was introduced by Davies and Simon in [4] and it has been studied extensively in recent years in the case of the symmetric diffusions (see e.g. [4] and [1]) and the symmetric α-stable process (see e.g. [2] and [7]). The concept of the intrinsic ultracontractivity for non-symmetric semigroups was introduced in [6].

If the Lévy measure of a symmetric Lévy process X_t is positive for any truncated cone with vertex at 0 (see (2.5)) and the transition density of the process X_t satisfies some regularity conditions, we prove the intrinsic ultracontractivity for semigroups generated by the killed process on exiting a bounded connected Lipschitz open set. Assuming additionally that the Lévy measure is positive for any open ball we show that the semigroup is intrinsically ultracontractive for any bounded open set. Our approach is based on ideas developed in [7].

The paper is organized in the following way. In Section 2 we recall some definitions and prove facts about continuity and strict positivity of a transition density of the process killed on exiting a bounded open set. In Section 3 we prove the intrinsic ultracontractivity.

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2. PRELIMINARIES

In $\mathbb{R}^d$, $d \geqslant 1$, we consider a symmetric Lévy process X_t (for definition see e.g. [8]). By ν we denote its Lévy measure which is assumed not to be zero. We also assume that the transition densities of X_t, which we denote by $p(t, x, y) = p(t, x - y)$, exist and are continuous for every $t > 0$. In addition, we assume that for any $\delta > 0$ there exists a constant $c = c(\delta)$ such that $p(t, x) \leqslant c$ for $t > 0$ and $|x| \geqslant \delta$.

The notation $C = C(\alpha, \beta, \gamma, \dots)$ means that the constant C depends on $\alpha, \beta, \gamma, \dots$ Values of constants may usually change from line to line, but they are always strictly positive and finite.

Denote an open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$ by $B(x, r)$ and define it as follows: $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$. By $\overline{B}(x, r)$ we denote a closed ball, that is $\overline{B}(x, r) = \{y \in \mathbb{R}^d : |x - y| \leqslant r\}$. For a bounded nonempty open set D we define:

$$\tau_D = \inf\{t > 0 : X_t \notin D\} \quad \text{and} \quad \eta_D = \inf\{t \geqslant 0 : X_t \notin D\}.$$

Next, we investigate the boundedness of the first moment of τ_D.

LEMMA 2.1. *For any bounded open set D there exists a constant $C = C(D)$ such that*

$$\sup_{x \in \mathbb{R}^d} E^x \tau_D \leqslant C.$$

P r o o f. The proof of this lemma follows by the same arguments as in the classical case for the Brownian motion (see e.g. the proof of Theorem 1.17 in [3]). The argument therein requires the existence of $t_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^d} P^x(X_{t_0} \in D) < 1.$$

The process is non-zero, and hence one can find $y \in R^d$, $y \neq 0$, such that the real-valued process $\langle y, X_t \rangle$ is a non-zero Lévy process. Since D is bounded, we can find r such that $D \subset \{z \in R^d : |\langle y, z \rangle| \leqslant r\}$. Hence, by Lemma 48.3 in [8], one can obtain

$$\sup_{x \in \mathbb{R}^d} P^x(X_t \in D) \leqslant \sup_{x \in \mathbb{R}^d} P^x(|\langle y, X_t \rangle| \leqslant r) = O(t^{-1/2}), \quad t \to \infty. \quad \blacksquare$$

In order to study the killed process on exiting of D we construct its transition densities by the classical formula

$$p_D(t, x, y) = p(t, x, y) - r_D(t, x, y),$$

where

$$r_D(t, x, y) = E^x[t > \tau_D;\ p(t - \tau_D, X_{\tau_D}, y)].$$

The arguments used for the Brownian motion (see e.g. the proof of Theorem 2.4 in [3]) will prevail in our case and one can easily show that $p_D(t,x,y),\ t>0$, satisfy the Chapman–Kolmogorov equation (semigroup property). Moreover, the transition density $p_D(t,x,y)$ is a symmetric function (x,y) a.s. Using the above assumptions on the transition densities of the (free) process one can actually show that $p_D(t,x,\cdot)$ and $p_D(t,\cdot,x)$ can be chosen as continuous functions on D. The semigroup generated by the process X_t killed on exiting of D will be denoted by $\{P_t^D\}_{t\geqslant 0}$. We define the Green function of D by $G_D(x,y)=\int_0^\infty p_D(t,x,y)dt$.

Now, we recall the Ikeda–Watanabe formula [5]. Assume D is a bounded nonempty open set and E is a Borel set such that $\operatorname{dist}(D,E)>0$. Then we have

$$P^x(X_{\tau_D}\in E)=\int_D G_D(x,y)\nu(E-y)dy. \tag{2.1}$$

Notice that the equation (2.1) is true for any open set $E\subset D^c$ (it can even happen that $\operatorname{dist}(D,E)=0$).

$\{P_t^D\}_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on $L^2(D)$. Because $p_D(t,x,y)$ is symmetric a.e., the operator P_t^D is selfadjoint. For D bounded we infer from continuity of $p(t,\cdot)$ that

$$p_D(t,x,y)\leqslant p(t,x-y)\leqslant \sup_{x\in B(0,\operatorname{diam}(D))} p(t,x)=C_1(t,D). \tag{2.2}$$

Therefore P_t^D is a Hilbert–Schmidt operator, hence it is also compact. So, it is well known that there exists an orthonormal basis of real-valued eigenfunctions $\{\varphi_n\}_{n=1}^\infty$ with corresponding eigenvalues $\{\exp(-\lambda_n t)\}_{n=1}^\infty$ satisfying $0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\ldots$, where all φ_n are continuous.

