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SHARP NORM INEQUALITIES FOR STOCHASTIC INTEGRALS IN WHICH THE INTEGRATOR IS A NONNEGATIVE SUPERMARTINGALE*

BY

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Abstract. The paper is devoted to sharp inequalities between moments of nonnegative supermartingales and their strong subordinates. Analogous estimates hold true for stochastic integrals with respect to a nonnegative right-continuous supermartingale. Similar inequalities are established for smooth functions on Euclidean domains.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geqslant 0}$ of sub-σ-fields of $\mathcal{F}$. Suppose, in addition, that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geqslant 0}$ be a right-continuous semimartingale with left limits, adapted to $(\mathcal{F}_t)$, and $H = (H_t)_{t\geqslant 0}$ be a predictable process taking values in $\mathbb{R}^\nu$ for some $\nu \geqslant 1$. Let

$$Y_t = H_0 X_0 + \int_0^t H_s dX_s$$

denote the stochastic integral.

The papers [2] and [4]–[10] by Burkholder contain some sharp estimates for the stochastic integrals under some additional assumptions on the integrator X and integrand H. Let us present some details. Suppose first that X is a martingale and H is bounded by 1. Then we have a weak type estimate

$$\lambda\mathbb{P}(Y^* \geqslant \lambda) \leqslant \lambda\mathbb{P}\big((|X| + |Y|)^* \geqslant \lambda\big) \leqslant 2\|X\|_1, \qquad \lambda > 0, \tag{1.1}$$

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and the norm inequality

$$\|Y\|_p \leqslant (p^* - 1)\|X\|_p, \qquad 1 < p < \infty. \tag{1.2}$$

Here p^* is defined as the maximum of p and its harmonic conjugate $p/(p-1)$, Y^* as the supremum $\sup_{t\geqslant 0}|Y_t|$ (with analogous definition of $(|X|+|Y|)^*$) and $\|X\|_p$ denotes the p-th norm of a martingale (see below). Both constants 2 and p^*-1 are the best possible (see [2]). We include the middle term in (1.1), since the inequality on the right leads to the related bound

$$\lambda\mathbb{P}(Y^* \geqslant \lambda) \leqslant (b-a)\|X\|_1$$

if the size condition on H is $a \leqslant H \leqslant b$, where $a \leqslant 0 \leqslant b$ are fixed. For details, see the proof of Theorem 3.7 in [7].

In the case where X is a nonnegative submartingale and H is bounded by 1, it is shown in [9] that the analogous sharp inequalities hold, with constant 2 in the weak type estimate replaced by 3 and p^*-1 appearing in the norm inequality replaced by $p^{**}-1$, $p^{**} = \max\{2p, p/(p-1)\}$. The paper [10] deals with L^p inequalities under the assumption that X is a nonnegative martingale and H is bounded by 1; the constant p^*-1 remains best possible for $1 < p \leqslant 2$, but for $p > 2$ it is replaced by smaller $[p2^{1-p}(p-1)^{p-1}]^{1/p}$.

In the paper we continue this line of research and study the norm inequalities where X is a nonnegative supermartingale and H is bounded by 1. This setting was already considered by Burkholder; the paper [9] contains sharp weak type and exponential estimates. Let us recall the weak type inequality.

THEOREM 1.1. *Suppose X is a nonnegative supermartingale and Y is the integral of H with respect to X, where H is a predictable process taking values in a closed unit ball of $\mathbb{R}^\nu$. For any $\lambda > 0$, we have*

$$\lambda\mathbb{P}(Y^* \geqslant \lambda) \leqslant \lambda\mathbb{P}\big((X+|Y|)^* \geqslant \lambda\big) \leqslant 2\|X\|_1. \tag{1.3}$$

The constant 2 is the best possible.

A natural question is whether any moment inequalities hold in this setting. It turns out that this is the case provided $p < 1$: define

$$\beta_p = \begin{cases} 2\big((2-p)/(2-2p)\big)^{1/p} & \text{if } p \in (-\infty,0)\cup(0,1), \\ 2e^{1/2} & \text{if } p = 0. \end{cases} \tag{1.4}$$

Let

$$\|Y\|_p = \sup_{t\geqslant 0}\|Y_t\|_p = \sup_{t\geqslant 0}(\mathbb{E}|Y_t|^p)^{1/p} \qquad \text{for } p \neq 0$$

and

$$\|Y\|_0 = \sup_{t\geqslant 0}\|Y_t\|_0 = \sup_{t\geqslant 0}\exp(\mathbb{E}\log|Y_t|).$$

The formulae above are not clear if $p \leqslant 0$ and $\mathbb{P}(|Y_t| = 0) > 0$. We set $\log 0 = -\infty$, $e^{-\infty} = 0$ and, for $p < 0$, $0^p = +\infty$ and $(+\infty)^p = 0$.

