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LAWS OF LARGE NUMBERS FOR TWO TAILED PARETO RANDOM VARIABLES

BY

ANDRÉ ADLER (CHICAGO, ILLINOIS)

Abstract. We sample m random variables from a two tailed Pareto distribution. A two tailed Pareto distribution is a random variable whose right tail is px^{-2} and whose left tail is qx^{-2}, where $p+q=1$. Next, we look at the largest of these random variables and establish various Weak and Strong Laws that can be obtained with weighted sums of these random variables. The case of $m=1$ is completely different from $m>1$.

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1. INTRODUCTION

In this paper we observe weighted sums of order statistics from a two tailed Pareto distribution. We sample m i.i.d. random variables with density

$$f(x)=\begin{cases} qx^{-2} & \text{if } x\leqslant -1,\\ 0 & \text{if } -1<x<1,\\ px^{-2} & \text{if } x\geqslant 1,\end{cases}$$

where $p+q=1$. We then observe the largest of these m random variables, $X_{(m)}$. What is striking is that the case of $m=1$ and $m>1$ are completely different. If $m=1$, we are just looking at one random variable and both tails are equally important. However, if $m>1$, then q has a much lesser importance. It only appears as $p=1-q$ in our limits. Likewise, in the case of $m=1$, if $p=q=1/2$, then our random variables are symmetrical and both our Strong and Weak Laws will have a limit of zero. But when m exceeds one, that is never the case.

Our goal is to establish laws of large numbers for weighted sums of these random variables. It should be noted that $E|X|=\infty$ in every case. We will show which sequences of constants $\{a_n, n\geqslant 1\}$ and $\{b_n, n\geqslant 1\}$ will allow our partial sums $\sum_{n=1}^{N} a_n X_{(m)n}/b_N$ to converge to a nonzero constant. As usual, we set $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$ and we use the constant C to denote a generic real number that is not necessarily the same in each appearance.

2. STRONG LAWS

We first present the case of $m = 1$. Here the larger of the two tails influences our limit. If $p > q$, then the limit is positive, and if $p < q$, the limit is negative.

THEOREM 2.1. *If $\{X_n, n \geqslant 1\}$ are i.i.d. two tailed Pareto random variables, then for all $\beta > 0$ we have*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \big((\lg n)^{\beta-2}/n\big) X_n}{(\lg N)^{\beta}} = \frac{p-q}{\beta} \quad \textit{almost surely.}$$

Proof. Let $a_n = (\lg n)^{\beta-2}/n$, $b_n = (\lg n)^{\beta}$ and $c_n = b_n/a_n = n(\lg n)^2$. We use the partition

$$\begin{aligned}\frac{1}{b_N}\sum_{n=1}^{N} a_n X_n = \frac{1}{b_N}\sum_{n=1}^{N} a_n [X_n I(|X_n| \leqslant c_n) - EXI(|X| \leqslant c_n)] \\ + \frac{1}{b_N}\sum_{n=1}^{N} a_n X_n I(|X_n| > c_n) + \frac{1}{b_N}\sum_{n=1}^{N} a_n EXI(|X| \leqslant c_n).\end{aligned}$$

The first term vanishes almost surely by the Khintchine–Kolmogorov Convergence Theorem (see [1], p. 113) and Kronecker's lemma since

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{c_n^2} EX^2 I(|X| \leqslant c_n) &= \sum_{n=1}^{\infty} \frac{1}{c_n^2}\Big[\int_{-c_n}^{-1} q\,dx + \int_{1}^{c_n} p\,dx\Big] \\ &= \sum_{n=1}^{\infty} \frac{1}{c_n^2}[q(c_n - 1) + p(c_n - 1)] \\ &= \sum_{n=1}^{\infty} \frac{c_n - 1}{c_n^2} \leqslant \sum_{n=1}^{\infty} \frac{1}{c_n} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.\end{aligned}$$