Since $p_D(t,x,\cdot)\in L^2(D)$, we can represent this function as

$$p_D(t,x,\cdot)=\sum_{n=1}^\infty\langle p_D(t,x,\cdot),\varphi_n\rangle\varphi_n.$$

But $\langle p_D(t,x,\cdot),\varphi_n\rangle=P_t^D\varphi_n(x)=\exp(-\lambda_n t)\varphi_n(x)$, so

$$p_D(t,x,y)=\sum_{n=1}^\infty\exp(-\lambda_n t)\varphi_n(x)\varphi_n(y). \tag{2.3}$$

Notice that φ_n is bounded. Indeed, by the Schwarz inequality we have

$$\begin{aligned}|\varphi_n|(x)&\leqslant\exp(\lambda_n t)\int_D|\varphi_n|(y)p_D(t,x,y)dy\\&\leqslant\exp(\lambda_n t)C_1(t,D)\int_D|\varphi_n|(y)dy\leqslant\exp(\lambda_n t)C_1(t,D)|D|^{1/2},\end{aligned} \tag{2.4}$$

where $C_1(t,D)$ is a constant from (2.2). Now, let us observe that the series from (2.3) is uniformly convergent on $D\times D$. Applying (2.4) for $t/3$ instead of t we have

$$\begin{aligned}\exp(-\lambda_n t)|\varphi_n(x)\varphi_n(y)|&\leqslant\exp(-\lambda_n t)\big(\exp(\lambda_n t/3)C_1(t/3,D)|D|^{1/2}\big)^2\\&=C_1(t/3,D)^2|D|\exp(-\lambda_n t/3).\end{aligned}$$

Next,

$$\sum_{n=1}^{\infty} \exp(-\lambda_n t/3) = \sum_{n=1}^{\infty} \exp(-\lambda_n t/3) \int_D \varphi_n(x)^2 dx$$
$$= \int_D p_D(t/3, x, x) dx \leqslant C_1(t/3, D)|D|.$$

Hence, we get $p_D(t, \cdot, \cdot) \in C(D \times D)$. Therefore $p_D(t, x, y) = p_D(t, y, x)$ for any $t > 0$ and $x, y \in D$.

Denote the unit sphere in $\mathbb{R}^d$ by S. Let $x \in S$; then by $\Gamma_\gamma(x)$ we denote a cone with vertex at 0 and the aperture 2γ, the axis of which goes through the point x, that is, $\Gamma_\gamma(x) = \{y \in \mathbb{R}^d : \langle x, y\rangle > |y| \cos\gamma\}$. For $d = 1$ we have $\Gamma_\gamma(1) = (0, \infty)$ and $\Gamma_\gamma(-1) = (-\infty, 0)$ for any $\gamma \in (0, \pi)$. Let $A(\rho, r) = B(0, r) \setminus B(0, \rho)$.

Now, we state our basic conditions on the Lévy measure ν of the process X_t:

(A1) For every $x \in S$, $\gamma \in (0, \pi/2]$ and $r > 0$,

$$\nu\big(\Gamma_\gamma(x) \cap B(0, r)\big) > 0. \tag{2.5}$$

(A2) For every $x \in \mathbb{R}^d$ and $r > 0$,

$$\nu\big(B(x, r)\big) > 0. \tag{2.6}$$

Observe that (A2) implies (A1).

Now, we show that $p_D(t, \cdot, \cdot)$ is strictly positive on $D \times D$ if D is a connected open bounded set or, if the condition (A2) holds, for any bounded open set D. To prove it we use the following proposition (see e.g. [4]).

PROPOSITION 2.1. *Let D be a bounded open set and $F \subset D$. Then the Green function $G_D(\cdot, \cdot)$ is strictly positive on $F \times F$ iff $p_D(t, \cdot, \cdot)$ is strictly positive on $F \times F$ for any $t > 0$.*

If $G_D(\cdot, \cdot)$ is strictly positive on $D \times D$, then $\varphi_1(\cdot)$ is strictly positive on D and $\lambda_1 < \lambda_2$. This follows from Jentzsch's theorem (see [9]).

PROPOSITION 2.2. *The transition density $p_D(t\cdot, \cdot)$, $t > 0$, is strictly positive on $D \times D$ in the following cases*:

(i) *D is a bounded connected open set.*

(ii) *D is a bounded open set and* (A2) *holds.*

P r o o f. Let D be a bounded connected open set. First, let us observe that for any $x \in D$ we have $p_D(t, x, x) > 0$. Indeed,

$$p_D(t, x, x) = \int_D p_D(t/2, x, y) p_D(t/2, y, x) dy = \int_D p_D^2(t/2, x, y) dy$$
$$\geqslant \Big(\int_D p_D(t/2, x, y) dy\Big)^2 / |D| = \big(P^x(\tau_D > t/2)\big)^2 / |D| > 0$$

for $t \leqslant t_0(x)$ (it is a consequence of the right-continuity of paths). Hence, from the formula (2.3) we conclude the strict positivity of $p_D(t,x,x)$ for all $t > 0$.