Here is one of the main results of the paper.

THEOREM 1.2. *Let $p \in (-\infty, 1)$. If X is a nonnegative supermartingale and Y is the integral of H with respect to X, where H is a predictable process taking values in a closed unit ball of $\mathbb{R}^\nu$, then*

$$\|Y\|_p \leqslant \beta_p \|X\|_p. \tag{1.5}$$

The constant β_p is the best possible. It is already best possible when X is assumed to be a nonnegative martingale, $\nu = 1$ and H takes values in the set $\{-1, 1\}$.

In the paper we will also establish some related inequalities in the discrete time case. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a discrete filtration $(\mathcal{F}_n)_{n \geqslant 0}$. Let $f = (f_n)$ and $g = (g_n)$ be two integrable $(\mathcal{F}_n)$-adapted processes, taking values in $\mathbb{R}^\mu$ and $\mathbb{R}^\nu$, respectively. We say that g is *differentially subordinate* to f (or just *subordinate* to f) if for any nonnegative integer n we have, with probability 1,

$$|dg_n| \leqslant |df_n|. \tag{1.6}$$

Here $df = (df_n)$ and $dg = (dg_n)$ stand for the difference sequences of the processes f and g, respectively, given by

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \ \ n = 1, 2, \ldots,$$

and a similar definition for dg. The process g is *strongly differentially subordinate* to f (or just *strongly subordinate* to f) if it is subordinate to f and, in addition, for any positive integer n we have, almost surely,

$$\big|\mathbb{E}(dg_n|\mathcal{F}_{n-1})\big| \leqslant \big|\mathbb{E}(df_n|\mathcal{F}_{n-1})\big|. \tag{1.7}$$

Given a predictable real sequence $v = (v_n)$, we say that g is the *transform* of f by v if for any nonnegative integer n we have $dg_n = v_n df_n$. In particular, if all v_n's are deterministic and belong to $\{-1, 1\}$, we say that g is a ± 1 *transform* of f. The notion of strong subordination generalizes martingale transforms: if v is bounded by 1 in absolute value and g is the transform of f by v, then g is strongly subordinate to f.

The discrete-time result of the paper can now be stated as follows:

THEOREM 1.3. *Let f be a nonnegative supermartingale and g be $\mathbb{R}^\nu$-valued process, strongly subordinate to f. Then for $p < 1$ we have the inequality*

$$\|g\|_p \leqslant \beta_p \|f\|_p \tag{1.8}$$

and the constant β_p is the best possible; it is already the best possible when f is assumed to be a nonnegative martingale and g to be its ± 1 transform.

This result will be established in the next section. We prove Theorem 1.2 in Section 3, while Section 4 is devoted to related norm inequalities for smooth functions.

2. DISCRETE-TIME CASE

In this section the proof of Theorem 1.3 will be presented. We will make heavy use of a technique invented by Burkholder [2]. It reduces the problem of proving a certain martingale inequality to the construction of a function which has some convex-type properties. The central role in our paper plays the function $W\colon \mathbb{R}_+ \times \mathbb{R}^\nu \to \mathbb{R}$ constructed in [9], given by

$$W(x,y) = \begin{cases} 2x - x^2 + |y|^2 & \text{if } x + |y| \leqslant 1, \\ 1 & \text{if } x + |y| > 1. \end{cases} \tag{2.1}$$

This is the special function corresponding to the weak type inequality (1.3). Before formulating its properties we put, here and below, the following notation: if $x \in \mathbb{R}_+$, then $\underline{x} = (x, 0, 0, \ldots, 0) \in \mathbb{R}^\nu$. Then we have

$$W(x,y) \leqslant W(x,\underline{x}) \quad \text{if } |y| \leqslant x, \tag{2.2}$$

and

$$\mathbb{E}W(f_k, g_k) \leqslant \mathbb{E}W(f_{k-1}, g_{k-1}), \qquad k = 1, 2, \ldots, \tag{2.3}$$

for any nonnegative supermartingale f and any g (see the proof of Theorem 8.1 in [9]). The inequalities (2.2) and (2.3) imply

$$\mathbb{E}W(f_n, g_n) \leqslant \mathbb{E}W(f_0, \underline{f_0}), \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

for any f and g as above. The special functions U_p, $p < 1$, corresponding to the norm inequalities, will be obtained by integrating the function W (or its slight modification) against the kernel $t \mapsto t^{p-1}$ (see (2.6) below). It is worth to mention that, quite surprisingly, these functions (or rather, the formulae for U_p's) appear in [4], [5] (cf. also [7]) as the special functions corresponding to the norm inequalities (1.2); however, the major difference is that in those papers the range of the exponent p is equal to $(1, \infty)$, while in our situation $p < 1$.