The second term of the partition vanishes, with probability one, by the Borel–Cantelli lemma since

$$\begin{aligned}\sum_{n=1}^{\infty} P\{|X| > c_n\} &= \sum_{n=1}^{\infty} \Big[\int_{-\infty}^{-c_n} qx^{-2}dx + \int_{c_n}^{\infty} px^{-2}dx\Big] \\ &= \sum_{n=1}^{\infty} \frac{p+q}{c_n} = \sum_{n=1}^{\infty} \frac{1}{c_n} = \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.\end{aligned}$$

The truncated first moment is

$$\begin{aligned}EXI(|X|\leqslant c_n) &= \int_{-c_n}^{-1} qx^{-1}dx + \int_{1}^{c_n} px^{-1}dx\\ &= -q\lg c_n + p\lg c_n = (p-q)\lg c_n \sim (p-q)\lg n.\end{aligned}$$

Therefore

$$\frac{\sum_{n=1}^{N} a_n EXI(|X|\leqslant c_n)}{b_N} \sim \frac{(p-q)\sum_{n=1}^{N}(\lg n)^{\beta-1}/n}{(\lg N)^{\beta}} \to \frac{p-q}{\beta},$$

which completes the proof. ■

Now we turn our attention to our order statistics. Here the results are quite surprising. We start with a random sample of m random variables with the density

$$f(x) = \begin{cases} qx^{-2} & \text{if } x \leqslant -1,\\ 0 & \text{if } -1 < x < 1,\\ px^{-2} & \text{if } x \geqslant 1.\end{cases}$$

The largest of these, $X_{(m)}$, has the density

$$f_{X_{(m)}}(x) = \begin{cases} mq^m(-x)^{-m-1} & \text{if } x \leqslant -1,\\ 0 & \text{if } -1 < x < 1,\\ mp(1-p/x)^{m-1}x^{-2} & \text{if } x \geqslant 1.\end{cases}$$

Then we repeat this n times to form the sequence $\{X_{(m)n}, n \geqslant 1\}$. Our next result shows how these order statistics behave over time.

THEOREM 2.2. *If* $\{X_{(m)n}, n \geqslant 1\}$ *is a sample of the largest order statistics from a two tailed Pareto distribution, then for all* $\beta > 0$ *we have*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N}\big((\lg n)^{\beta-2}/n\big)X_{(m)n}}{(\lg N)^{\beta}} = \frac{mp}{\beta} \quad \textit{almost surely}.$$

P r o o f. As before, let $a_n = (\lg n)^{\beta-2}/n$, $b_n = (\lg n)^{\beta}$, $c_n = b_n/a_n = n(\lg n)^2$ and

$$\begin{aligned}&\frac{1}{b_N}\sum_{n=1}^{N} a_n X_{(m)n}\\ &= \frac{1}{b_N}\sum_{n=1}^{N} a_n[X_{(m)n}I(|X_{(m)n}|\leqslant c_n) - EX_{(m)n}I(|X_{(m)n}|\leqslant c_n)]\\ &\quad + \frac{1}{b_N}\sum_{n=1}^{N} a_n X_{(m)n}I(|X_{(m)n}| > c_n) + \frac{1}{b_N}\sum_{n=1}^{N} a_n EX_{(m)n}I(|X_{(m)n}|\leqslant c_n).\end{aligned}$$

The first term is almost surely negligible since

$$\sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_{(m)n}^2 I(|X_{(m)n}| \leqslant c_n)$$
$$= \sum_{n=1}^{\infty} \frac{1}{c_n^2} \Big[\int_{-c_n}^{-1} mq^m (-1)^{m-1} x^{-m+1} dx + \int_{1}^{c_n} mp(1-p/x)^{m-1} dx \Big]$$
$$\leqslant C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_{1}^{c_n} dx \leqslant C \sum_{n=1}^{\infty} \frac{1}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

Next we have

$$\sum_{n=1}^{\infty} P\{|X_{(m)n}| > c_n\}$$
$$= \sum_{n=1}^{\infty} \Big[\int_{-\infty}^{-c_n} mq^m (-1)^{m-1} x^{-m-1} dx + \int_{c_n}^{\infty} mp(1-p/x)^{m-1} x^{-2} dx \Big]$$
$$\leqslant C \sum_{n=1}^{\infty} \int_{c_n}^{\infty} x^{-2} dx = C \sum_{n=1}^{\infty} \frac{1}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

