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NESTED SUBCLASSES OF SOME SUBCLASS OF THE CLASS OF TYPE G SELFDECOMPOSABLE DISTRIBUTIONS ON $\mathbb{R}^d$

BY

TAKAHIRO AOYAMA (NODA)

Abstract. Nested subclasses, denoted by $M_n(\mathbb{R}^d)$, $n = 1, 2, \ldots$, of the class $M(\mathbb{R}^d)$, a subclass of the class of type G and selfdecomposable distributions on $\mathbb{R}^d$ are studied. An analytic characterization in terms of Lévy measures and a probabilistic characterization by stochastic integral representations for $M(\mathbb{R}^d)$ are known. In this paper, analytic characterizations for $M_n(\mathbb{R}^d)$, $n = 1, 2, \ldots$, are given in terms of Lévy measures as well as probabilistic characterizations by stochastic integral representations are shown. A relationship with stable distributions is given.

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1. INTRODUCTION

Throughout this paper, $I(\mathbb{R}^d)$ (resp., $I_{\mathrm{sym}}(\mathbb{R}^d)$) stands for the class of all infinitely divisible (resp., all symmetric infinitely divisible) distributions on $\mathbb{R}^d$. The characteristic function $\widehat{\mu}(z), z \in \mathbb{R}^d$, of an infinitely divisible distribution $\mu \in I(\mathbb{R}^d)$ has the so-called Lévy–Khintchine representation in the form:

$$\widehat{\mu}(z) = \exp\Big[-2^{-1}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \big(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle(1+|x|^2)^{-1}\big)\nu(dx)\Big], \quad z \in \mathbb{R}^d,$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d$ (called the Lévy measure) satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

The triplet (A, ν, γ) is called the *generating triplet* of $\mu \in I(\mathbb{R}^d)$. Consider a polar decomposition of ν given by

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \tag{1.1}$$

where S is the unit sphere in $\mathbb{R}^d$, λ is a measure on S with $0 < \lambda(S) \leqslant \infty$ and $\{\nu_\xi : \xi \in S\}$ is a family of measures on $(0, \infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}\big((0,\infty)\big)$, $0 < \nu_\xi\big((0,\infty)\big) \leqslant \infty$ for each $\xi \in S$. Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of a measurable function $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$. λ is called the *spherical component* of ν and ν_ξ the *radial component*. We will say that ν has the *polar decomposition* (λ, ν_ξ). (See, e.g., [3] and [7].) Let $C_\mu(z) = \log \widehat{\mu}(z)$, $z \in \mathbb{R}^d$, be the cumulant of $\mu \in I(\mathbb{R}^d)$.

We can characterize five classes of infinitely divisible distributions in terms of the radical component ν_ξ.

(i) Class $U(\mathbb{R}^d)$ (*Jurek class*, see [5]): $\nu_\xi(dr) = l_\xi(r)dr$ and $l_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing in r for λ-a.e. ξ.

(ii) Class $B(\mathbb{R}^d)$ (*Goldie–Steutel–Bondesson class*, see, e.g., [3]): $\nu_\xi(dr) = l_\xi(r)dr$ and $l_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in r for λ-a.e. ξ.

(iii) Class $L(\mathbb{R}^d)$ (*class of selfdecomposable distributions*, see, e.g., [8]): $\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ and $k_\xi(r)$ is measurable in $\xi \in S$ and nonincreasing in r for λ-a.e. ξ.

(iv) Class $T(\mathbb{R}^d)$ (*Thorin class*, see, e.g., [3]): $\nu_\xi(dr) = k_\xi(r)r^{-1}dr$ and $k_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in r for λ-a.e. ξ.

(v) Class $G(\mathbb{R}^d)$ (*class of type G distributions*, see, e.g., [4]): $\mu \in I_{\mathrm{sym}}(\mathbb{R}^d)$, $\nu_\xi(dr) = g_\xi(r^2)dr$ and $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in r for λ-a.e. ξ.

We have introduced a class named $M(\mathbb{R}^d)$ in the previous paper [2], which is a subclass of type G and selfdecomposable distributions on $\mathbb{R}^d$. Its definition is the following.