P r o o f o f t h e i n e q u a l i t y (1.8). Let f be a nonnegative supermartingale and g be strongly subordinate to f. We start from the observation that $\|f\|_p = \|f_0\|_p$ for $p \leqslant 1$. For $0 \leqslant p < 1$ this follows from the fact that the functions $t \mapsto t^p$, $t \mapsto \log t$, $t \geqslant 0$, are concave and increasing: by Jensen's inequality, for any $n = 1, 2, \ldots$ we may write $\|f_n\|_p \leqslant \|\mathbb{E}(f_n|\mathcal{F}_0)\|_p \leqslant \|f_0\|_p$. If $p < 0$, then we use the fact that the function $t \mapsto t^p$, $t \geqslant 0$, is convex and decreasing to obtain the inequalities $\mathbb{E}|f_n|^p \geqslant \mathbb{E}\big|\mathbb{E}(f_n|\mathcal{F}_0)\big|^p \geqslant \mathbb{E}|f_0|^p$ (note that the expectations may be equal to $+\infty$), and hence $\|f_n\|_p \leqslant \|f_0\|_p$ for any $n = 1, 2, \ldots$

Therefore, in order to establish (1.8), it suffices to prove that for any nonnegative integer n

$$\|g_n\|_p \leqslant \beta_p \|f_0\|_p. \tag{2.5}$$

Let nonzero $(0, a) \in \mathbb{R}^\nu \times \mathbb{R}$ be a fixed vector. Consider processes $f^a = |a| + f$, $g^a = (g, a) \in \mathbb{R}^\nu \times \mathbb{R}$. Then f^a is a nonnegative supermartingale and g^a is differentially subordinate to f^a. This modification guarantees the integrability of negative powers of these processes; see the inequality (2.11) below.

For $p < 1$, $p \neq 0$, consider the functions $U_p,\ V_p$ on $\{(x, y) \in (0, \infty) \times \mathbb{R}^\nu : |y| > 0\}$ given by

$$\begin{aligned} U_p(x, y) &= p\big(x - (p-1)|y|\big)(x + |y|)^{p-1}, \\ V_p(x, y) &= p(1-p)|y|^p, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} U_0(x, y) &= \log(x + |y|) + \frac{x}{x + |y|}, \\ V_0(x, y) &= \log|y|. \end{aligned}$$

Note that we have the majorization

$$U_p \geqslant V_p. \tag{2.7}$$

This is clear for $p = 0$. For $p \neq 0$, calculate the partial derivative

$$\frac{\partial U_p}{\partial x}(x, y) = p^2 (x + |y|)^{p-2}\big(x + (2-p)|y|\big) > 0,$$

and since $U_p(0+, y) = V_p(x, y)$, the inequality (2.7) is established.

The key fact about U_p is that the following remarkable identities hold true: for $0 < p < 1$

$$U_p(x, y) = \frac{p^2(p-1)(p-2)}{2} \int\limits_0^\infty t^{p-1} W(x/t, y/t)\, dt, \tag{2.8}$$

$$U_0(x, y) - U_0(x', y') = \int\limits_0^\infty t^{-1}\big(W(x/t, y/t) - W(x'/t, y'/t)\big) dt, \tag{2.9}$$

while for $p < 0$

$$U_p(x, y) = \frac{p^2(p-1)(p-2)}{2} \int\limits_0^\infty t^{p-1}\big(W(x/t, y/t) - 1\big) dt. \tag{2.10}$$

These identities, together with (2.4), yield

$$\mathbb{E}U_p(f_n^a, g_n^a) \leqslant \mathbb{E}U_p(f_0^a, \underline{f_0^a}), \qquad n = 0, 1, 2, \dots \tag{2.11}$$

(both expectations are finite due to the definition of f^a and g^a). Combining this inequality with (2.7) we obtain, for $p \neq 0$ and any n,

$$\begin{aligned} p(1-p)\mathbb{E}|g_n^a|^p = \mathbb{E}V_p(f_n^a, g_n^a) &\leqslant \mathbb{E}U_p(f_n^a, g_n^a) \\ &\leqslant \mathbb{E}U_p(f_0^a, \underline{f_0^a}) = p(2-p)2^{p-1}\mathbb{E}(f_0^a)^p \end{aligned} \tag{2.12}$$

and, similarly, for $p = 0$

$$\begin{aligned} \mathbb{E}\log|g_n^a| = \mathbb{E}V_0(f_n^a, g_n^a) &\leqslant \mathbb{E}U_0(f_n^a, g_n^a) \\ &\leqslant \mathbb{E}U_0(f_0^a, \underline{f_0^a}) = \log 2 + \frac{1}{2} + \mathbb{E}\log f_0^a. \end{aligned} \tag{2.13}$$

