PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 28, Fasc. 1 (2008), pp. 107–120

SOME REMARKS ON THE MAXIMUM OF A ONE-DIMENSIONAL DIFFUSION PROCESS

BY

MARIO ABUNDO (ROME)

Abstract. For a certain class of one-dimensional diffusions $X(t)$, we study the distribution of $\max_{t\in[0,T]} X(t)$ and the distribution of the first instant at which $X(t)$ attains the maximum by reducing $X(t)$ to Brownian motion. Moreover, for T fixed or random, we study the asymptotics of threshold crossing probability, i.e. the rate of decay of $P\big(\max_{s\in[0,T]} X(s) > z\big)$ as z goes to infinity. Some examples are also reported.

2000 AMS Mathematics Subject Classification: Primary: 60J60, 60J65; Secondary: 60H10.

Key words and phrases: Diffusion process, Brownian motion, first-crossing time, random time-change.

1. INTRODUCTION

In this note, we consider a temporally homogeneous one-dimensional diffusion X and we study the distribution of the maximum process $S_t = \max_{s\in[0,t]} X(s)$ or, equivalently, the distribution of the first-crossing time τ_z of $X(t)$ through a given threshold value z. The knowledge of the distribution of S_t is very useful in a variety of applications ranging from biology and engineering to mathematical finance: for instance, when modelling neural activity, in queueing theory, in ruin problems, in modelling option prices.

Really, unlike the case of Brownian motion (BM), for general stochastic processes closed formulae for $P(S_t \leqslant z)$ are not available; so in certain applications one is satisfied with the determination of the tail behaviour of $P(S_T > z)$ for some fixed $T > 0$. The evaluation of the tail probability $P(S_T > z)$ for a fixed z is a key point in many statistical problems (for some examples in parametric and nonparametric statistics, in imaging processing and in genetical problems, see the references quoted in [10]).

For instance, a numerical method to compute the distribution of the maximum of a Gaussian random process was developed in [10]. Some theoretical results on Gaussian random fields were proved e.g. in [13], [14], [6]; in particular, if $X(t)$ is a Gaussian process with stationary increments, under certain conditions on the variance and covariance functions of the process, in [7] and [20] the following estimate was shown for $z \to +\infty$:

$$P(S_T > z) \sim \text{const} \cdot z^{\beta} \Psi\left(\frac{z}{\sigma_X(T)}\right), \tag{1.1}$$

where β is a positive constant, $\Psi(x) = P(W > x)$ is the tail distribution of a standard Gaussian random variable W, and $\sigma_X^2(t)$ is the variance function of $X(t)$ (here $f(x) \sim g(x)$ means that $\lim_{x\to+\infty} f(x)/g(x) = 1$). In particular, as well known, if $X(t)$ is (standard) BM, for any $z \geqslant 0$ the following equality holds:

$$P(S_T > z) = 2\Psi\left(\frac{z}{\sqrt{T}}\right). \tag{1.1'}$$

A different type of asymptotics was obtained in [3] for a certain class of diffusions $X(t)$ by studying the behaviour of $P(S_T \leqslant z)$ for fixed $z > 0$, as $T \to \infty$. By a variable change, $X(t)$ transforms into a local martingale $Y(t)$ (see Section 2); if its quadratic variation is bounded from above and below by two deterministic increasing functions $\beta(t)$ and $\alpha(t)$, then (see [3])

$$\liminf_{T\to\infty} P(S_T \leqslant z)\sqrt{\beta(T)} \geqslant c_-, \qquad \limsup_{T\to\infty} P(S_T \leqslant z)\sqrt{\alpha(T)} \leqslant c_+, \tag{1.2}$$

where c_- and c_+ are positive constants.

Another interesting aspect of the problem is to estimate $P(S_T > z)$ in the case when T is a random variable independent of X. The asymptotics of $P(S_T > z)$, as $z \to +\infty$, can be reduced, e.g., to those of an overflow probability in queueing theory (see [23]); another interpretation is that of a ruin probability for a certain risk process (see [23]).

Let $X(t)$ be a Gaussian process with stationary increments, and assume that T is a nonnegative random variable, independent of X, with regularly varying tail distribution at ∞ with index $\nu > 0$, i.e. $P(T > x) = L(x)x^{-\nu}$, $L(\cdot)$ being a function slowly varying at ∞. Moreover, suppose that the variance function $\sigma_X^2(t)$ is continuous on $[0, +\infty)$ and it is regularly varying at infinity with index $\alpha \in (0, 2]$. Then, under an additional condition on the behaviour of $\sigma_X(t)$ at $t = 0$, the following holds (see [12]):

$$P(S_T > z) \sim C \cdot P\big(T > \sigma_X^{-1}(z)\big) \quad \text{as } z \to +\infty, \tag{1.3}$$

where $C = E\left(\max_{t\in[0,1]} B_\alpha(t)\right)^{\nu/\alpha}$ and $B_\alpha(t)$ denotes a fractional Brownian motion (FBM) with Hurst parameter $\alpha \in (0, 1]$ (i.e. a centered Gaussian process with stationary increments, continuous sample paths, $B_\alpha(0) = 0$ and $\mathrm{Var}\big(B_\alpha(t)\big) = t^{2\alpha}$). In the special case when $X(t)$ is FBM itself, formula (1.3) is nothing but the classical result of Breiman [8].