DEFINITION 1.1 (*Class $M(\mathbb{R}^d)$*). $\mu \in M(\mathbb{R}^d)$ if $\mu \in I_{\mathrm{sym}}(\mathbb{R}^d)$ and

$$\nu_\xi(dr) = g_\xi(r^2)r^{-1}dr, \tag{1.2}$$

where $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in r for λ-a.e. ξ. We call $g_\xi(r)$ in (1.2) the *g-function* of ν (or μ).

Denote by $\mathcal{L}(X)$ the law of a random variable X on $\mathbb{R}^d$, and for $\mu \in I(\mathbb{R}^d)$ let $\{X_t^{(\mu)}\}$ stand for a Lévy process with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

As to the definition of stochastic integrals of nonrandom functions with respect to Lévy processes $\{X_t\}$ on $\mathbb{R}^d$, we follow the definition in [9] and [10], whose idea is to define integrals with respect to $\mathbb{R}^d$-valued independently scattered random measure induced by a Lévy process on $\mathbb{R}^d$. This idea was used in [11] and [6] for the case $d = 1$. See also [3].

Let

$$I_{\log}(\mathbb{R}^d) = \Big\{\mu \in I(\mathbb{R}^d)\colon \int\limits_{|x|>1} \log|x|\mu(dx) < \infty\Big\},$$

$$\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2), \quad m(x) = \int\limits_x^\infty \phi(u)u^{-1}du, \quad x > 0,$$

and let us denote the inverse of $m(x)$ by $m^*(t)$, that is, $t = m(x)$ if and only if $x = m^*(t)$. In [2], we have shown that the stochastic integral $\int_0^\infty m^*(t)dX_t^{(\mu)}$ exists and is finite a.s. for any $\mu \in I_{\log}(\mathbb{R}^d)$. Thus we can define the following mapping.

DEFINITION 1.2 ($\mathcal{M}$*-mapping*). For any $\mu \in I_{\log}(\mathbb{R}^d)$, we define the mapping $\mathcal{M}$ by

$$\mathcal{M}(\mu) = \mathcal{L}\Big(\int\limits_0^\infty m^*(t)dX_t^{(\mu)}\Big).$$

One of the results in [2] was the following

PROPOSITION 1.1. *We have*

$$M(\mathbb{R}^d) = \mathcal{M}\big(I_{\log}(\mathbb{R}^d)\big).$$

It is trivial by the definition that $M(\mathbb{R}^d)$ is a subclass of the class of type G and selfdecomposable distributions. However, we have more. Namely, $M(\mathbb{R}^d)$ is a proper subclass. Actually, in [2] we gave an example of μ which belongs to $L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$ but does not belong to $M(\mathbb{R}^d)$.

2. NESTED SUBCLASSES OF $M(\mathbb{R}^d)$ AND THEIR LÉVY MEASURES

In this section, we construct nested subclasses of $M(\mathbb{R}^d)$ as follows. Write $M_0(\mathbb{R}^d) = M(\mathbb{R}^d)$. We start with the following

PROPOSITION 2.1 (Aoyama et al. [2]). *Let ν and ν_0 be the Lévy measures of $\mu \in I_{\log}(\mathbb{R}^d)$ and $\mu_0 := \mathcal{M}(\mu) \in M_0(\mathbb{R}^d)$, respectively. Then*

$$\nu_0(B) = \int\limits_0^\infty \nu(u^{-1}B)\phi(u)u^{-1}du, \quad B \in \mathcal{B}_0(\mathbb{R}^d). \tag{2.1}$$

We define nested subclasses of $M(\mathbb{R}^d)$ in terms of their Lévy measures.

DEFINITION 2.1 (*Class $M_n(\mathbb{R}^d)$*). For any $n \in \mathbb{N}$, define

$$M_n(\mathbb{R}^d) = \{\mu_0 \in M_0(\mathbb{R}^d)\colon \nu \text{ in (2.1) is the Lévy measure of some distribution in } M_{n-1}(\mathbb{R}^d)\}.$$

$M_\infty(\mathbb{R}^d)$ is defined by $\bigcap_{n=0}^\infty M_n(\mathbb{R}^d)$.