Now we tend with $|a| \to 0$ and obtain the inequality (2.5) by a standard use of Lebesgue's convergence theorems. ∎

S h a r p n e s s o f t h e i n e q u a l i t y (1.8). This will be clear by the following example. Let δ be a fixed positive number and the interval $[0, 1]$ with Lebesgue measure be the underlying probability space. For convenience, we identify an interval with its indicator function. Let $x_{-1} = 1$, $x_k = \big(2(1 + k\delta)\big)^{-1}$ for $k \geqslant 0$ and set

$$f_n = \frac{1}{2}x_{n-1}^{-1}[0, x_{n-1}], \qquad dg_n = (-1)^n df_n, \qquad n = 0, 1, 2, \dots$$

Note that f is a nonnegative martingale with respect to the natural filtration and g is its ± 1 transform. As one easily checks by induction,

$$\begin{aligned} g_0 &= \frac{1}{2}[0, 1], \\ g_{2n} &= \sum_{k=0}^{n-1}(2x_{2k})^{-1}\{(x_{2k}, x_{2k-1}] - (x_{2k+1}, x_{2k}]\} + \delta[0, x_{2n-1}], \;\; n = 1, 2, \dots, \\ g_{2n+1} &= \sum_{k=0}^{n-1}(2x_{2k})^{-1}\{(x_{2k}, x_{2k-1}] - (x_{2k+1}, x_{2k}]\} + (2x_{2n})^{-1}(x_{2n}, x_{2n-1}] \end{aligned}$$

for $n = 0, 1, 2, \dots$ Fix $0 < p < 1$. We have

$$\begin{aligned} \mathbb{E}|g_{2n+1}|^p &\geqslant \frac{1}{2} + \sum_{k=1}^{n-1}(2x_{2k})^{-p}(x_{2k-1} - x_{2k+1}) \\ &= \frac{1}{2} + \sum_{k=1}^{n-1}(1 + 2k\delta)^p\frac{\delta}{(1 + 2k\delta)^2 - \delta^2} \geqslant \frac{1}{2} + \frac{1}{2}\Big[2\delta\sum_{k=1}^{n-1}(1 + 2k\delta)^{p-2}\Big]. \end{aligned}$$

The expression in the square brackets is a Riemann sum which, clearly, by the proper choice of n and δ, can be made arbitrarily close to $\int_1^\infty t^{p-2}dt = (1-p)^{-1}$. In other words, for a fixed $\varepsilon > 0$, there exist n and δ such that

$$\mathbb{E}|g_{2n+1}|^p \geqslant \frac{1}{2} + \frac{1}{2(1-p)} - \varepsilon = \frac{2-p}{2-2p} - \varepsilon,$$

which implies, together with $\mathbb{E}f_0^p = 2^{-p}$, that

$$\beta_p \geqslant 2\left(\frac{2-p}{2-2p}\right)^{1/p}.$$

For $p \leqslant 0$ we need a slight modification of the martingale f: fix a positive integer N, leave f_n unchanged for $n \leqslant 2N+1$, and set

$$f_{2N+2} = f_{2N+3} = \ldots = (x_{2N})^{-1}\left(\frac{1}{2}x_{2N}, x_{2N}\right].$$

Then

$$\begin{aligned} g_{2N+2} = &\sum_{k=0}^{N-1}(2x_{2k})^{-1}\{(x_{2k}, x_{2k-1}] - (x_{2k+1}, x_{2k}]\} \\ &+ (2x_{2N})^{-1}\left\{\left(\frac{1}{2}x_{2N}, x_{2N-1}\right] - \left[0, \frac{1}{2}x_{2N}\right]\right\}. \end{aligned}$$