In this paper, motivated by the intention of extending the results above to Itô processes, we consider a class of temporally homogeneous one-dimensional diffusion processes $X(t),\ t \geqslant 0$, characterized by drift $b(x)$ and infinitesimal variance $\sigma^2(x)$, where $b(\cdot)$ and $\sigma(\cdot)$ are regular enough functions. Therefore, our aim is to study the rate of decay of $P(S_T > z)$ as $z \to \infty$, and to show that results on the maximum process S_t, analogous to (1.1) and (1.3), hold for $X(t)$. Moreover, we show that the distribution of the first instant at which $X(t)$ attains the maximum in the interval $[0, T]$ follows a compound arc-sine law. All this is done by reducing the process $X(t)$ to BM, by using the arguments considered in [3] and [5], that is, by combining a deterministic transformation of the process X with a random time-change.

2. NOTATION AND MAIN RESULTS

Let $X(t) \in I = (r_1, r_2)$ $(-\infty \leqslant r_1 \leqslant 0 < r_2 \leqslant +\infty)$ be the solution of the stochastic differential equation (SDE):

$$dX(t) = b\big(X(t)\big)dt + \sigma\big(X(t)\big)dB_t, \qquad X(0) = 0, \tag{2.1}$$

where B_t is (standard) BM. Throughout the paper we will suppose that the usual conditions (see e.g. [15], [17]) for the existence and uniqueness of the solution of (2.1) are satisfied. Moreover, we require that $X(t) \in I$ for all $t \geqslant 0$ (for conditions implying this see e.g. [2], [16], [17]).

For $x, y \in I = (r_1, r_2)$, let $\tau_y(x) \doteq \inf\{t > 0 : X(t) = y \mid X(0) = x\}$ be the first-hitting time of X to y when starting from x. We recall that the diffusion $X(t)$ is said to be *regular* if for any $x,\ y \in I$ the condition $P\big(\tau_y(x) < \infty\big) > 0$ holds, while it is said to be *recurrent* if for any $x,\ y \in I$ the condition $P\big(\tau_y(x) < \infty\big) = 1$ is satisfied (see e.g. [16], [3]). Let us consider now the infinitesimal generator L associated with the diffusion (2.1):

$$Lh(x) = b(x)h'(x) + \frac{1}{2}h''(x)\sigma^2(x), \qquad h \in C^2(I). \tag{2.2}$$

The *scale* function $u(x)$ is the solution of the problem:

$$Lu(x) = 0,\ x \in I; \qquad u(0) = 0,\ u'(0) = 1; \tag{2.3}$$

u is strictly increasing and it is explicitly given by

$$u(x) = \int_0^x \exp\bigg(-\int_0^t \frac{2b(z)}{\sigma^2(z)}dz\bigg)dt. \tag{2.4}$$

If the boundaries r_1 and r_2 of I are unattainable (see e.g. [15], [16]), the recurrence of $X(t)$ is equivalent to $u(x) \to \infty$ as $x \to r_i$. For instance, BM is recurrent, being in this case $u(x) = x$. The process $Y(t) \doteq u\big(X(t)\big)$ turns out to be a local martingale, since by Itô's formula we have $dY(t) = u'\Big(u^{-1}\big(Y(t)\big)\Big)\sigma\Big(u^{-1}\big(Y(t)\big)\Big)dB_t$. The quadratic variation of the process $Y(t)$ will be denoted by

$$\langle Y\rangle_t = \int_0^t \big[u'\big(X(s)\big)\sigma\big(X(s)\big)\big]^2 ds.$$

Finally, we say that the diffusion process $X(t) \in I$, which is the solution of the SDE (2.1), is *conjugated to BM* if there exists an increasing differentiable function $v : I \longrightarrow \mathbb{R}$ with $v(0) = 0$, such that the process $Z(t) \doteq v\big(X(t)\big)$ is BM. Notice that if $X(t)$ is conjugated to BM, then X is recurrent.

Now, we go to investigate the distribution of $S_T = \max_{s\in[0,T]} X(s)$, where $X(t)$ is the solution of (2.1). The maximum of the process $X(t)$ is naturally related to $\tau_z \doteq \inf\{t > 0 : X(t) \geqslant z\}$; in order to make the first-crossing time problem meaningful, we shall assume that $X(t)$ is recurrent.

THEOREM 2.1. *Let $T > 0$ be given, let us assume that the solution $X(t)$ of* (2.1) *is recurrent, and that $\langle Y\rangle_\infty = \infty$, where $Y(t) = u\big(X(t)\big)$. Moreover, we suppose that there exist two deterministic, continuous increasing functions $\alpha(t)$ and $\beta(t)$, with $\alpha(0) = \beta(0) = 0$, such that, for every $t < T$, $\alpha(t) \leqslant \langle Y\rangle_t \leqslant \beta(t)$. Then, for any $z > 0$,*

$$2\Psi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right) \leqslant P(S_T > z) \leqslant 2\Psi\left(\frac{u(z)}{\sqrt{\beta(T)}}\right), \tag{2.5}$$

where

$$\Phi(x) = 1 - \Psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt,$$

and $u(\cdot)$ is given by (2.4).