For nonnegative integer n and $x > 0$, let $\eta_n(x)$ be the probability density functions of $2^{-(n+1)}|Z_0 Z_1 \dots Z_n|$, where Z_i are independent standard normal random variables.

REMARK 2.1. (1) $\lim_{x\to+0} \eta_n(x)x^{-1} = \infty$ *and* $\lim_{x\to\infty} \eta_n(x)x = 0$.
(2) $\eta_0(x) = \phi(x)$ *and for* $n \in \mathbb{N}$

$$\eta_n(x) = \int_0^\infty \phi(xu^{-1})\eta_{n-1}(u)u^{-1}du. \tag{2.2}$$

(3) $\eta_n(x)$ *can be written as follows*:

$$\eta_n(x) = \int_0^\infty \phi(u_1)u_1^{-1}du_1 \dots \int_0^\infty \phi(u_{n-1})u_{n-1}^{-1}du_{n-1} \int_0^\infty \phi\Big(x\big(\prod_{i=1}^n u_i\big)^{-1}\Big)\phi(u_n)u_n^{-1}du_n.$$

Proof. We have (2) and (3) inductively. (1) can be shown as follows. For $0 < x \leqslant 1$,

$$\begin{aligned}\eta_n(x)x^{-1} &\geqslant x^{-1}\int_0^\infty \phi(u_1)u_1^{-1}du_1 \\ &\quad\dots \int_0^\infty \phi(u_{n-1})u_{n-1}^{-1}du_{n-1} \int_0^\infty \phi\Big(\big(\prod_{i=1}^n u_i\big)^{-1}\Big)\phi(u_n)u_n^{-1}du_n \\ &\to \infty \quad (\text{as } x \to +0),\end{aligned}$$

and for any $x > 0$,

$$\begin{aligned}\eta_n(x)x &\leqslant x\int_0^\infty \phi(u_1)u_1^{-1}du_1 \\ &\quad\dots \int_0^\infty \phi(u_{n-1})u_{n-1}^{-1}du_{n-1} \int_0^\infty 2x^{-2}\big(\prod_{i=1}^n u_i\big)^2\phi(u_n)u_n^{-1}du_n \\ &= 2(2\pi)^{-n/2}x^{-1} \to 0 \quad (\text{as } x \to \infty). \ \blacksquare\end{aligned}$$

Then we have the following

THEOREM 2.1 (A characterization of the Lévy measures of $\mu_n \in M_n(\mathbb{R}^d)$). *Let* $\mu_n \in I_{\mathrm{sym}}(\mathbb{R}^d)$, $n = 1, 2, \dots$, *and denote its Lévy measure by* ν_n. *Then* $\mu_n \in M_n(\mathbb{R}^d)$ *if and only if*

$$\nu_n(B) = \int_0^\infty \nu_0(u^{-1}B)\eta_{n-1}(u)u^{-1}du, \quad B \in \mathcal{B}_0(\mathbb{R}^d), \tag{2.3}$$

where ν_0 *is the Lévy measure of some* $\mu_0 \in M_0(\mathbb{R}^d)$.

Proof. (i) The "only if" part. Let $n = 1$ and suppose $\mu_1 \in M_1(\mathbb{R}^d)$. Then, by the definition,

$$\nu_1(B) = \int_0^\infty \nu_0(u^{-1}B)\phi(u)u^{-1}du$$

for some Lévy measure ν_0 whose distribution is in $M_0(\mathbb{R}^d)$. We are going to show the assertion by induction. Suppose that the assertion is true for some $n \in \mathbb{N}$. Namely, suppose the Lévy measure ν_n of $\mu_n \in M_n(\mathbb{R}^d)$ is given by

$$\nu_n(B) = \int_0^\infty \nu_0(u^{-1}B)\eta_{n-1}(u)u^{-1}du.$$

Suppose $\mu_{n+1} \in M_{n+1}(\mathbb{R}^d)$ and denote its Lévy measure by ν_{n+1}. Then