Fix $\varepsilon > 0$. As $(2x_{2N})^{-1} \geqslant 1$, we have

$$\begin{aligned} \mathbb{E}\log|g_{2N+2}| &\geqslant \sum_{k=1}^{N-1}\log(2x_{2k})^{-1}(x_{2k-1} - x_{2k+1}) \\ &\geqslant \frac{1}{2}\cdot 2\delta\sum_{k=1}^{N-1}\frac{\log(1+2k\delta)}{(1+2k\delta)^2} \geqslant \frac{1}{2}\int_1^\infty t^{-2}\log t\,dt - \varepsilon = \frac{1}{2} - \varepsilon \end{aligned}$$

if only N and δ are chosen properly. Since $\mathbb{E}\log f_0 = -\log 2$, this shows that $\log\beta_0 \geqslant \log 2 + \frac{1}{2}$ or $\beta_0 \geqslant 2e^{1/2}$. Finally, for $p < 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{E}|g_{2N+2}|^p &\leqslant \frac{1}{2}\left[1 + \frac{\delta}{1+\delta}\right] + \frac{1}{2}\cdot 2\delta\sum_{k=1}^{N-1}\frac{(1+2k\delta)^p}{(1+2k\delta)^2 - \delta^2} + \frac{(1+2N\delta)^p}{2\big(1+(2N-1)\delta\big)} \\ &< \frac{1}{2}\left[1+\frac{\delta}{1+\delta}\right] + \frac{1}{2}\cdot 2\delta\sum_{k=1}^{N-1}\big(1+(2k-1)\delta\big)^{p-2} + \frac{(1+2N\delta)^p}{2\big(1+(2N-1)\delta\big)} \\ &\leqslant \frac{1}{2} + \frac{1}{2}\int_1^\infty t^{p-2}dt + \varepsilon = \frac{2-p}{2-2p} + \varepsilon \end{aligned}$$

for proper N and δ. As $\mathbb{E}f_0^p = 2^{-p}$, this proves the bound for β_p. ■

A boundary value problem. There is an alternative way to deduce that the inequality (1.8) is sharp. We will present it in the case $0 < p < 1$, but similar arguments can be applied to the case $p \leqslant 0$ as well.

The main tool we use is described in the next lemma. For related results and discussion, the reader is referred to Sections 5 and 7 in [3]. See also Section 2 in [7].

LEMMA 2.1. *Let* $0 < p < 1$ *and* $S = \{(x, y) \in \mathbb{R}^2 : x + y \geqslant 0\}$. *Suppose that* $\beta \in [1, \infty)$ *satisfies*

$$\|g\|_p \leqslant \beta \|f_0\|_p \tag{2.14}$$

for all pairs (f, g), *where* f *is a nonnegative martingale and* g *is a* ± 1 *transform of* f. *Then there exists a biconcave function* $u\colon S \to \mathbb{R}$, *which has the following properties*:

(i) *It majorizes the function* $v(x, y) = |x - y|^p$ *on* S.
(ii) *It is homogeneous of order* p.
(iii) *It satisfies the symmetry condition* $u(x, y) = u(y, x)$.
(iv) *We have* $\big(u(1, 0)\big)^{1/p} \leqslant \beta$.

In fact, both conditions are equivalent; the existence of the function u satisfying the properties (i)–(iv) implies that the inequality (2.14) holds for all f and g as in the statement of the lemma. However, we need only one implication and, for the sake of completeness, we provide its proof. The function u (or rather one of the functions u, since it is not unique) has a description in terms of zigzag martingales. Let us recall that an $\mathbb{R}^2$-valued martingale $Z = (Z^1, Z^2)$ is called *zigzag* if for any $n \geqslant 0$ we have $\mathbb{P}(Z^1_n = Z^1_{n+1}) = 1$ or $\mathbb{P}(Z^2_n = Z^2_{n+1}) = 1$. The martingale Z is called *simple* if for any n the random variable Z_n is simple and there exists N such that $Z_N = Z_{N+1} = Z_{N+2} = \ldots$ almost surely. Let $\mathbf{Z}(x, y)$ denote the set of all simple zigzag martingales taking values in S and starting from $(x, y) \in S$.

Proof of Lemma 2.1. Let us define

$$u(x, y) = \sup\{\mathbb{E}|Z^1_\infty - Z^2_\infty|^p : Z = (Z^1, Z^2) \in \mathbf{Z}(x, y)\}.$$

The finiteness and biconcavity of the function described by the formula above can be shown exactly in the same manner as in the proof of Theorem 2.1 in [7]. Since the constant martingale $Z \equiv (x, y)$ belongs to $\mathbf{Z}(x, y)$, the property (i) is satisfied. The homogeneity and symmetry follow immediately from the definition. For the property (iv), fix $\varepsilon > 0$ and take a martingale $Z \in \mathbf{Z}(1, 0)$ such that

$$u(1, 0) - \varepsilon < \mathbb{E}|Z^1_\infty - Z^2_\infty|^p.$$

It suffices to note that since Z is zigzag, the martingale $g = Z^1 - Z^2$ is a ± 1 transform of $f = Z^1 + Z^2$. Therefore, by (2.14), $\big(u(1, 0) - \varepsilon\big)^{1/p} < \|g\|_p \leqslant \beta \|f_0\|_p = \beta$. This proves the claim. ■