Proof. For $z > 0$ we have

$$P(S_T \leqslant z) = P\Big(\max_{t\in[0,T]} u\big(X(t)\big) \leqslant u(z)\Big) = P\big(\max_{t\in[0,T]} Y(t) \leqslant u(z)\big).$$

Since $\langle Y\rangle_\infty = \infty$, we can use a random time-change (see e.g. [21]), which implies that there exists a Wiener process $\tilde B_t$ such that a.s. $Y(t) = \tilde B_{\langle Y\rangle_t}$. Thus

$$P(S_T \leqslant z) = P\big(\max_{t\in[0,T]} \tilde B_{\langle Y\rangle_t} \leqslant u(z)\big) = P\big(\max_{t\in[0,\langle Y\rangle_T]} \tilde B_t \leqslant u(z)\big).$$

Since $\alpha(t) \leqslant \langle Y\rangle_t \leqslant \beta(t)$, we obtain

$$\max_{t\in[0,\alpha(T)]} \tilde B_t \leqslant \max_{t\in[0,\langle Y\rangle_T]} \tilde B_t \leqslant \max_{t\in[0,\beta(T)]} \tilde B_t,$$

that is,

$$P\big(\max_{t\in[0,\beta(T)]}\tilde{B}_t \leqslant u(z)\big) \leqslant P\big(\max_{t\in[0,\langle Y\rangle_T]}\tilde{B}_t \leqslant u(z)\big)$$
$$\leqslant P\big(\max_{t\in[0,\alpha(T)]}\tilde{B}_t \leqslant u(z)\big).$$

Thus, (2.5) follows by $(1.1')$. ∎

If the quadratic variation $\langle Y\rangle_t$ of $Y(t)$ is deterministic, (2.5) becomes

$$P(S_T > z) = 2\Psi\left(\frac{u(z)}{\sqrt{\alpha(T)}}\right)$$

(note the affinity with (1.1)). Moreover, if $X(t)$ is conjugated to BM by means of the function v, then $\langle Y\rangle_t = \alpha(t) = \beta(t) = t$, so we obtain

$$P(S_T > z) = 2\Psi\left(\frac{v(z)}{\sqrt{T}}\right).$$

Let T be a nonnegative random variable whose distribution $F(t) = P(T \leqslant t)$ has regularly varying tails with index $\nu \geqslant 0$, that is, $P(T > t) = L(t)t^{-\nu}$, $L(\cdot)$ being a function slowly varying at $+\infty$ (i.e. $\lim_{x,y\to+\infty} L(x)/L(y) = 1$). We will write $T \in RV(\nu)$, following the notation of [12]. A slightly weaker tail behaviour arises in the case when two positive constants a and b exist such that, as $t \to +\infty$, $aL(t)t^{-\nu} \leqslant 1 - F(t) \leqslant bL(t)t^{-\nu}$. In this case we will write $T \in V(\nu)$.

Before studying the case of a diffusion, we consider BM with drift $\mu\in\mathbb{R}$. For T deterministic, the following holds (see [11]).

PROPOSITION 2.1. *Let T be given and fixed and let $\mu < 0$; then, as $z \to \infty$,*

$$P\big(\max_{t\in[0,T]}(B_t + \mu t) > z\big) \sim \Psi\left(\frac{z-\mu T}{\sqrt{T}}\right). \tag{2.6}$$

For T random we have (see [23])

THEOREM 2.2. *Let $T \in RV(\nu)$. Then $B_T + \mu T$ and $\max_{t\in[0,T]}(B_t+\mu t)$ are tail-equivalent, i.e.*

$$P\big(\max_{t\in[0,T]}(B_t + \mu t) > z\big) \sim P(B_T + \mu T > z) \quad \textit{as } z \to +\infty. \tag{2.7}$$

Now, let us go to consider a diffusion process which is the solution of the SDE (2.1); we obtain the following preliminary results:

PROPOSITION 2.2. *Let $T \in RV(\nu)$ and let us suppose that all the assumptions of Theorem 2.1 are satisfied. Then*

$$L\big(\alpha^{-1}(z)\big)\big(\alpha^{-1}(z)\big)^{-\nu} \leqslant P(\langle Y\rangle_T > z) \leqslant L\big(\beta^{-1}(z)\big)\big(\beta^{-1}(z)\big)^{-\nu}, \tag{2.8}$$

where $\langle Y\rangle_t,\ \alpha(t),\ \beta(t)$ are defined in Theorem 2.1.

P r o o f. From the inequalities $\alpha(t) \leqslant \langle Y \rangle_t \leqslant \beta(t)$ we obtain

$$P\big(\alpha(T) > z\big) \leqslant P(\langle Y \rangle_T > z) \leqslant P\big(\beta(T) > z\big),$$

i.e.

$$P\big(T > \alpha^{-1}(z)\big) \leqslant P(\langle Y \rangle_T > z) \leqslant P\big(T > \beta^{-1}(z)\big).$$

Since $T \in RV(\nu)$, the assertion (2.8) immediately follows. ∎

PROPOSITION 2.3. *Under the assumptions of Proposition* 2.2, *if* $\alpha^{-1}(t)$ *and* $\beta^{-1}(t)$ *are regularly varying at* $+\infty$ *with index* $\gamma > 0$, *i.e. there exist constants* $c_\alpha,\ c_\beta > 0$ *such that* $\alpha^{-1}(t) \sim c_\alpha t^\gamma$ *and* $\beta^{-1}(t) \sim c_\beta t^\gamma$ *as* $t \to \infty$, *then* $\langle Y \rangle_T \in V(\gamma\nu)$. *In the case when* $c_\alpha = c_\beta$, *we have* $\langle Y \rangle_T \in RV(\gamma\nu)$.