Let $K \subset D$ be a compact connected set. By continuity of $p_D(t,\cdot,\cdot)$ we infer that for any $x \in K$ there is a radius r_x such that

$$p_D(t,z,y) > 0 \quad \text{for } z,y \in B(x,2r_x).$$

Since K is compact, there are $x_1,\ldots,x_k \in K$ such that $K \subset \bigcup_{i=1}^k B(x_k, r_{x_k})$. Now, we use the fact that K is connected to imply from the Chapman–Kolmogorov equation that $p_D(kt,x,y) > 0$ for any $x,y \in K$. Hence, by the Chapman–Kolmogorov equation, we have $p_D(s,x,y) > 0$ for $s \geqslant kt$ and $x,y \in K$, so $G_D(x,y) > 0$ for $x,y \in K$. By Proposition 2.1 we obtain $p_D(t,\cdot,\cdot) > 0$ on $K \times K$ for $t > 0$. This implies that $p_D(t,\cdot,\cdot)$ is strictly positive on $D \times D$ for any $t > 0$.

Next, we prove the second part of the proposition. Let $x \neq y \in D$. Then there is a radius r such that $B(x,2r), B(y,2r) \subset D$ and $B(x,2r) \cap B(y,2r) = \emptyset$. By the continuity and strict positivity of $p_{B(x,(3/2)r)}(t,\cdot,\cdot)$ we infer that there exists a constant $c_1 = c_1(r)$ such that

$$\begin{aligned} c_1 \leqslant \int_0^\infty \inf_{z \in B(x,r)} p_{B(x,(3/2)r)}(t,x,z)dt &\leqslant \inf_{z \in B(x,r)} G_{B(x,(3/2)r)}(x,z) \\ &\leqslant \inf_{z \in B(x,r)} G_D(x,z). \end{aligned}$$

Since $G_D(x,\cdot)$ is a harmonic function on $B(y,2r)$, that is, it has the mean value property:

$$G_D(x,z) = E^z G_D\big(x, X(\tau_U)\big) \quad \text{for any open set } U \text{ such that } \overline{U} \subset B(y,2r),$$

by the Ikeda–Watanabe formula (2.1) we get

$$\begin{aligned} G_D(x,y) &= E^y G_D(x, X_{\tau_{B(y,(3/2)r)}}) \\ &\geqslant E^y\{X_{\tau_{B(y,(3/2)r)}} \in B(x,r); G_D(x, X_{\tau_{B(y,(3/2)r)}})\} \\ &\geqslant c_1 P^y\big(X_{\tau_{B(y,(3/2)r)}} \in B(x,r)\big) \\ &= c_1 \int_{B(y,(3/2)r)} G_{B(y,(3/2)r)}(y,w)\nu\big(B(x,r)-w\big)dw \\ &\geqslant c_1^2 \int_{B(y,r)} \nu\big(B(x-w,r)\big)dw > 0. \end{aligned}$$

Hence, by Proposition 2.1, the transition density is strictly positive. ■

LEMMA 2.2. *Let D be a bounded open set. Then for any $x \in D$ and $t > 0$ we have*

$$p_D(t,x,y) \leqslant C(t,D) E^x \tau_D E^y \tau_D.$$

P r o o f. By the Chapman–Kolmogorov equation we obtain for $t > 0$

$$p_D(t,x,y) = \int_D p_D(t/2,x,z)p_D(t/2,z,y)dz \leqslant C_1(t/2,D)P^x(\tau_D > t/2),$$

where C_1 is a constant from (2.2). Applying again the Chapman–Kolmogorov equation together with the above inequality we get

$$p_D(t,x,y) \leqslant cP^x(\tau_D > t/4)\int_D p_D(t/2,z,y)dz = cP^x(\tau_D > t/4)P^y(\tau_D > t/2).$$

The application of Chebyshev's inequality completes the proof. ■

In the next three lemmas we prove some properties of the Lévy measure, which are essential for the proofs in the next section.

LEMMA 2.3. *Suppose that* (A1) *holds. Then for every* $r > 0$ *and* $\gamma \in (0,\pi/4]$ *there is a constant* $\rho_0 = \rho(r,\gamma) > 0$ *such that*

$$\inf_{x,y\in S} \nu\Big(\big(\Gamma_\gamma(x) + \rho y\big) \cap B(0,r)\Big) > 0 \quad \textit{for } \rho \leqslant \rho_0.$$

P r o o f. We fix $r > 0$ and $\gamma \in (0,\pi/4]$.

First, we show that there exists a radius ρ_1 such that

$$\inf_{x\in S} \nu\big(\Gamma_\gamma(x) \cap A(\rho_1,r)\big) > 0. \tag{2.7}$$

We consider an open cover of the unit sphere $\{\Gamma_{\gamma/2}(x) \cap S\}_{x\in S}$. By compactness of S there are $x_1, x_2, \ldots, x_k$ such that $S \subset \Gamma_{\gamma/2}(x_1) \cup \ldots \cup \Gamma_{\gamma/2}(x_k)$. As a consequence of (2.5) there exists $\widetilde{\rho}_i = \widetilde{\rho}_i(r,\gamma)$ satisfying

$$\nu\big(\Gamma_{\gamma/2}(x_i) \cap A(\widetilde{\rho}_i,r)\big) > 0, \quad 1 \leqslant i \leqslant k.$$

Let us put

$$\rho_1 = \min_i \widetilde{\rho}_i > 0 \quad \text{and} \quad c = \min_i \nu\big(\Gamma_{\gamma/2}(x_i) \cap A(\rho_1,r)\big) > 0.$$

For any $x \in S$, there exists a number i such that $x \in \Gamma_{\gamma/2}(x_i)$. Then we have $\Gamma_{\gamma/2}(x_i) \subset \Gamma_\gamma(x)$. Therefore

$$\nu\big(\Gamma_\gamma(x) \cap A(\rho_1,r)\big) \geqslant c,$$

and (2.7) is proved.