$$\begin{aligned}
\text{(2.4)} \quad \nu_{n+1}(B) &= \int_0^\infty \nu_n(u^{-1}B)\phi(u)u^{-1}du \quad \text{(by the definition of } M_{n+1}(\mathbb{R}^d))\\
&= \int_0^\infty \phi(u)u^{-1}du \int_0^\infty \nu_0(u^{-1}v^{-1}B)\eta_{n-1}(v)v^{-1}dv\\
&= \int_0^\infty \eta_{n-1}(v)v^{-1}dv \int_0^\infty \nu_0(y^{-1}B)\phi(yv^{-1})y^{-1}dy\\
&= \int_0^\infty \nu_0(y^{-1}B)y^{-1}dy \int_0^\infty \eta_{n-1}(v)\phi(yv^{-1})v^{-1}dv\\
\text{(2.5)} \quad &= \int_0^\infty \nu_0(y^{-1}B)\eta_n(y)y^{-1}dy \quad \text{(by (2.2))}.
\end{aligned}$$

This shows that the assertion is also true for $n + 1$.

(ii) The "if" part. The assertion is true for $n = 1$. Namely, by the definition of $M_1(\mathbb{R}^d)$, if

$$\nu_1(B) = \int_0^\infty \nu_0(u^{-1}B)\phi(u)u^{-1}du$$

for some ν_0, the Lévy measure of some $\mu_0 \in M_0(\mathbb{R}^d)$, then μ_1 whose Lévy measure is ν_1 belongs to $M_1(\mathbb{R}^d)$. Suppose that the assertion is true for some $n \in \mathbb{N}$ and suppose that $\mu_{n+1} \in I_{\text{sym}}(\mathbb{R}^d)$ have the Lévy measure

$$\nu_{n+1}(B) = \int_0^\infty \nu_0(u^{-1}B)\eta_n(u)u^{-1}du.$$

Then from the calculation from (2.4) to (2.5) we have

$$\nu_{n+1}(B) = \int_0^\infty \phi(u)u^{-1}du \int_0^\infty \nu_0(v^{-1}B)\eta_{n-1}(v)v^{-1}dv = \int_0^\infty \phi(u)u^{-1}\nu_n(u^{-1}B)du$$

and μ_n with the Lévy measure ν_n belongs to $M_n(\mathbb{R}^d)$ by the induction hypothesis. Thus $\mu_{n+1} \in M_{n+1}(\mathbb{R}^d)$ follows from Definition 2.1. This completes the proof. ■

The following is a characterization of the Lévy measures of distributions in $M_n(\mathbb{R}^d)$ in terms of the g-function of the Lévy measure.

THEOREM 2.2. *Let $n \in \mathbb{N}$. A measure $\mu_n \in I_{\mathrm{sym}}(\mathbb{R}^d)$ belongs to $M_n(\mathbb{R}^d)$ if and only if its Lévy measure ν_n is either zero or it can be represented as*

$$\nu_n(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) g_{n,\xi}(r^2) r^{-1} dr, \qquad B \in \mathcal{B}_0(\mathbb{R}^d),$$

where $g_{n,\xi}(r)$ is represented as

$$g_{n,\xi}(s) = \int_0^\infty \eta_{n-1}(s^{1/2} y^{-1}) g_\xi(y^2) y^{-1} dy. \tag{2.6}$$

Here $g_\xi(r)$ is measurable in $\xi \in S$ and completely monotone in r for λ-a.e. ξ.

Proof. Recall from (1.1) and (1.2) that

$$\nu_0(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) g_\xi(r^2) r^{-1} dr.$$

We see by Theorem 2.1 that $\mu_n \in M_n(\mathbb{R}^d)$ if and only if ν_n is represented as

$$\begin{aligned}
\nu_n(B) &= \int_0^\infty \nu_0(u^{-1}B) \eta_{n-1}(u) u^{-1} du \\
&= \int_0^\infty \eta_{n-1}(u) u^{-1} du \int_S \lambda(d\xi) \int_0^\infty 1_{u^{-1}B}(y\xi) g_\xi(y^2) y^{-1} dy \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-1} dr \int_0^\infty \eta_{n-1}(ry^{-1}) g_\xi(y^2) y^{-1} dy \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) g_{n,\xi}(r^2) r^{-1} dr.
\end{aligned}$$