P r o o f. Since $\alpha^{-1}(t)$ and $\beta^{-1}(t)$ are regularly varying at $+\infty$ with index γ, it follows that as $z \to \infty$, $\big(\alpha^{-1}(z)\big)^{-\nu} \sim c_\alpha z^{-\gamma\nu}$, $\big(\beta^{-1}(z)\big)^{-\nu} \sim c_\beta z^{-\gamma\nu}$, and so $\langle Y \rangle_T \in V(\gamma\nu)$. The other assertion can be trivially verified. ∎

REMARK 2.1. If $\langle Y \rangle_t$ is deterministic, then $\langle Y \rangle_t = \alpha(t) \equiv \beta(t)$, so Proposition 2.2 implies that $P(\langle Y \rangle_T > z) = L\big(\alpha^{-1}(z)\big)\big(\alpha^{-1}(z)\big)^{-\nu}$, while if $c = c_\alpha = c_\beta$, Proposition 2.3 implies that $P(\langle Y \rangle_T > z) \sim c\tilde{L}(z) z^{-\gamma\nu}$.

Now, we can obtain

THEOREM 2.3. *Let* $T \in RV(\nu)$ *and let us suppose that all the assumptions of Theorem* 2.1 *are satisfied. Moreover, let us assume that the functions* $\alpha^{-1}(t)$ *and* $\beta^{-1}(t)$ *are regularly varying at* $+\infty$ *with index* $\gamma > 0$. *Then for* $z \to +\infty$ *it follows that*

$$L\big(\alpha^{-1}(z^2)\big)\big(\alpha^{-1}(z^2)\big)^{-\nu} \leqslant P(S_T > z) \leqslant L\big(\beta^{-1}(z^2)\big)\big(\beta^{-1}(z^2)\big)^{-\nu}, \tag{2.9}$$

where L *is a function slowly varying at* ∞.

P r o o f. We recall from [12] that if $\tilde{B}_t$ is BM and Λ is a nonnegative random variable such that $\Lambda \in RV(\mu)$, then as $z \to \infty$

$$P(\max_{s \in [0,\Lambda]} \tilde{B}_s > z) = P(\sqrt{\Lambda} \max_{s \in [0,1]} \tilde{B}_s > z) \sim E(\max_{s \in [0,1]} \tilde{B}_s)^{2\mu} P(\Lambda > z^2). \tag{2.10}$$

Moreover,

$$E(\max_{s \in [0,1]} \tilde{B}_s)^{2\mu} = \frac{2^\mu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \mu\right) \doteq \mathcal{E}(\mu).$$

Then

$$\begin{aligned} P(S_T > z) &= P\big(\max_{s \in [0,T]} Y(s) > u(z)\big) \\ &= P\big(\max_{s \in [0,T]} \tilde{B}_{\langle Y \rangle_s} > u(z)\big) = P\big(\max_{s \in [0,\rho(T)]} \tilde{B}_s > u(z)\big). \end{aligned}$$

Consequently, if $\langle Y\rangle_t$ is deterministic, we have $\langle Y\rangle_T \in RV(\gamma\nu)$. Therefore, by (2.10) with $\Lambda = \langle Y\rangle_T$ and $\mu = \gamma\nu$, we get, for $z \to \infty$,

$$P(S_T > z) \sim E(\max_{s\in[0,1]} \tilde{B}_s)^{2\gamma\nu} P(\langle Y\rangle_T > z^2) = \mathcal{E}(\gamma\nu) L(z^2) z^{-2\gamma\nu}.$$

If $\langle Y\rangle_t$ is non-deterministic, by the same arguments as those used in the proof of Proposition 2.3, we see that $a, b > 0$ exist such that

$$aE(\max_{s\in[0,1]} \tilde{B}_s)^{2\gamma\nu} P(\langle Y\rangle_T > z^2) \leqslant P(S_T > z) \leqslant bE(\max_{s\in[0,1]} \tilde{B}_s)^{2\gamma\nu} P(\langle Y\rangle_T > z^2).$$

Since $P\big(T > \alpha^{-1}(z^2)\big) \leqslant P(\langle Y\rangle_T > z^2) \leqslant P\big(T > \beta^{-1}(z^2)\big)$, the assertion (2.9) easily follows. ∎

REMARK 2.2. A trivial case where $\langle Y\rangle_t$ is deterministic occurs when $X(t) \equiv Y(t) = \int_0^t \bar{\sigma}(s) dB_s$, where $\bar{\sigma}(\cdot) > 0$ is a deterministic function and $\langle Y\rangle_t = \int_0^t \bar{\sigma}^2(s) ds$ behaves like $t^\gamma,\ t \to \infty$. For such a process, if T is given and fixed, it follows that

$$P(S_T > z) = 2\Psi\big(z/\sqrt{\textstyle\int_0^T \bar{\sigma}^2(s) ds}\big).$$

Instead, if $T \in RV(\nu)$, let $\rho^{-1}(s)$ be the inverse function of $\rho(t) \doteq \langle Y\rangle_t$ and let us suppose that $\rho^{-1}(s)$ behaves like s^γ $(s \to \infty)$. By using also (2.9), for $z \to \infty$ we obtain

$$\begin{aligned} P(S_T > z) &= P(\max_{s\in[0,T]} B_{\rho(s)} > z) \\ &= P(\max_{s\in[0,\rho(T)]} B_s > z) \sim \mathcal{E}(\gamma\nu) P\big(\rho(T) > z^2\big) \\ &= \mathcal{E}(\gamma\nu) P\Big(\int_0^T \bar{\sigma}^2(s) ds > z^2\Big) = \mathcal{E}(\gamma\nu) L(z^2) z^{-2\gamma\nu}. \end{aligned}$$