Next, observe that for $\rho_2 = \rho_1/2$ we have, for any $x \in S$,

$$\Gamma_\gamma(x) \cap A(\rho_1,r) \subset \big(\Gamma_{2\gamma}(x) + \rho_2 x\big) \cap A(\rho_1,r).$$

This implies

$$\inf_{x\in S} \nu\Big(\big(\Gamma_{2\gamma}(x)+\rho_2 x\big)\cap B(0,r)\Big) > 0.$$

Let us observe that $B(x/\sin 2\gamma, 1) \subset \Gamma_{2\gamma}(x)$. Therefore for $\rho \leqslant \rho_0$ such that $\rho_0 = \rho_2 \sin 2\gamma$ we have

$$\Gamma_{2\gamma}(x)+\rho_2 x \subset \Gamma_{2\gamma}(x)+\frac{\rho}{\sin 2\gamma}x \subset \Gamma_{2\gamma}(x)+\rho y.$$

Hence

$$\inf_{x,y\in S} \nu\Big(\big(\Gamma_{2\gamma}(x)+\rho y\big)\cap B(0,r)\Big) > 0. \quad \blacksquare$$

The proof of the following remark is omitted, because it is an easy consequence of Lemma 2.3 and the proof of the second part of Proposition 2.2.

REMARK 2.1. *Suppose that* (A1) *holds and* $d=1$. *Then for* $x_0 \in (a,b)$ *there exists a radius* $r = r(x_0,\nu)$ *such that*

$$p_{(a,b)\setminus[x_0-r,x_0+r]}(x,y) > 0 \quad \textit{for } x,y \in (a,b)\setminus[x_0-r,x_0+r].$$

By $\delta_D(x)$ we denote a distance between a point x and the boundary of D, that is $\delta_D(x) = \operatorname{dist}(\{x\},\partial D)$. For $\rho>0$ we define $D_\rho = \{x\in D : \delta_D(x) \geqslant \rho\}$.

LEMMA 2.4. *Suppose that* (A1) *holds.*

(i) *If D is a bounded Lipschitz open set*, *then there exists* $\rho_0 = \rho_0(D)$ *such that for every* $\rho \leqslant \rho_0$

$$\inf_{y\in D\setminus D_\rho} \nu(D_\rho - y) > 0. \tag{2.8}$$

(ii) *For* $r>0$ *there is* $\rho=\rho(r)$ *such that*

$$\inf_{r\leqslant |y|\leqslant r+\rho} \nu\big(B(y,r-\rho)\big) > 0, \tag{2.9}$$

$$\inf_{r-\rho\leqslant |y|\leqslant r} \nu\big(\overline{B}(y,r)^c\big) > 0. \tag{2.10}$$

Proof. Let us observe that it is enough to prove (2.8) for some $\rho_0>0$, because then for $\rho\leqslant\rho_0$ this is true as well.

Since D is a bounded Lipschitz open set, there are constants $\gamma=\gamma(D) < \pi/4$ and $R_0 = R_0(D)$ such that, for any $Q\in\partial D$, there exists $x\in S$ such that a cone $\Gamma_\gamma(x)$ satisfies $\big(\Gamma_\gamma(x)+Q\big)\cap B(Q,R_0)\subset D$. By Lemma 2.3 there exists $0<\rho_1<R_0/4$ such that, for $\rho\leqslant \rho_1(1+\sin\gamma)/\sin\gamma$,

$$\inf_{x,z\in S} \nu\Big(\big(\Gamma_\gamma(x)+\rho z\big)\cap B(0,R_0/2)\Big) > 0. \tag{2.11}$$

Let $y \in D \setminus D_{\rho_1}$ and $Q \in \partial D$ so that $\delta_D(y) = |y - Q|$. Then we have

$$\big(\Gamma_\gamma(x) + Q + x\rho_1/\sin\gamma\big) \cap B(y, R_0/2) \subset D_{\rho_1},$$

because $B(y, R_0/2) \subset B\big(Q, \frac{3}{4} R_0\big)$. That is,

$$\nu(D_{\rho_1} - y) \geqslant \nu\Big(\big(\Gamma_\gamma(x) + \rho_2 z\big) \cap B(0, R_0/2)\Big), \tag{2.12}$$

where

$$z = \big(Q - y + (\rho_1/\sin\gamma)x\big)/|Q - y + (\rho_1/\sin\gamma)x|$$

and

$$\rho_2 = |Q - y + (\rho_1/\sin\gamma)x| \leqslant \rho_1(1 + 1/\sin\gamma).$$

This together with (2.12) and (2.11) implies

$$\inf_{y \in D \setminus D_{\rho_1}} \nu(D_{\rho_1} - y) > 0.$$

Let $r > 0$. Then by Lemma 2.3 there is a constant $\rho_3 > 0$ such that

$$\inf_{x \in S} \nu\Big(\big(\Gamma_{\pi/4}(x) + 2\rho_3 x\big) \cap B(0, r)\Big) > 0.$$