With the above result, we can establish the sharpness of (1.8). Consider a function $w : (-1, \infty) \to \mathbb{R}$ given by $w(t) = u(1, t)$. The symmetry and homogeneity of u imply that for $t > 0$ we have

$$w(t) = t^p w(1/t). \tag{2.15}$$

The function w is concave, and hence both one-sided derivatives $w'_-(1)$, $w'_+(1)$ exist. By (2.15), we obtain

$$\begin{aligned} w'_+(1) &= \lim_{x \downarrow 1} \frac{w(x) - w(1)}{x - 1} = \lim_{x \downarrow 1} \frac{x^p w(1/x) - w(1)}{x - 1} \\ &= -\lim_{x \downarrow 1} x^{p-1} \cdot \frac{w(1/x) - w(1)}{x^{-1} - 1} + \lim_{x \downarrow 1} w(1) \cdot \frac{x^p - 1}{x - 1} \\ &= -w'_-(1) + pw(1). \end{aligned}$$

By concavity, we have $w'_-(1) \geqslant w'_+(1)$, which leads to

$$pw(1) \leqslant 2w'_-(1) \leqslant 2 \cdot \frac{w(1) - w(-1)}{1 - (-1)} = w(1) - w(-1)$$

or

$$w(-1) \leqslant (1 - p)w(1).$$

Using the property (iv) from Lemma 2.1 and concavity of w, we have

$$\beta^p \geqslant w(0) \geqslant \frac{w(-1) + w(1)}{2} \geqslant \frac{2 - p}{2 - 2p} w(-1).$$

Now it suffices to use the inequality

$$w(-1) = u(1, -1) \geqslant v(1, -1) = 2^p$$

to obtain the desired bound for β_p.

Norm inequalities in the case of α-strong subordination, $\alpha \in [0, 1]$. The strong differential subordination can be generalized as follows (cf. Choi [11]). Fix $\alpha \in [0, \infty)$. Let f and g be integrable adapted processes, taking values in $\mathbb{R}^\mu$ and $\mathbb{R}^\nu$, respectively. Then g is *α-strongly subordinate* to f if it is subordinate to f and for any $n \geqslant 1$ we have, almost surely,

$$\big|\mathbb{E}(dg_n | \mathcal{F}_{n-1})\big| \leqslant \alpha \big|\mathbb{E}(df_n | \mathcal{F}_{n-1})\big|. \tag{2.16}$$

Therefore, strong subordination considered in this paper is just 1-strong subordination in this new terminology.

The following theorem is an immediate consequence of the previous results.

THEOREM 2.1. *Let $\alpha \in [0,1]$. Let f be a nonnegative supermartingale and g be α-strongly subordinate to f.*

(i) *For any $\lambda > 0$, we have*

$$\lambda \mathbb{P}(\sup_{n \geqslant 0} |g_n| \geqslant \lambda) \leqslant \lambda \mathbb{P}\big(\sup_{n \geqslant 0}(f_n + |g_n|) \geqslant \lambda\big) \leqslant 2\|f\|_1 \tag{2.17}$$

and the constant 2 is the best possible.

(ii) *For any $p < 1$ we have*

$$\|g\|_p \leqslant \beta_p \|f\|_p \tag{2.18}$$

and the constant β_p is the best possible.

Proof. As for $\alpha \in [0,1]$ the α-strong subordination implies 1-strong subordination, the inequalities (2.17) and (2.18) follow from Theorem 8.1 in [9] and Theorem 1.3. But in both these theorems, the constants are the best possible even in the case when f is assumed to be martingale and g to be its ± 1 transform. It suffices to note that if f, g are martingales, then (2.16) is automatically satisfied, as both conditional expectations vanish. ∎

This should be compared to the case when f is a nonnegative submartingale and g is α-subordinate to f: then the best constants in the weak and norm estimates do depend on α. The following are the results of Choi [11], [12].

THEOREM 2.2. *Let $\alpha \in [0,1]$. Let f be a nonnegative submartingale and g be α-strongly subordinate to f.*

(i) *For any $\lambda > 0$, we have*

$$\lambda \mathbb{P}(\sup_{n \geqslant 0} |g_n| \geqslant \lambda) \leqslant \lambda \mathbb{P}\big(\sup_{n \geqslant 0}(f_n + |g_n|) \geqslant \lambda\big) \leqslant (\alpha + 2)\|f\|_1 \tag{2.19}$$

and the constant $\alpha + 2$ is the best possible.