THE ARC-SINE LAW

Let us consider now, for T given and fixed, the first instant θ at which $X(t)$ attains its maximum value in the interval $[0, T]$, i.e.

$$X(\theta) = \max_{t\in[0,T]} X(t) = S_T.$$

Notice that θ is not a stopping time. As is well known (see [19]), when $X(t) \equiv B_t$, the distribution of θ follows the arc-sine law, that is:

$$P(\theta \leqslant t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{T}}, \quad 0 < t < T. \tag{2.11}$$

Let $X(t)$ be a diffusion process satisfying the assumptions of Theorem 2.1; first we suppose that the quadratic variation $\langle Y \rangle_t = \rho(t)$ is deterministic. Then

$$u(S_T) = \max_{t \in [0,T]} u\big(X(t)\big) = \max_{t \in [0,T]} \tilde{B}_{\rho(t)} = \max_{s \in [0,\rho(T)]} \tilde{B}_s,$$

and so $\tilde{B}_{\rho(\theta)} = u\big(X(\theta)\big) = \max_{t \in [0,\rho(T)]} \tilde{B}_t$. Therefore, $\rho(\theta)$ follows the arc-sine law, i.e.

$$P\big(\rho(\theta) \leqslant t\big) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{\rho(T)}}, \quad t \in [0, \rho(T)].$$

Thus,

$$P(\theta \leqslant t) = P\big(\rho(\theta) \leqslant \rho(t)\big) = \frac{2}{\pi} \arcsin \sqrt{\frac{\rho(t)}{\rho(T)}}, \quad t \in [0, T].$$

In particular, if $X(t)$ is conjugated to BM, then $\rho(t) = t$, and so θ follows the arc-sine law. If $\rho(t)$ is not deterministic, recalling that $\alpha(t) \leqslant \rho(t) \leqslant \beta(t)$, we get $\max_{t \in [0,\alpha(T)]} \tilde{B}_t \leqslant u(S_T) \leqslant \max_{t \in [0,\beta(T)]} \tilde{B}_t$. If we denote by $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$ the first instant at which $\tilde{B}_t$ attains its maximum in the interval $[0, \alpha(T)]$ and in the interval $[0, \beta(T)]$, respectively, we obtain $\tilde{\theta}_\alpha \leqslant \rho(\theta) \leqslant \tilde{\theta}_\beta$. Thus,

$$\frac{2}{\pi} \arcsin \sqrt{\frac{t}{\beta(T)}} \leqslant P\big(\rho(\theta) \leqslant t\big) \leqslant \frac{2}{\pi} \arcsin \sqrt{\frac{t}{\alpha(T)}}, \quad t \in \big(0, \alpha(T)\big),$$

and therefore

$$\frac{2}{\pi} \arcsin \sqrt{\frac{\alpha(t)}{\beta(T)}} \leqslant P(\theta \leqslant t) \leqslant \frac{2}{\pi} \arcsin \sqrt{\frac{\beta(t)}{\alpha(T)}}, \quad 0 < t < \beta^{-1}\big(\alpha(T)\big).$$

In the special case when $X(t)$ is an integral process with deterministic integrand, by combining the result of [4] and that of [22], we are able to obtain

THEOREM 2.4. *Let $X(t) = \int_0^t \bar{\sigma}(s) dB_s$, where $\bar{\sigma}(\cdot)$ is a deterministic positive function, and let us suppose that $\rho(\infty) = \infty$, where $\rho(t) = \int_0^t \bar{\sigma}^2(s) ds$. Then the following holds:*

$$\inf_{\tau \in [0,1]} E[S_T - X(\tau)]^2 = \inf_{\tau \in [0,1]} E[X(\theta) - X(\tau)]^2 = \rho(1) \cdot \inf_{\tau \in [0,1]} \left[E|\tilde{\theta} - \tau| + \frac{1}{2} \right],$$

where the infimum is taken over all stopping times $\tau \in [0,1]$ of $X(t)$ and $\tilde{\theta}$ is the first instant at which $\tilde{B}_t$ attains its maximum in the interval $[0,1]$.

Moreover, let us consider the following two optimal stopping problems:

$$\text{(i)} \quad \inf_{\tau \in (0,1)} E\left[X(\theta) - X(\tau)\right]^2 \quad \textit{and} \quad \text{(ii)} \quad \inf_{\tau \in (0,1)} E\left[|\theta - \tau|\right]^2.$$

Then the optimal stopping times in (i) *and* (ii) *are the same and they are equal to*

$$\tau^* = \inf\{0 < t < \rho(1) : \max_{s\in[0,t]} B_s - B_t \geqslant z_0^* \sqrt{\rho(1) - t}\},$$

where z_0^* *is the unique positive root of the equation* $4\Phi(z) - 2z\phi(z) - 3 = 0$, *and* $\phi(z)$ *and* $\Phi(z)$ *are the density and the distribution function*, *respectively*, *of a standard Gaussian variable. If one considers the optimal stopping problem* (i) *or* (ii) *in the interval* $[0, T]$, *in the above formulae* $\rho(1)$ *has to be replaced with* $\rho(T)$.