Notice that

$$\{\Gamma_{\pi/4}(y/|y|) + (|y| + \rho_3 - r)y/|y|\} \cap B(0, r) \subset B(y, r - \rho_3)$$

for $r \leqslant |y| \leqslant r + \rho_3$ whereas for $r - 2\rho_3 \leqslant |y| \leqslant r$ we have

$$\{\Gamma_{\pi/4}(-y/|y|) - (r - |y|)y/|y|\} \cap B(0, r) \subset \overline{B}(y, r)^c.$$

Therefore (2.9) and (2.10) are proved. ■

LEMMA 2.5. *Suppose that* (A2) *holds. Then for any* $r, \rho > 0$ *we have*

$$\inf_{x \in \overline{B}(0,r)} \nu\big(B(x, \rho)\big) > 0.$$

Proof. Suppose that $\inf_{x \in \overline{B}(0,r)} \nu\big(B(x, \rho)\big) = 0$. Then there is a sequence $\{x_n\}_{n \geqslant 1} \subset \overline{B}(0, r)$ such that

$$\lim_{n \to \infty} \nu\big(B(x_n, \rho)\big) = 0. \tag{2.13}$$

Since $\overline{B}(0, r)$ is a compact set, we can assume that $\lim_{n\to\infty} x_n = y$ for some $y \in \overline{B}(0, r)$. Hence for n large enough we have $B(y, \rho/2) \subset B(x_n, \rho)$. Consequently, $\nu\big(B(x_{n_k}, \rho)\big) \geqslant \nu\big(B(y, \rho/2)\big) > 0$ because (A2) holds. Thus we get a contradiction to (2.13). ■

REMARK 2.2. *Suppose that the Lévy measure satisfies the assumption of Lemma* 2.5. *Then for any bounded open set D there is a ρ_0 such that for $\rho \leqslant \rho_0$*

$$\inf_{y \in D \setminus D_\rho} \nu(D_\rho - y) > 0.$$

P r o o f. There exists ρ_0 such that the set D_{ρ_0} is nonempty. Let $x_0 \in D_{\rho_0}$ and $B(x_0, \rho) \subset D_{\rho_0}$. Then

$$\inf_{y \in D \setminus D_\rho} \nu(D_\rho - y) \geqslant \inf_{|y - x_0| \leqslant \mathrm{diam}(D)} \nu\big(B(x_0 - y, \rho)\big) > 0$$

by Lemma 2.5. ∎

Now, we define, according to [4], the intrinsic ultracontractivity.

DEFINITION 2.1. The semigroup $\{P_t^D\}_{t \geqslant 0}$ is said to be *intrinsically ultracontractive* if, for any $t > 0$, there exists a constant c_t such that

$$p_D(t, x, y) \leqslant c_t \varphi_1(x) \varphi_1(y), \quad x, y \in D.$$

PROPOSITION 2.3. *Let D be a bounded nonempty open set. If there is a constant C such that $E^x \tau_D \leqslant C \varphi_1(x)$, $x \in D$, then $\{P_t^D\}_{t \geqslant 0}$ is intrinsically ultracontractive.*

P r o o f. Suppose that $E^x \tau_D \leqslant C \varphi_1(x)$. By Lemma 2.2 we have

$$p_D(t, x, y) \leqslant C_t E^x \tau_D E^y \tau_D,$$

which completes the proof. ∎

Actually the condition $E^x \tau_D \leqslant C \varphi_1(x)$ is not necessary for the intrinsic ultracontractivity and it is equivalent to the following lower bound for the Green function.

PROPOSITION 2.4. *Let D be a bounded open set and let $G_D(\cdot, \cdot)$ be strictly positive on $D \times D$. There exists a constant C such that $E^x \tau_D \leqslant C \varphi_1(x)$, $x \in D$, iff there is a constant c such that $c E^x \tau_D E^y \tau_D \leqslant G_D(x, y)$, $x, y \in D$.*

P r o o f. If $E^x \tau_D \leqslant C \varphi_1(x)$, $\{P_t^D\}$ is intrinsically ultracontractive. Therefore Theorem 3.2 in [4] implies, for any $t > 0$,

$$c_t \varphi_1(x) \varphi_1(y) \leqslant p_D(t, x, y)$$

for some constant c_t. Hence

$$\begin{aligned} G_D(x, y) &= \int_0^\infty p_D(t, x, y) dt \\ &\geqslant \int_0^\infty c_t \varphi_1(x) \varphi_1(y) dt = c \varphi_1(x) \varphi_1(y) \geqslant c C^{-2} E^x \tau_D E^y \tau_D. \end{aligned}$$

If $E^x\tau_D E^y\tau_D \leqslant cG_D(x,y)$, we obtain

$$E^x\tau_D \int_D \varphi_1(y)E^y\tau_D dy \leqslant c\int_D G_D(x,y)\varphi_1(y)dy. \tag{2.14}$$

Next

$$\begin{aligned}\int_D G_D(x,y)\varphi_1(y)dy &= \int_0^\infty \int_D p_D(t,x,y)\varphi_1(y)dydt \\ &= \int_0^\infty \exp(-\lambda_1 t)\varphi_1(x)dt = \lambda_1^{-1}\varphi_1(x).\end{aligned} \tag{2.15}$$

This combined with (2.14) implies $E^x\tau_D \leqslant C\varphi_1(x)$. ∎

3. MAIN RESULTS

First, we prove the intrinsic ultracontractivity for the semigroup $\{P_t^D\}_{t\geqslant 0}$ generated by the symmetric Lévy process for which the Lévy measure satisfies the conditions (A1) and (2.8).