(ii) *For any $1 < p < \infty$ we have*

$$\|g\|_p \leqslant \gamma_p \|f\|_p, \tag{2.20}$$

where $\gamma_p = \max\{(\alpha+1)p, p/(p-1)\} - 1$. The constant γ_p is the best possible.

Lack of norm inequalities for $p \geqslant 1$. We will prove that $\beta_p = \infty$ for $p \geqslant 1$. This is true for $p = 1$ since then the norm inequality fails to hold even in the case when f is a nonnegative martingale and g is its ± 1 transform: otherwise, the Haar basis would be unconditional in L^1 (cf. [7]). Assume $p > 1$ and consider the following processes f, g on the interval $[0,1]$ with Lebesgue measure. For $n = 0, 1, 2, \ldots$ set

$$f_{2n} = 2^{n/(p-1)}[0, 2^{-np/(p-1)}], \qquad f_{2n+1} = \frac{1}{2} f_{2n}, \qquad dg_n = (-1)^n df_n.$$

It is easy to check that f is a nonnegative supermartingale and

$$\|f_{2n}\|_p = 1, \quad \|f_{2n+1}\|_p = \frac{1}{2} \quad \text{for any } n,$$

which gives $\|f\|_p = 1$. Note that

$$g_{2n+2} = g_{2n} + f_{2n+2} - 2f_{2n+1} + f_{2n} = g_{2n} + f_{2n+2},$$

which implies

$$g_{2n} = \sum_{k=0}^{n} f_{2k} = \sum_{k=0}^{n} 2^{k/(p-1)}[0, 2^{-kp/(p-1)}]$$
$$\geqslant \sum_{k=0}^{n} 2^{k/(p-1)}[2^{-(k+1)p/(p-1)}, 2^{-kp/(p-1)}]$$

and

$$\mathbb{E}g_{2n}^p \geqslant \sum_{k=0}^{n} (1 - 2^{-p/(p-1)}) = (n+1)(1 - 2^{-p/(p-1)}).$$

This shows that $\|g\|_p = \infty$ and proves the claim.

3. INEQUALITY FOR STOCHASTIC INTEGRALS

In this section the proof of Theorem 1.2 is given. We do not try to deduce it from Theorem 1.3 by discretization argument. Instead, we again exploit the special function W and the identities (2.8)–(2.10).

Proof of the inequality (1.5). Let X, H, Y be as in Theorem 1.2. By the supermartingale property, it suffices to show that for any $t > 0$ we have

$$\|Y_t\|_p \leqslant \beta_p \|X_0\|_p. \tag{3.1}$$

Consider the family $\mathbf{Y}$ of all Y of the form

$$Y_t = H_0 X_0 + \sum_{k=1}^{n} h_k [X(T_k \wedge t) - X(T_{k-1} \wedge t)],$$

where n is a positive integer, the coefficients h_k belong to the closed unit ball of $\mathbb{R}^\nu$ and $0 = T_0 \leqslant T_1 \leqslant \ldots \leqslant T_n$ is a sequence of stopping times, which take only a finite number of finite values. Suppose that $Y \in \mathbf{Y}$. Consider a nonnegative sequence

$$f = \big(X(T_0), X(T_1), \ldots, X(T_n), X(T_n), \ldots\big)$$

and let g stand for its transform by $(H_0, h_1, \ldots, h_n, 0, 0, \ldots)$. By Doob's optional sampling theorem, f is a supermartingale. For nonzero real a, let $f^a = |a| + f$ and

$g^a = (g, a) \in \mathbb{R}^\nu \times \mathbb{R}$. The process g^a is strongly subordinate to f^a, so, if T_n is bounded from above by t, then by (2.4)

$$\begin{aligned} \text{(3.2)} \qquad \mathbb{E}W\big(|a| + X_t, (Y_t, a)\big) &= \mathbb{E}W(f_n^a, g_n^a) \\ &\leqslant \mathbb{E}W(f_0^a, \underline{f_0^a}) = \mathbb{E}W(|a| + X_0, \underline{|a| + X_0}). \end{aligned}$$

We will extend this inequality to the case of general Y. Note that if $\nu = 1$, then the processes X and H satisfy the conditions of Proposition 4.1 of Bichteler [1]: therefore, if Y is the integral of H with respect to X, then there exists a sequence $Y^j \in \mathbf{Y}$ such that

$$\lim_{j \to \infty} (Y^j - Y)^* = 0 \text{ a.s.}$$

By additivity of the integral, the statement above can be extended to any $\nu \geqslant 1$. Combining it with (3.2) and Fatou's lemma, we obtain

$$\text{(3.3)} \qquad \mathbb{E}W\big(|a| + X_t, (Y_t, a)\big) \leqslant \mathbb{E}W(|a| + X_0, \underline{|a| + X_0})$$

for all Y as in the statement of the theorem. Now we complete the proof exactly in the same manner as the proof of (1.8). Namely, the integral identities (2.8)–(2.10) yield

$$\mathbb{E}U_p(|a| + X_t, a + Y_t) \leqslant \mathbb{E}U_p(|a| + X_0, \underline{|a| + X_0})$$