P r o o f. Let $(\mathcal{F}_t^{\tilde{B}})_{0\leqslant t\leqslant 1}$ be the filtration generated by $\tilde{B}_t$. Then by Lemma 1 of [22] which is true for BM, for any $(\mathcal{F}_t^{\tilde{B}})$-stopping time $\tau \in [0, 1]$, we have

$$E[\tilde{B}_{\tilde{\theta}} - B_\tau]^2 = E[|\tilde{\theta} - \tau|] + \frac{1}{2}.$$

By reducing $X(t)$ to BM, we obtain

$$\begin{aligned}\inf_{\tau\in[0,1]} E\,[X(\theta) - X(\tau)]^2 &= \inf_{\tau\in[0,1]} E[\tilde{B}_{\rho(\theta)} - \tilde{B}_{\rho(\tau)}]^2 \\ &= \inf_{\tau'\in[0,\rho(1)]} E[\tilde{B}_{\theta'} - \tilde{B}_{\tau'}]^2,\end{aligned}$$

where we have set $\theta' = \rho(\theta)$ and $\tau' = \rho(\tau) \in [0, \rho(1)]$. By the scaling property of BM, $\tilde{B}_{\rho(1)\cdot t}/\sqrt{\rho(1)} \doteq W_t$ is also BM. Thus, setting $s = \tau'/\rho(1)$, we can write the last infimum as

$$\rho(1) \inf_{s\in[0,1]} E[\max_{r\in[0,1]} W_r - W_s]^2 = \rho(1) \inf_{\tau\in[0,1]} E[\tilde{B}_{\tilde{\theta}} - \tilde{B}_\tau]^2,$$

which implies the main assertion. The other assertions are obtained by using Theorem 2.2 of [4]. ∎

3. SOME EXAMPLES

EXAMPLE 3.1 (*Ornstein–Uhlenbeck process*). For $b, \sigma > 0$, let us consider the process $X(t)$ such that $dX(t) = -bX(t)dt + \sigma dB_t,\ X(0) = x_0$. The explicit solution is $X(t) = e^{-bt}U(t)$, where $U(t) = x_0 + \int_0^t \sigma e^{bs} dB_s$. Setting $Y(t) = \int_0^t \sigma e^{bs} dB_s$ and using a random time-change, we obtain $U(t) = x_0 + B_{\rho(t)}$, where the quadratic variation $\langle Y\rangle_t = (\sigma^2/2b)(e^{2bt} - 1)$ is deterministic. Thus, we have

$$\begin{aligned}P(S_T > z) &= P\big(\max_{s\in[0,T]} e^{-bs}U(s) > z\big) \leqslant P\big(\max_{s\in[0,T]} U(s) > z\big) \\ &= P(\max_{s\in[0,T]} B_{\rho(s)} > z - x_0) = P(\max_{t\in[0,\rho(T)]} B_t > z - x_0).\end{aligned}$$

If T is given and fixed, we obtain

$$P(S_T > z) \leqslant 2\Psi\left(\frac{z - x_0}{\sqrt{\rho(T)}}\right).$$

Instead, if T is random and $P(T > z) = L(z)\big(\rho(z)\big)^{-\nu}$ $(z \to \infty)$, then we have $P\big(\rho(T) > z\big) = P\big(T > \rho^{-1}(z)\big) = L\big(\rho^{-1}(z)\big)z^{-\nu}$, and so $\rho(T) \in RV(\nu)$. By using Theorem 2.3, we obtain

$$P\big(\max_{s\in[0,T]} X(s) > z\big) \leqslant \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} + \nu\right)L\Big(\rho^{-1}\big((z - x_0)^2\big)\Big)(z - x_0)^{-2\nu}.$$

EXAMPLE 3.2 (*Feller process*). Let us consider the process $X(t) \in [0, +\infty)$ such that $dX(t) = \frac{1}{4}dt + \sqrt{X(t) \vee 0}\, dB_t,\ X(0) = 0$. The process $X(t)$ is conjugated to Brownian motion by means of the function $v(x) = 2\sqrt{x}$. Thus, if T is given and fixed, we obtain

$$\begin{aligned} P(S_T > z) &= P\Big(\max_{s\in[0,T]} v\big(X(s)\big) > v(z)\Big) \\ &= P\big(\max_{s\in[0,T]} B_s > 2\sqrt{z}\big) = 2\Psi\left(\frac{2\sqrt{z}}{\sqrt{T}}\right). \end{aligned}$$

If $T \in RV(\nu)$, for $z \to \infty$ we get

$$\begin{aligned} P(S_T > z) &= P\big(\max_{s\in[0,T]} B_s > 2\sqrt{z}\big) \sim \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} + \nu\right)P(T > 4z) \\ &= \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} + \nu\right)L(4z)(4z)^{-\nu}. \end{aligned}$$

For what concerns θ, it follows the arc-sine law, since $X(t)$ is conjugated to BM.

EXAMPLE 3.3 (*Geometrical Brownian motion*). For $b, \sigma > 0$, let us consider the diffusion equation $dX(t) = bX(t) + \sigma X(t)dB_t,\ X(0) = x_0$. In the framework of the Black–Scholes model in mathematical finance, it describes the time course of the price $X(t)$ of risky assets. The explicit solution is given by $X(t) = x_0 \exp\big((b - \sigma^2/2)t + \sigma B_t\big)$. If $b \leqslant \sigma^2/2$, we get

$$P(S_T > z) \leqslant P\left(\max_{s\in[0,T]} \exp(\sigma B_s) > \frac{z}{x_0}\right) = P\left(\max_{s\in[0,T]} B_s > \frac{\log(z/x_0)}{\sigma}\right).$$