The truncated first moment is

$$EX_{(m)n} I(|X_{(m)n}| \leqslant c_n)$$
$$= mq^m (-1)^{m-1} \int_{-c_n}^{-1} x^{-m} dx + mp \int_{1}^{c_n} (1-p/x)^{m-1} x^{-1} dx$$
$$= mq^m (-1)^{m-1} \int_{-c_n}^{-1} x^{-m} dx + mp \sum_{j=0}^{m-1} \binom{m-1}{j} (-p)^j \int_{1}^{c_n} x^{-j-1} dx$$
$$= mq^m (-1)^{m-1} \int_{-c_n}^{-1} x^{-m} dx + mp \int_{1}^{c_n} x^{-1} dx$$
$$+ mp \sum_{j=1}^{m-1} \binom{m-1}{j} (-p)^j \int_{1}^{c_n} x^{-j-1} dx$$
$$\sim mp \int_{1}^{c_n} x^{-1} dx = mp \lg c_n \sim mp \lg n.$$

Putting this all together we have

$$\frac{\sum_{n=1}^{N} a_n EX_{(m)n} I(|X_{(m)n}| \leqslant c_n)}{b_N} \sim \frac{mp \sum_{n=1}^{N} (\lg n)^{\beta-1}/n}{(\lg N)^{\beta}} \to \frac{mp}{\beta},$$

which completes this proof. ■

We see how our parameters are involved in the final answer. By observing just the maximum of two random variables, our left tail loses nearly all of its influence. The only contribution we have from q in that case is $p = 1 - q$.

3. WEAK LAWS

We next look at Weak Laws of these same random variables. Once again the case of $m = 1$ and $m > 1$ are completely different. Here, we have a little more flexibility in our weights, $a_n = n^{\alpha} L(n)$, where $L(\cdot)$ is any slowly varying function. In our Strong Laws we cannot play with our constants in this same manner that we can in our Weak Laws. We can change the coefficients in our Strong Laws from powers of logarithms to any regularly varying function of exponent -1 (see [2], p. 275), but that will greatly affect the norming sequence $\{b_n, n \geqslant 1\}$. In the corresponding Weak Laws that kind of change does affect the norming sequence, but in a very straightforward and simplistic manner. The new norming sequence is just increased by that same quantity $L(\cdot)$. The following theorem only considers $m = 1$.

THEOREM 3.1. *If $\{X_n, n \geqslant 1\}$ are i.i.d. two tailed Pareto random variables, then for all $\alpha > -1$ and any slowly varying function $L(\cdot)$ we have as $N \to \infty$*

$$\frac{\sum_{n=1}^{N} n^{\alpha} L(n) X_n}{N^{\alpha+1} L(N) \lg N} \xrightarrow{P} \frac{p-q}{\alpha+1}.$$

P r o o f. This theorem is a consequence of the Degenerate Convergence Theorem which can be found on page 356 of [1]. As usual, set $a_n = n^{\alpha} L(n)$ and $b_n = n^{\alpha+1} L(n) \lg n$. By choosing N sufficiently large we have b_N / a_n as large as we wish. Thus for all $\epsilon > 0$ we have

$$\begin{aligned}
\sum_{n=1}^{N} P\{|X| \geqslant \epsilon b_N / a_n\} &= \sum_{n=1}^{N} \Big[\int_{-\infty}^{-\epsilon b_N / a_n} q x^{-2} dx + \int_{\epsilon b_N / a_n}^{\infty} p x^{-2} dx \Big] \\
&= \sum_{n=1}^{N} \left[\frac{q a_n}{\epsilon b_N} + \frac{p a_n}{\epsilon b_N} \right] = \frac{1}{\epsilon b_N} \sum_{n=1}^{N} a_n \\
&< \frac{C \sum_{n=1}^{N} n^{\alpha} L(n)}{N^{\alpha+1} L(N) \lg N} < \frac{C}{\lg N} \to 0
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{N} \frac{a_n^2}{b_N^2} E X^2 I(|X| \leqslant b_N / a_n) &= \frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \Big[\int_{-b_N / a_n}^{-1} q\, dx + \int_{1}^{b_N / a_n} p\, dx \Big] \\
&< \frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \left[\frac{q b_N}{a_n} + \frac{p b_N}{a_n} \right] = \frac{1}{b_N} \sum_{n=1}^{N} a_n \to 0.
\end{aligned}$$