(both expectations are finite). Applying (2.7) leads to

$$\mathbb{E}V_p(|a| + X_t, a + Y_t) \leqslant \mathbb{E}U_p(|a| + X_0, \underline{|a| + X_0})$$

or

$$\|a + Y_t\|_p \leqslant \beta_p \||a| + X_0\|_p.$$

Now let $a \to 0$; a standard use of Lebesgue's convergence theorems completes the proof of (3.1). ■

S h a r p n e s s. This follows immediately by the discrete-time case. ■

4. INEQUALITIES FOR SMOOTH FUNCTIONS

As it is exposed in [7], [9], [10], the inequalities for (sub-, super-) martingales and their (strong) subordinates usually have analogues in harmonic analysis. The purpose of this section is to study such harmonic extension of the inequality (1.8).

Let n be a fixed positive integer and D an open connected subset of $\mathbb{R}^n$. Let $\xi \in D$ be fixed and consider two functions $u : D \to \mathbb{R}_+$ and $v : D \to \mathbb{R}^\nu$ with continuous first and second partial derivatives. We impose the following conditions on u and v:

$$\text{(4.1)} \qquad |v(\xi)| \leqslant |u(\xi)|,$$

$$(4.2) \qquad |\nabla v| \leqslant |\nabla u|,$$

$$(4.3) \qquad |\Delta v| \leqslant |\Delta u|.$$

Here the gradient and laplacian of v are defined as follows. We have

$$v = (v^1, v^2, \dots, v^\nu) \quad \text{with } v^k\colon D \to \mathbb{R},\ \ k = 1, 2, \dots, \nu,$$

and

$$|\nabla v|^2 = \sum_{k=1}^{\nu} |\nabla v^k|^2, \quad |\Delta v|^2 = \sum_{k=1}^{\nu} |\Delta v^k|^2.$$

The three assumptions (4.1), (4.2) and (4.3) play an analogous role to strong subordination in the martingale setting. Suppose D_0 is a bounded subdomain of D with $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$ and set

$$\|u\|_p = \sup_{D_0} \Big[\int_{\partial D_0} |u|^p d\mu \Big]^{1/p},\ \ p \neq 0, \qquad \|u\|_0 = \exp\Big\{ \sup_{D_0} \Big[\int_{\partial D_0} \log|u| d\mu \Big] \Big\},$$

where the supremum is taken over all such D_0 (with the symbols $\log 0$, 0^p for $p < 0$, etc., being understood as in the martingale setting). Here $\mu = \mu^{\xi}_{D_0}$ stands for the harmonic measure on ∂D_0 with respect to ξ.

The moment inequality for smooth functions can be stated as follows.

THEOREM 4.1. *Let $p \in (-\infty, 1)$. If u and v are as above and u is nonnegative and superharmonic on D, then*

$$(4.4) \qquad \|v\|_p \leqslant \beta_p \|u\|_p.$$

P r o o f. It is analogous to the proofs of Theorems 1.2 and 1.3 and is based on Burkholder's function W. First we take a nonzero vector $a \in \mathbb{R}^\nu$ which is orthogonal to the image of v (with no loss of generality we may assume its existence, adding one dimension to $\mathbb{R}^\nu$ if necessary). The function $W(u + |a|, v + a)$ is superharmonic (here (4.2) and (4.3) are used; see the proof of Theorem 13.2 in [9]), therefore, for any fixed D_0, we have

$$\int_{\partial D_0} W(u + |a|, v + a) d\mu \leqslant W\big(u(\xi) + |a|, v(\xi) + a\big).$$

This can be further bounded from above by $W\big(u(\xi) + |a|, \overline{u(\xi) + |a|}\big)$, due to (2.2) and (4.1). Applying the integral identities (2.8)–(2.10) and the inequality (2.7), we obtain

$$\Big[\int_{\partial D_0} |v + a|^p d\mu \Big]^{1/p} \leqslant \beta_p\big(u(\xi) + |a|\big) = \beta_p \|u + |a|\|_p$$

(the last equality holds since u is superharmonic and nonnegative) and the similar inequality for $p = 0$. Now we take $a \to 0$ and apply Lebesgue's dominated convergence theorem. Since D_0 was arbitrary, the proof is complete. ■

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