Next, we use Lemma 2.4 and Remark 2.2 to get the intrinsic ultracontractivity in two cases: firstly when the Lévy measure satisfies (A1) and D is a bounded Lipschitz set, and secondly when the Lévy measure satisfies (A2) without any restrictions on D.

The idea of proof of the main theorem and the notation are similar to that in the paper [7]. Assume that the set D is bounded and open.

We fix $x_0 \in D$ and $r > 0$ such that $\overline{B}(x_0, 2r) \subset D$. We put $K = \overline{B}(x_0, r)$, $L = B(x_0, 2r)$, $M = D\backslash K$ and $N = D\backslash L$. Define stopping times S_n and T_n:

$$\begin{aligned} S_1 &= 0, \\ T_n &= S_n + \eta_M \circ \theta_{S_n}, \\ S_n &= T_{n-1} + \eta_L \circ \theta_{T_{n-1}}. \end{aligned}$$

Recall that $D_\rho = \{x\in D\colon \delta_D(x)\geqslant\rho\}$. Now, we prove several lemmas needed to follow the approach used in [7].

LEMMA 3.1. *Let D be a connected bounded open set and let the Lévy measure satisfy the condition* (A1). *Then for every $\rho > 0$ there is a constant $c = c(D, x_0, \rho)$ such that for any $x \in D_\rho$*

$$P^x\big(X(\tau_M) \in K\big) \geqslant c.$$

P r o o f. Observe that it is enough to consider $x \in D_\rho \setminus B(x_0, r)$.

By Lemma 2.4 we find ρ_1 such that

$$\inf_{r\leqslant|y|\leqslant r+\rho} \nu\big(B(y, r-\rho)\big) = c_1 > 0 \tag{3.1}$$

for any $\rho \leqslant \rho_1$. Write $J = D \setminus \overline{B}(x_0, r-\rho)$. From the Ikeda–Watanabe formula (2.1) we obtain

$$\begin{aligned} P^x\big(X(\tau_M) \in K\big) &\geqslant P^x\big(X(\tau_J) \in B(x_0, r-\rho)\big) \\ &= \int_J G_J(x,y)\nu\big(B(x_0, r-\rho) - y\big)dy \\ &\geqslant \int_{D_\rho \setminus B(x_0,r)} G_J(x,y)\nu\big(B(x_0 - y, r-\rho)\big)dy. \end{aligned}$$

Notice that $p_J(t,\cdot,\cdot)$ is a continuous and positive function on $J \times J$ (for $d > 1$ the set J is open and connected and for $d = 1$ this is a consequence of Remark 2.1). Moreover, the set $\big(D_\rho \setminus B(x_0,r)\big) \times \big(D_\rho \setminus B(x_0,r)\big)$ is a compact subset of $J \times J$, hence we obtain $\inf_{x,y \in D_\rho \setminus B(x_0,r)} p_J(t,x,y) > 0$. Consequently,

$$\inf_{x,y \in D_\rho \setminus B(x_0,r)} G_J(x,y) \geqslant \int_0^\infty \inf_{x,y \in D_\rho \setminus B(x_0,r)} p_J(t,x,y)dt = c_2 > 0.$$

Therefore, by (3.1),

$$\begin{aligned} P^x\big(X(\tau_M) \in K\big) &\geqslant c_2 \int_{B(x_0,r+\rho)\setminus B(x_0,r)} \nu\big(B(x_0 - y, r-\rho)\big)dy \\ &\geqslant c_2 c_1 \int_{B(x_0,r+\rho)\setminus B(x_0,r)} dy > 0. \end{aligned}$$

Of course, for $\rho > \rho_1$ we have $D_\rho \subset D_{\rho_1}$ and the lemma is true in this case as well. ■

LEMMA 3.2. *Let D be a connected bounded open set and let the Lévy measure satisfy the conditions* (A1) *and* (2.8). *Then there exists a constant $c = c(D, x_0)$ such that*

$$P^x\big(X(\eta_M) \in K\big) \geqslant cE^x\eta_M \quad \textit{for all } x \in \mathbb{R}^d.$$

P r o o f. As a consequence of the condition (2.8) there exist $\rho \leqslant r$ and a constant c_1 such that

$$\inf_{y \in D \setminus D_{2\rho}} \nu(D_{2\rho} - y) = c_1. \tag{3.2}$$

From Lemma 3.1 it follows that there is a constant c_2 such that

$$P^x(X_{\tau_M} \in K) \geqslant c_2 \quad \text{for } x \in D_\rho. \tag{3.3}$$

By Lemma 2.1

$$E^x\tau_M \leqslant c_3$$

for some constant c_3. Taking $c_4 = c_2/c_3$ we obtain

$$P^x(X_{\tau_M} \in K) \geqslant c_4 E^x \tau_M \quad \text{for } x \in D_\rho. \tag{3.4}$$

Next, let $x \in D\backslash D_\rho$. Then from the strong Markov property and (3.4) we get

$$\begin{aligned} P^x\big(X(\tau_M) \in K\big) &= E^x\Big(P^{X(\tau_{D\backslash D_\rho})}\big(X(\tau_M) \in K\big)\Big) \\ &\geqslant c_4 E^x(E^{X(\tau_{D\backslash D_\rho})}\tau_M) = c_4(E^x\tau_M - E^x\tau_{D\backslash D_\rho}). \end{aligned} \tag{3.5}$$