While

$$\sum_{n=1}^{N}\frac{a_n}{b_N}EXI(|X|\leqslant b_N/a_n)=\sum_{n=1}^{N}\frac{a_n}{b_N}\Big[\int\limits_{-b_N/a_n}^{-1}qx^{-1}dx+\int\limits_{1}^{b_N/a_n}px^{-1}dx\Big]$$

$$=\frac{p-q}{b_N}\sum_{n=1}^{N}a_n[\lg(b_N)-\lg(a_n)]$$

$$=\frac{p-q}{N^{\alpha+1}L(N)\lg N}\sum_{n=1}^{N}n^{\alpha}L(n)\big[\lg\big(N^{\alpha+1}L(N)\lg N\big)-\lg\big(n^{\alpha}L(n)\big)\big]$$

$$=(p-q)\Bigg[\frac{(\alpha+1)\sum_{n=1}^{N}n^{\alpha}L(n)}{N^{\alpha+1}L(N)}+\frac{\sum_{n=1}^{N}n^{\alpha}L(n)\lg\big(L(N)\big)}{N^{\alpha+1}L(N)\lg N}$$

$$+\frac{\sum_{n=1}^{N}n^{\alpha}L(n)\lg_2 N}{N^{\alpha+1}L(N)\lg N}-\frac{\alpha\sum_{n=1}^{N}n^{\alpha}L(n)\lg n}{N^{\alpha+1}L(N)\lg N}-\frac{\sum_{n=1}^{N}n^{\alpha}L(n)\lg\big(L(n)\big)}{N^{\alpha+1}L(N)\lg N}\Bigg].$$

The first term converges to one since

$$\frac{(\alpha+1)\sum_{n=1}^{N}n^{\alpha}L(n)}{N^{\alpha+1}L(N)}\to\frac{\alpha+1}{\alpha+1}=1.$$

The second term converges to zero since

$$\frac{\sum_{n=1}^{N}n^{\alpha}L(n)\lg\big(L(N)\big)}{N^{\alpha+1}L(N)\lg N}<\frac{C\lg\big(L(N)\big)}{\lg N}\to 0$$

using the fact that $L(\cdot)$ is slowly varying. Similarly, the third term is bounded above by the function

$$\frac{C\lg_2 N}{\lg N}\to 0.$$

However, the fourth term

$$\frac{-\alpha\sum_{n=1}^{N}n^{\alpha}L(n)\lg(n)}{N^{\alpha+1}L(N)\lg N}\to\frac{-\alpha}{\alpha+1}.$$

Lastly, we have

$$\frac{\sum_{n=1}^{N}n^{\alpha}L(n)\lg\big(L(n)\big)}{N^{\alpha+1}L(N)\lg N}<\frac{C\lg\big(L(N)\big)}{\lg N}\to 0.$$

Collecting all our terms we have

$$\sum_{n=1}^{N}\frac{a_n}{b_N}EXI(|X|\leqslant b_N/a_n)\to(p-q)\left[1-\frac{\alpha}{\alpha+1}\right]=\frac{p-q}{\alpha+1},$$

which completes this proof. ■

We conclude with a Weak Law for our largest order statistic from these two tailed Pareto random variables. Hence, we are once again taking repeat samples of the largest of our m random variables.