This completes the proof. ■

3. STOCHASTIC INTEGRAL CHARACTERIZATIONS OF $M_n(\mathbb{R}^d), n \in \mathbb{N}$

In this section, we characterize distributions in $M_n(\mathbb{R}^d)$ by stochastic integral representations. Let $I_{\log^n}(\mathbb{R}^d) = \left\{\mu \in I(\mathbb{R}^d) : \int_{|x|>1} (\log|x|)^n \mu(dx) < \infty\right\}$ and $m_n(x) = \int_x^\infty \eta_n(u) u^{-1} du, x > 0$. Since $m_n(x)$ is strictly monotone, we can define its inverse by $m_n^*(t)$, that is, $t = m_n(x)$ if and only if $x = m_n^*(t)$.

LEMMA 3.1. *For each* $n \in \mathbb{N}$ *there exists* $C_i > 0\,(i = 1, 2, 3)$ *such that for every* $0 < u < 1$

$$\int_u^\infty \eta_n(s)s^{-1}ds \leqslant C_1\big(\log(u^{-1})^{n+1} + 1\big), \tag{3.1}$$

$$\int_0^u \eta_n(s)ds \leqslant C_2 u, \tag{3.2}$$

and

$$\int_0^u s\eta_n(s)ds \leqslant C_3 u^{-2}. \tag{3.3}$$

Proof. We have (3.2) and (3.3) by standard calculations. For $n \in \mathbb{N}$ and $0 < u < 1$, we have

$$\begin{aligned}
&\int_u^\infty \eta_n(s)s^{-1}ds \\
&\quad = \int_0^\infty \phi(u_1)u_1^{-1}du_1 \dots \int_0^\infty \phi(u_n)u_n^{-1}du_n \int_u^\infty \phi\Big(s\big(\prod_{i=1}^n u_i\big)^{-1}\Big)ds \\
&\quad = \int_0^\infty \phi(u_1)u_1^{-1}du_1 \dots \int_0^\infty \phi(u_n)u_n^{-1}du_n \big(\int_u^1 + \int_1^\infty\big)\phi\Big(s\big(\prod_{i=1}^n u_i\big)^{-1}\Big)ds \\
&\quad \leqslant C\big(\int_u^1 (\log s^{-1})s^{-1}ds\big) + C \\
&\quad \leqslant C\big((\log u^{-1})^{n+1} + 1\big),
\end{aligned}$$

where and in what follows C will denote an absolute positive constant which may be different from one to another. Thus we have (3.1). This completes the proof. ■

THEOREM 3.1. *For each* $n \in \mathbb{N}$ *the stochastic integral*

$$\int_0^\infty m_n^*(t)dX_t^{(\mu)}$$

exists for every $\mu \in I_{\log^{n+1}}(\mathbb{R}^d)$.

Proof. For the proof, we need the following lemma, which is a special case of Proposition 5.5 of [10].

LEMMA 3.2. *Let* $\{X_t^{(\mu)}\}$ *be a Lévy process on* $\mathbb{R}^d$ *and* $f(t)$ *a real-valued measurable function on* $[0, \infty)$. *Let* (A, ν, γ) *be the triplet of* μ. *Then* $\int_0^\infty f(t)dX_t^{(\mu)}$ *exists if the following conditions are satisfied*:

$$\int_0^\infty f(t)^2dt < \infty, \tag{3.4}$$

and

(3.5) $$\int_0^\infty dt \int_{\mathbb{R}^d} \big(|f(t)x|^2 \wedge 1\big)\nu(dx) < \infty,$$

(3.6) $$\int_0^\infty \Big| f(t)\gamma + f(t) \int_{\mathbb{R}^d} x\Big(\big(1+|f(t)x|^2\big)^{-1} - (1+|x|^2)^{-1}\Big)\nu(dx)\Big| dt < \infty.$$

For the proof, it is enough to show that $f(t) = m_n^*(t)$ satisfies (3.4)–(3.6) in Lemma 3.2 for every $\mu \in I_{\log^{n+1}}(\mathbb{R}^d)$. Note that $m_n(+0) = \infty$ and $m_n(\infty) = 0$. Since

$$\int_0^\infty m_n^*(t)^2 dt = \int_0^\infty s^2 \eta_n(s) s^{-1} ds$$
$$= 2^{-(n+1)} E(|Z_0 Z_1 \dots Z_n|) = (2\pi)^{-(n+1)/2} < \infty,$$

we have (3.4).