Using (3.3) and the strong Markov property, we have

$$\begin{aligned} P^x\big(X(\tau_M) \in K\big) &= E^x\Big(X(\tau_{D\backslash D_\rho}) \in D_\rho, P^{X(\tau_{D\backslash D_\rho})}\big(X(\tau_M) \in K\big)\Big) \\ &\geqslant c_2 P^x\big(X(\tau_{D\backslash D_\rho}) \in D_\rho\big). \end{aligned} \tag{3.6}$$

Next, by the Ikeda–Watanabe formula (2.1) and finally by (3.2) we get

$$\begin{aligned} P^x\big(X(\tau_{D\backslash D_\rho}) \in D_\rho\big) &\geqslant P^x\big(X(\tau_{D\backslash D_\rho}) \in \operatorname{int}(D_\rho)\big) \\ &= \int_{D\backslash D_\rho} G_{D\backslash D_\rho}(x,y)\nu\big(\operatorname{int}(D_\rho) - y\big)dy \\ &\geqslant \int_{D\backslash D_\rho} G_{D\backslash D_\rho}(x,y)\nu(D_{2\rho} - y)dy \\ &\geqslant c_1 \int_{D\backslash D_\rho} G_{D\backslash D_\rho}(x,y)dy = c_1 E^x\tau_{D\backslash D_\rho}. \end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) we obtain

$$\begin{aligned} P^x\big(X(\tau_M) \in K\big) &= \Big(\frac{1}{2} + \frac{1}{2}\Big)P^x\big(X(\tau_M) \in K\big) \\ &\geqslant \frac{c_4}{2}(E^x\tau_M - E^x\tau_{D\backslash D_\rho}) + \frac{c_1c_2}{2}(E^x\tau_{D\backslash D_\rho}) \\ &\geqslant \frac{c_4 \wedge c_1c_2}{2}E^x\tau_M. \end{aligned}$$

For $x \in D \setminus K$, we have $E^x\tau_M = E^x\eta_M$, and the claim of the lemma for $x \in D^c$ is obvious, which completes the proof. ■

We now apply the above lemma to a bounded and connected open Lipschitz set D. By Lemma 2.4 the condition (2.8) holds if the Lévy measure satisfies (A1). Hence we get $P^x\big(X(\eta_M) \in K\big) \geqslant cE^x\eta_M$. Moreover, by Remark 2.2 we have this inequality for any bounded connected open set if the Lévy measure satisfies (A2). However, if (A2) holds, we can relax the connectedness assumption.

LEMMA 3.3. *Let D be a bounded open set. Suppose that* (A2) *holds. Then*

$$P^x\big(X(\eta_M) \in K\big) \geqslant cE^x\eta_M, \quad x \in \mathbb{R}^d.$$

P r o o f. Let $x \in M$. By the Ikeda–Watanabe formula (2.1) we get

$$\begin{aligned} P^x\big(X(\tau_M) \in K\big) &\geqslant P^x\big(X(\tau_M) \in B(x_0, r)\big) \\ &= \int_M G_M(x,y)\nu\big(B(x_0,r) - y\big)dy \\ &\geqslant \inf_{|y| \leqslant \operatorname{diam}(D)} \nu\big(B(y,r)\big) \int_M G_M(x,y)dy = cE^x\tau_M, \end{aligned}$$

where the constant c exists by Lemma 2.5. ■

LEMMA 3.4. *Suppose that D is a bounded open set and the Lévy measure satisfies the condition* (A1). *Then for all $x \in \mathbb{R}^d$ there exists a random variable Z such that for all $n \geqslant Z$ we have $T_n = \eta_D$ almost surely P^x.*

P r o o f. By the Borel–Cantelli lemma it is enough to show that there exists a constant $\beta < 1$ such that

$$P^x(T_n < \eta_D) \leqslant \beta^{n-1} \quad \text{for all } x \in \mathbb{R}^d \text{ and } n \geqslant 1. \tag{3.8}$$

Let us write $r_0 = \operatorname{diam}(D)$. By Lemma 2.4, we find $0 < \rho \leqslant r_0 - \frac{3}{2}r$ such that

$$\inf_{r_0 - \rho \leqslant |y| \leqslant r_0} \nu\big(\overline{B}(y, r_0)^c\big) \geqslant c_1 \tag{3.9}$$

for some constant c_1. Let $A = B(x_0, r_0) \setminus K$. Then, for $d > 1$, the set A is connected. Moreover, the set $\overline{B}(x_0, r_0 - \rho/2) \setminus B(x_0, 2r)$ is compact, so there exists a constant c_2 such that

$$\inf_{x,y \in \overline{B}(x_0, r_0-\rho/2) \setminus B(x_0,2r)} G_A(x,y) = c_2.$$

The above equation is true for $d = 1$ as well – in this case we use Remark 2.1. From the Ikeda–Watanabe formula (2.1) and (3.9) we obtain for $x \in N$

$$\begin{aligned} P^x\big(X(\eta_M) \in D^c\big) &\geqslant P^x\big(X(\tau_A) \in \overline{B}(x_0, r_0)^c\big) \\ &\geqslant \int_{B(x_0, r_0-\rho/2) \setminus B(x_0,2r)} G_A(x,y)\nu\big(\overline{B}(x_0 - y, r_0)^c\big)dy \\ &\geqslant c_1 c_2 \int_{B(0, r_0-\rho/2) \setminus B(0, r_0 - \rho)} dy \\ &= 1 - \beta > 0. \end{aligned}$$