THEOREM 3.2. *If $\{X_{(m)n}, n \geqslant 1\}$ is a sample of the largest order statistics from a two tailed Pareto distribution, then for all $\alpha > -1$ and any slowly varying function $L(\cdot)$ we have as $N \to \infty$*

$$\frac{\sum_{n=1}^{N} n^{\alpha} L(n) X_{(m)n}}{N^{\alpha+1} L(N) \lg N} \xrightarrow{P} \frac{mp}{\alpha+1}.$$

Proof. Set $a_n = n^{\alpha} L(n)$ and $b_n = n^{\alpha+1} L(n) \lg n$. By choosing N sufficiently large we have b_N/a_n as large as we wish. Thus for all $\epsilon > 0$ we have

$$\begin{aligned}
&\sum_{n=1}^{N} P\{|X_{(m)n}| \geqslant \epsilon b_N/a_n\} \\
&= \sum_{n=1}^{N} \Big[\int_{-\infty}^{-\epsilon b_N/a_n} mq^m (-1)^{m-1} x^{-m-1} dx + \int_{\epsilon b_N/a_n}^{\infty} mp(1-p/x)^{m-1} x^{-2} dx \Big] \\
&< C \sum_{n=1}^{N} \Big[\int_{-\infty}^{-\epsilon b_N/a_n} x^{-m-1} dx + \int_{\epsilon b_N/a_n}^{\infty} x^{-2} dx \Big] \\
&< \frac{C}{b_N} \sum_{n=1}^{N} a_n = \frac{C \sum_{n=1}^{N} n^{\alpha} L(n)}{N^{\alpha+1} L(N) \lg N} < \frac{C}{\lg N} \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{a_n^2}{b_N^2} E X_{(m)n}^2 I(|X_{(m)n}| \leqslant b_N/a_n) \\
&= \frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \Big[\int_{-b_N/a_n}^{-1} mq^m (-1)^{m-1} x^{-m+1} dx + \int_{1}^{b_N/a_n} mp(1-p/x)^{m-1} dx \Big] \\
&= \frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \Big[\int_{-b_N/a_n}^{-1} mq^m (-x)^{-m+1} dx + \int_{1}^{b_N/a_n} mp(1-p/x)^{m-1} dx \Big] \\
&= \frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \Big[\int_{1}^{b_N/a_n} mq^m u^{-m+1} du + \int_{1}^{b_N/a_n} mp(1-p/x)^{m-1} dx \Big] \\
&< \frac{C}{b_N^2} \sum_{n=1}^{N} a_n^2 \Big[\int_{1}^{b_N/a_n} x^{-m+1} dx + \int_{1}^{b_N/a_n} dx \Big] \\
&< \frac{C}{b_N^2} \sum_{n=1}^{N} a_n^2 [b_N/a_n] = \frac{C}{b_N} \sum_{n=1}^{N} a_n \to 0.
\end{aligned}$$

Then our limit is

$$\sum_{n=1}^{N} \frac{a_n}{b_N} EX_{(m)n} I(|X_{(m)n}| \leqslant b_N/a_n)$$

$$= \sum_{n=1}^{N} \frac{a_n}{b_N} \Big[\int_{-b_N/a_n}^{-1} mq^m(-1)^{m-1} x^{-m} dx + \int_{1}^{b_N/a_n} mp(1-p/x)^{m-1} x^{-1} dx \Big]$$

$$= \sum_{n=1}^{N} \frac{a_n}{b_N} \Bigg[mq^m(-1)^{m-1} \int_{-b_N/a_n}^{-1} x^{-m} dx + mp \int_{1}^{b_N/a_n} x^{-1} dx$$

$$+ mp \sum_{j=1}^{m-1} \binom{m-1}{j} \int_{1}^{b_N/a_n} (-p/x)^j x^{-1} dx \Bigg]$$

$$\sim mp \sum_{n=1}^{N} \frac{a_n}{b_N} \int_{1}^{b_N/a_n} x^{-1} dx = \frac{mp}{b_N} \sum_{n=1}^{N} a_n \lg(b_N/a_n) \to \frac{mp}{\alpha+1}$$

as in the proof of Theorem 2.2. ∎

Note that in Theorem 3.2, as in Theorem 2.2, we see that the right tail dominates the left tail as long as we are observing the maximum of at least two random variables from our underlying distribution.

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Department of Mathematics
Illinois Institute of Technology
Chicago, Illinois, 60616, U.S.A.
E-mail: adler@iit.edu

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