Consequently, if T is given and fixed, we have $P(S_T > z) \leqslant 2\Psi\Big(\frac{\log(z/x_0)}{\sigma\sqrt{T}}\Big)$; if $T \in RV(\nu)$, for $z \to \infty$ we obtain

$$P(S_t > z) \leqslant \frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} + \nu\right)L\left(\frac{\log^2(z/x_0)}{\sigma^2}\right)\left(\frac{\log(z/x_0)}{\sigma}\right)^{-2\nu}.$$

EXAMPLE 3.4 (*Wright & Fisher-like process*). Let $X(t)$ be such that

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2}X(t)\right)dt + \sqrt{X(t)\big(1 - X(t)\big) \vee 0}\, dB_t, \qquad X(0) = 0.$$

It is a particular case of the Wright & Fisher diffusion equation for population genetics, and it is also used in certain diffusion models for neural activity (see [18]). It can be shown (see e.g. [1] and [2]) that $X(t) \in [0, 1]$ for all $t \geqslant 0$, and it is conjugated to BM by means of the function $v(x) = 2 \arcsin \sqrt{x}$. For T given and fixed, we have

$$P(S_T > z) = 2\Psi\left(\frac{2 \arcsin \sqrt{z}}{\sqrt{T}}\right).$$

Clearly, θ follows the arc-sine law.

EXAMPLE 3.5. The usefulness of inequalities (2.5) for T fixed, and (2.9) for $T \in RV(\nu)$, relies on the fact that the function $\alpha(t)$ is close enough to $\beta(t)$. Here we show an example of diffusion satisfying all the assumptions of Section 2, for which this holds. Let $\sigma > 0$ and $\epsilon > 0$, and consider the SDE

$$dX(t) = \frac{\epsilon\sigma^2 \sin\big(2X(t)\big)}{2\big(1 + \epsilon \cos^2 X(t)\big)} dt + \sigma dB_t.$$

We have

$$u(x) = \frac{1}{1 + \epsilon}\left[\left(1 + \frac{\epsilon}{2}\right)x + \frac{\epsilon}{4}\sin(2x)\right].$$

As easily seen, $X(t)$ is recurrent, and

$$\langle Y \rangle_t = \int_0^t \frac{\big(1 + \epsilon \cos^2 X(s)\big)^2}{(1 + \epsilon)^2} \sigma^2 ds$$

(note that $\langle Y \rangle_t$ is non-deterministic). Thus

$$\alpha(t) \doteq \frac{\sigma^2 t}{(1 + \epsilon)^2} \leqslant \langle Y \rangle_t \leqslant \sigma^2 t \doteq \beta(t).$$

Evidently, if $\epsilon \simeq 0$, then $\beta(t) \simeq \alpha(t)$.

EXAMPLE 3.6 (*A temporally non-homogeneous SDE*). For $t \in [0, 1]$, let us consider the SDE

$$dZ(t) = -\frac{Z(t)}{1 - t} dt + dB_t, \qquad Z(0) = Z(1) = 0,$$

whose solution is

$$Z(t) = (1-t)\int_0^t \frac{1}{1-s}\,dB_s.$$

The diffusion $Z(t)$ is the Brownian bridge, i.e. BM conditioned to take the value 0 at time $t=1$. Now, set $X(t) = Z(t)/(1-t)$. The quadratic variation of $X(t)$ is $\langle X\rangle_t = \rho(t) = t/(1-t),\ 0 \leqslant t \leqslant 1$. Consequently, by a random time-change we obtain $X(t) = \tilde{B}\big(t/(1-t)\big)$ for a suitable BM $\tilde{B}$. If $T \in (0,1)$ is given and fixed and $z > 0$, we have

$$P(S_T > z) = P\big(\max_{t\in[0,T]} X(t) > z\big) = 2\Psi\left(z\Big/\sqrt{\frac{T}{1-T}}\right).$$

If τ^Z denotes the first-passage time of $Z(t)$ over the straight line $y = z(1-t)$, we get $P(S_T > z) = P(\tau^Z \leqslant T)$. The distribution of the first instant θ at which $X(t)$ attains its maximum is

$$P(\theta \leqslant t) = \frac{2}{\pi}\arcsin\sqrt{\frac{t(1-T)}{T(1-t)}}, \qquad 0 \leqslant t < T < 1.$$

4. CONCLUSION AND FINAL REMARKS

For a certain class of one-dimensional diffusions $X(t)$, we have studied the rate of decay of $P\big(\max_{s\in[0,T]} X(s) > z\big)$ as $z \to \infty$, both in the case when T is fixed (Theorem 2.1) and when T is a random variable independent of the process X (Theorem 2.3). Moreover, we have studied the distribution of the first instant at which $X(t)$ attains its maximum. By combining a deterministic transformation of the process $X(t)$ with a random time-change, the quantities under investigation have been related to those regarding a suitable Brownian motion. It is worthy to note that, by using the Donsker's approximating procedure, it is possible to approximate B_t by means of a binomial random walk. Precisely, let $\{U_n\}_{n\geqslant 1}$ be a sequence of independent identically distributed random variables such that $P(U_n=-1)=P(U_n=1)=\frac{1}{2}$. Setting $V_0=0$ and $V_n=U_1+\ldots+U_n$, $n \geqslant 1$, it is well known that the sequence of processes defined by

$$B_t^{(n)} = \big(V_{[nt]} + (nt-[nt])U_{[nt]+1}\big)/\sqrt{n}$$

converges in distribution to a standard BM as $n \to \infty$ (see e.g. [21]).