Consequently, for any $x \in \mathbb{R}^d$ and $n \geqslant 1$, we get

$$\begin{aligned}
&P^x(T_n < \eta_D, T_{n+1} = \eta_D)\\
&\qquad = P^x(T_n < \eta_D, S_{n+1} = \eta_D)\\
&\qquad\quad + P^x\big(T_n < \eta_D, S_{n+1} < \eta_D, X(T_{n+1}) \in D^c\big)\\
&\qquad = P^x(T_n < \eta_D, S_{n+1} = \eta_D)\\
&\qquad\quad + P^x\big(T_n < \eta_D, X(S_{n+1}) \in N, X(\eta_M) \circ \theta_{S_{n+1}} \in D^c\big)\\
&\qquad = P^x(T_n < \eta_D, S_{n+1} = \eta_D)\\
&\qquad\quad + E^x\Big(T_n < \eta_D, X(S_{n+1}) \in N, P^{X(S_{n+1})}\big(X(\eta_M) \in D^c\big)\Big)\\
&\qquad \geqslant (1-\beta)P^x(T_n < \eta_D, S_{n+1} = \eta_D)\\
&\qquad\quad + (1-\beta)P^x(T_n < \eta_D, S_{n+1} < \eta_D)\\
&\qquad = (1-\beta)P^x(T_n < \eta_D).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
P^x(T_{n+1} < \eta_D) &= P^x(T_n < \eta_D) - P^x(T_n < \eta_D, T_{n+1} = \eta_D)\\
&\leqslant P^x(T_n < \eta_D) - (1-\beta)P^x(T_n < \eta_D)\\
&= \beta P^x(T_n < \eta_D).
\end{aligned}$$

This proves (3.8), and hence the lemma. ■

The following lemma is crucial for the proof of the main result.

PROPOSITION 3.1. *Let D be a bounded open set and let H be a nonempty open subset of D. Suppose that* (A2) *holds or D is connected and the Lévy measure satisfies the conditions* (A1) *and* (2.8). *Then there is $c = c(H)$ such that*

$$E^x \int_0^{\tau_D} \mathbf{1}_H(X_t)dt \geqslant cE^x\tau_D.$$

Proof. If the set D is connected and the Lévy measure satisfies (A1) and (2.8), then these conditions cover the assumptions of Lemmas 3.2 and 3.4. These lemmas allow us to repeat the proof of Theorem 8 in [7] to get the proposition.

When (A2) holds, we use Lemmas 3.3 and 3.4 to complete the proof. ■

The following lemma have already appeared in [7] under some additional assumptions; we provide the proof for reader's convenience.

LEMMA 3.5. *Let D be a bounded open set. Suppose that* (A2) *holds or D is connected and the Lévy measure satisfies the conditions* (A1) *and* (2.8). *Then there exists a constant C such that*

$$E^x\tau_D \leqslant C\varphi_1(x) \quad \textit{for all } x \in D.$$

P r o o f. We have, for all $t > 0$,

$$\exp(-\lambda_1 t)\varphi_1(x) = \int_D p_D(t,x,y)\varphi_1(y)dy.$$

Integrating with respect to dt we get

$$\varphi_1(x) = \lambda_1 \int_D G_D(x,y)\varphi_1(y)dy.$$

Since φ_1 is continuous and positive by Proposition 2.2, it follows that there is a constant $\varepsilon > 0$ such that the set $H = \{x : \varphi_1(x) > \varepsilon\}$ is nonempty and open. By Proposition 3.1 we have

$$\begin{aligned} E^x\tau_D &\leqslant c^{-1}\int_H G_D(x,y)dy \leqslant (c\varepsilon)^{-1}\int_H G_D(x,y)\varphi_1(y)dy \\ &\leqslant (c\varepsilon)^{-1}\int_D G_D(x,y)\varphi_1(y)dy = (c\varepsilon\lambda_1)^{-1}\varphi_1(x). \quad \blacksquare \end{aligned}$$

Now we prove our main theorem.

THEOREM 3.1. *Let D be a bounded open set. The semigroup $\{P_t^D\}$ is intrinsically ultracontractive in the following two cases*:

(a) *The Lévy measure satisfies* (A1) *and D is a connected Lipschitz set.*

(b) *The Lévy measure satisfies* (A2).

P r o o f. Suppose that D is a connected Lipschitz set and the Lévy measure satisfies (A1). Then (2.8) holds by Lemma 2.4, so Lemma 3.5 implies that

$$E^x\tau_D \leqslant c\varphi_1(x).$$

Applying Proposition 2.3 we get the intrinsic ultracontractivity in this case.

Now, assume that (A2) holds. Then using Lemma 3.5 and Proposition 2.3 we complete the proof. ■

Finally, Proposition 2.4 yields the following theorem.

THEOREM 3.2. *Let D be a bounded open set. If the Lévy measure satisfies* (A1) *and D is a connected Lipschitz set or if the Lévy measure satisfies* (A2), *then there exists a constant $c = c(D,\nu)$ such that*

$$cE^x\tau_D E^y\tau_D \leqslant G_D(x,y) \quad \textit{for } x,y \in D.$$

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