As to (3.5), we have

$$\int_0^\infty dt \int_{\mathbb{R}^d} \big(|m_n^*(t)x|^2 \wedge 1\big)\nu(dx) = -\int_0^\infty dm_n(s) \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(dx)$$
$$= \int_0^\infty \eta_n(s)s^{-1}ds\big(\int_{|x|\leqslant 1/s} |sx|^2\nu(dx) + \int_{|x|>1/s} \nu(dx)\big) =: I_1 + I_2,$$

say. Here

$$I_1 = \int_{\mathbb{R}^d} |x|^2\nu(dx) \int_0^{1/|x|} s\eta_n(s)ds$$
$$= \big(\int_{|x|\leqslant 1} + \int_{|x|>1}\big)|x|^2\nu(dx) \int_0^{1/|x|} s\eta_n(s)ds =: I_{11} + I_{12},$$

say, and

$$I_{11} \leqslant \int_{|x|\leqslant 1} |x|^2\nu(dx) \int_0^\infty s\eta_n(s)ds < \infty.$$

We have the finiteness of I_{12} by (3.3) in Lemma 3.1. Also,

$$I_2 = \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty \eta_n(s)s^{-1}ds = \big(\int_{|x|\leqslant 1} + \int_{|x|>1}\big)\nu(dx) \int_{1/|x|}^\infty \eta_n(s)s^{-1}ds$$
$$=: I_{21} + I_{22},$$

say. As to I_{21}, we have

$$I_{21} \leqslant \int_{|x|\leqslant 1} \nu(dx) \int_0^\infty \phi(u_1)u_1^{-1}du_1 \ldots \int_0^\infty \phi(u_n)u_n^{-1}du_n \big(\prod_{i=1}^n u_i\big)^2 \int_{1/|x|}^\infty 2s^{-3}ds$$
$$\leqslant C \int_{|x|\leqslant 1} |x|^2\nu(dx) < \infty.$$

We have the finiteness of I_{22} by (3.1) in Lemma 3.1.

For (3.6), we have

$$\int_0^\infty \Big| m_n^*(t)\gamma + m_n^*(t) \int_{\mathbb{R}^d} x\Big(\big(1+|m_n^*(t)x|^2\big)^{-1} - (1+|x|^2)^{-1}\Big)\nu(dx)\Big| dt$$
$$\leqslant -|\gamma| \int_0^\infty s dm_n(s)$$
$$-\int_0^\infty \big| s \int_{\mathbb{R}^d} x\big((1+|sx|^2)^{-1} - (1+|x|^2)^{-1}\big)\nu(dx)\big| dm_n(s) =: I_3 + I_4,$$

say, where

$$I_3 \leqslant |\gamma| \int_0^\infty \eta_n(s)ds < \infty,$$
$$I_4 \leqslant \int_0^\infty \eta_n(s)ds \Big| \int_{\mathbb{R}^d} \Big((x|x|^2|s^2-1|)\big((1+|sx|^2)(1+|x|^2)\big)^{-1}\Big)\nu(dx)\Big|$$
$$\leqslant \int_0^\infty |s^2-1|\eta_n(s)ds \int_{\mathbb{R}^d} |x|^3 \left((1+|sx|^2)(1+|x|^2)\right)^{-1} \nu(dx)$$
$$= \int_0^\infty |s^2-1|\eta_n(s)ds \big(\int_{|x|\leqslant 1} + \int_{|x|>1} \big)|x|^3 \left((1+|sx|^2)(1+|x|^2)\right)^{-1}\nu(dx)$$
$$=: I_{41} + I_{42},$$

say. Here

$$I_{41} \leqslant \int_0^\infty |s^2-1|\eta_n(s)ds \int_{|x|\leqslant 1} |x|^3(1+|x|^2)^{-1}\nu(dx) < \infty,$$