Now, let $X(t)$ be a diffusion belonging to the class considered in Theorem 2.1 or Theorem 2.3, for which $\rho(t)=\langle Y\rangle_t$ is deterministic. Using the arguments and the notation of Section 2, we are able to infer that the sequence of processes defined by

$$X^{(n)}(t) = u^{-1}\Big(\Big(V_{[n\rho(t)]} + \big(n\rho(t) - [n\rho(t)]\big)U_{[n\rho(t)]+1}\Big)/\sqrt{n}\Big)$$

converges in distribution to $X(t)$ as $n \to \infty$ (note, however, that this approximation scheme becomes somehow ambiguous if $\rho(t)$ is a random process). Thus, since quantities of the form $E[f(B_t, \max_{s\in[0,t]} B_s)]$ have been estimated, e.g. in [9], by using the Donsker's approximating procedure, analogous estimations could be obtained for quantities of the form $E\big[f\big(X(t), \max_{s\in[0,t]} X(s)\big)\big]$, by using $X^{(n)}(t)$.

We conclude noting that the arguments of this paper can be used also to obtain information about the distribution of $s(T) \doteq \min_{t\in[0,T]} X(t)$, by using the distribution of the minimum of BM, that is,

$$P(\min_{t\in[0,T]} B_t \leqslant x) = 1 \quad \text{if } x \geqslant 0,$$
$$P(\min_{t\in[0,T]} B_t \leqslant x) = 2\Phi\left(\frac{x}{\sqrt{T}}\right) \quad \text{if } x < 0.$$

Acknowledgements. I would like to express particular thanks to the referee for useful comments and suggestions which have improved the presentation of the whole paper.

REFERENCES

[1] M. Abundo, *Diffusion approximation of discrete Markov chains with binomial-like transition probabilities*, in: *Cybernetics and Systems '96*, R. Trappl (Ed.), Austrian Society for Cybernetics Studies, Vienna 1996, pp. 508–513.

[2] M. Abundo, *On some properties of one-dimensional diffusion processes on an interval*, Probab. Math. Statist. 17 (2) (1997), pp. 277–310.

[3] M. Abundo, *On the first-passage time of diffusion processes over a one-sided stochastic boundary*, Stochastic Anal. Appl. 21 (1) (2003), pp. 1–23.

[4] M. Abundo, *Stopping a stochastic integral process as close as possible to the ultimate value of a functional*, Sci. Math. Jpn. 60 (3) (2004), pp. 475–479.

[5] M. Abundo, *Limit at zero of the first-passage time density and the inverse problem for one-dimensional diffusions*, Stochastic Anal. Appl. 24 (2006), pp. 1119–1145.

[6] J. M. Azaïs and C. Delmas, *Asymptotic expansions for the distribution of the maximum of a Gaussian random field*, Extremes 2 (2002), pp. 181–212.

[7] S. M. Berman, *An asymptotic formula for the distribution of the maximum of Gaussian process with stationary increments*, J. Appl. Probab. 22 (1985), pp. 454–460.

[8] L. Breiman, *On some limit theorems similar to the arc-sin law*, Theory Probab. Appl. 10 (1965), pp. 323–331.

[9] R. Carbone, *Binomial approximation of Brownian motion and its maximum*, Statist. Probab. Lett. 69 (2004), pp. 271–285.

[10] C. Cierco-Ayrolles, A. Croquette and C. Delmas, *Computing the distribution of the maximum of Gaussian random processes*, Methodol. Comput. Appl. Probab. 5 (2003), pp. 427–438.

[11] K. Dębicki and T. Rolski, *A note on transient Gaussian fluid models*, Queueing Syst. 41 (2002), pp. 321–342.

[12] K. Dębicki, B. Zwart and S. Borst, *The supremum of a Gaussian process over a random interval*, Statist. Probab. Lett. 68 (2004), pp. 221–234.

[13] C. Delmas, *An asymptotic expansion for the distribution of the maximum of a class of Gaussian fields*, C.R. Acad. Sci. Paris Sér. I Math. 327 (1998), pp. 393–397.

[14] C. Delmas, *Distribution du maximum d'un champ aléatoire et applications statistiques*, Ph.D. Thesis, University Toulouse III, France, 2001.
[15] I. I. Gihman and A. V. Skorohod, *Stochastic Differential Equations*, Springer, New York 1972.
[16] R. Z. Has'minskij, *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, 1980.
[17] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland 1981.
[18] V. Lanska, P. Lansky and C. E. Smiths, *Synaptic transmission in a diffusion model for neural activity*, J. Theor. Biol. 166 (1994), pp. 393–406.
[19] P. Lévy, *Processus stochastiques et mouvement brownien*, Gauthier-Villars, Paris 1965.
[20] V. I. Piterbarg and V. Prisyazhnyuk, *Asymptotic behavior of the probability of a large excursion for a nonstationary Gaussian process*, Teor. Veroyatnost. Mat. Statist. 18 (1978), pp. 121–133.
[21] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, Berlin–Heidelberg 1991.
[22] M. A. Urosov, *On a property of the moment at which Brownian motion attains its maximum and some optimal stopping problems*, Theory Probab. Appl. 49 (1) (2005), pp. 169–176.
[23] B. Zwart, S. Borst and K. Dębicki, *Subexponential asymptotics of hybrid fluid and ruin models*, Ann. Appl. Probab. 15 (1A) (2005), pp. 500–517.

Dipartimento di Matematica
Università Tor Vergata
via della Ricerca Scientifica
00133 Roma, Italy
E-mail: abundo@mat.uniroma2.it

Received on 7.9.2006;
revised version on 1.6.2007