and

$$I_{42} \leqslant \int_{|x|>1} |x|^3(1+|x|^2)^{-1}\nu(dx) \int_0^\infty (s^2+1)(1+|sx|^2)^{-1}\eta_n(s)ds$$
$$= \int_{|x|>1} |x|^3(1+|x|^2)^{-1}\nu(dx) \big(\int_0^1 + \int_1^\infty \big)(s^2+1)(1+|sx|^2)^{-1}\eta_n(s)ds$$
$$=: I_{421} + I_{422},$$

say. Furthermore,

$$I_{421} = \int_{|x|>1} |x|^3(1+|x|^2)^{-1}\nu(dx)\int_0^1(s^2+1)(1+|sx|^2)^{-1}\eta_n(s)ds$$
$$= \int_{|x|>1} |x|^3(1+|x|^2)^{-1}\nu(dx)\Big(\int_0^{1/|x|} + \int_{1/|x|}^1\Big)(s^2+1)(1+|sx|^2)^{-1}\eta_n(s)ds$$
$$=: I_{4211} + I_{4212},$$

say. We have

$$I_{4211} \leqslant \int_{|x|>1} |x|\nu(dx)\int_0^{1/|x|}\eta_n(s)ds \leqslant C\int_{|x|>1}\nu(dx) < \infty$$

by (3.2) in Lemma 3.1, and

$$I_{4212} \leqslant \int_{|x|>1}\nu(dx)\int_{1/|x|}^1\big(|sx|(s^2+1)\big)(1+|sx|^2)^{-1}\eta_n(s)s^{-1}ds$$
$$\leqslant \int_{|x|>1}\nu(dx)\int_{1/|x|}^1\eta_n(s)s^{-1}ds \leqslant \int_{|x|>1}\nu(dx)\int_{1/|x|}^\infty\eta_n(s)s^{-1}ds < \infty$$

by (3.1) in Lemma 3.1. Also

$$I_{422} = \int_{|x|>1} |x|^3(1+|x|^2)^{-1}\nu(dx)\int_1^\infty(s^2+1)(1+|sx|^2)^{-1}\eta_n(s)ds$$
$$\leqslant \int_{|x|>1} |x|^3(1+|x|^2)^{-2}\nu(dx)\int_1^\infty(s^2+1)\eta_n(s)ds < \infty.$$

Thus we have (3.6). This completes the proof. ∎

Let $\mathcal{M}_1 = \mathcal{M}^1 = \mathcal{M}$.

DEFINITION 3.1. Let $n \in \mathbb{N}$. Define the mapping $\mathcal{M}_{n+1}$ by

$$\mathcal{M}_{n+1}(\mu) = \mathcal{L}\Big(\int_0^\infty m_n^*(t)dX_t^{(\mu)}\Big), \quad \mu \in I_{\log^{n+1}}(\mathbb{R}^d),$$

and let $\mathcal{M}^{n+1}$ be the $(n+1)$ times iteration of $\mathcal{M}$. That is, $\mathcal{M}^{n+1}(\mu)$ can be defined with $\mathcal{M}^{n+1}(\mu) = \mathcal{M}\big(\mathcal{M}^n(\mu)\big)$ if and only if $\mathcal{M}^n(\mu)$ is defined and belongs to $I_{\log}(\mathbb{R}^d)$.

THEOREM 3.2. *For* $n \in \mathbb{N}$

$$M_n(\mathbb{R}^d) = \mathcal{M}\big(M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)\big).$$

Proof. The proof is almost the same as that of Theorem 2.4 (i) in [2]. Let $\mu_{n-1} \in M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)$ and $\mu_n = \mathcal{M}(\mu_{n-1})$. Also, let ν_{n-1} and ν_n be the Lévy measures of μ_{n-1} and μ_n, respectively. Then, by Proposition 2.1, we have $\nu_n(B) = \int_0^\infty \nu_{n-1}(s^{-1}B)\phi(s)s^{-1}ds$. Thus $\mu_n \in M_n(\mathbb{R}^d)$ by Definition 2.1, and $\mathcal{M}\big(M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)\big) \subset M_n(\mathbb{R}^d)$.

Conversely, suppose that $\mu_n \in M_n(\mathbb{R}^d)$. Then, by the definition of $M_n(\mathbb{R}^d)$ and Proposition 2.1 again, we see that $\mu_n = \mathcal{L}\big(\int_0^\infty m^*(t)dX_t^{(\mu)}\big)$ for some $\mu \in M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)$. This means that $\mu_n \in \mathcal{M}\big(M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)\big)$, and

$$M_n(\mathbb{R}^d) \subset \mathcal{M}\big(M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)\big),$$

completing the proof. ∎

COROLLARY 3.1. *For* $n \in \mathbb{N}$

$$M_n(\mathbb{R}^d) = \mathcal{M}^{n+1}\big(I_{\log^{n+1}}(\mathbb{R}^d)\big).$$

We next show

THEOREM 3.3. *For* $n \in \mathbb{N}$

$$\mathcal{M}_{n+1}\big(I_{\log^{n+1}}(\mathbb{R}^d)\big) = \mathcal{M}^{n+1}\big(I_{\log^{n+1}}(\mathbb{R}^d)\big).$$

Proof. We note that $\widetilde{\mu} \in \mathcal{M}_{n+1}\big(I_{\log^{n+1}}(\mathbb{R}^d)\big)$ if and only if

$$\widetilde{\mu} = \mathcal{L}\Big(\int_0^\infty m_n^*(t)dX_t^{(\mu)}\Big), \quad \mu \in I_{\log^{n+1}}(\mathbb{R}^d),$$

and that $\widetilde{\mu} \in \mathcal{M}^{n+1}\big(I_{\log^{n+1}}(\mathbb{R}^d)\big)$ if and only if

$$\widetilde{\mu} = \mathcal{L}\Big(\int_0^\infty m^*(t)dX_t^{(\mu)}\Big), \quad \mu \in M_{n-1}(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d).$$

We next claim that, for any $\mu \in I_{\log^{n+1}}(\mathbb{R}^d)$,

$$\int_0^\infty \phi(u)u^{-1}du \int_0^\infty |C_\mu(uvz)|\eta_{n-1}(v)v^{-1}dv < \infty, \quad z \in \mathbb{R}^d. \tag{3.7}$$

If it is proved, we can exchange the order of the integrals in the calculation of cumulants below.

The proof of (3.7) is as follows. The idea is from Barndorff–Nielsen et al. [3]. If the generating triplet of μ is (A,ν,γ), then

$$|C_\mu(z)| \leqslant 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$\begin{aligned}|C_\mu(uvz)| \leqslant{}& 2^{-1}(\mathrm{tr}A)u^2v^2|z|^2 + |\gamma||u||v||z| + \int_{\mathbb{R}^d} |g(z,uvx)|\nu(dx)\\ &+ \int_{\mathbb{R}^d} |g(uvz,x) - g(z,uvx)|\nu(dx) =: J_1 + J_2 + J_3 + J_4,\end{aligned}$$

say. The finiteness of $\int_0^\infty \phi(u)u^{-1}du\int_0^\infty (J_1+J_2)\eta_{n-1}(v)v^{-1}dv$ is easily to be shown by the same calculation as in the proof of Theorem 3.1.

Noting that $|g(z,x)| \leqslant C_z|x|^2(1+|x|^2)^{-1}$ with a positive constant C_z depending on z, we have

$$\begin{aligned}&\int_0^\infty \phi(u)u^{-1}du\int_0^\infty J_3\eta_{n-1}(v)v^{-1}dv\\ &\leqslant C_z\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty \phi(u)u^{-1}du\int_0^\infty |uvx|^2(1+|uvx|^2)^{-1}\eta_{n-1}(v)v^{-1}dv\\ &= C_z\int_{\mathbb{R}^d}\nu(dx)\int_0^\infty |sx|^2(1+|sx|^2)^{-1}\eta_n(s)s^{-1}ds\\ &= C_z\Big(\int_{|x|\leqslant 1}\nu(dx) + \int_{|x|>1}\nu(dx)\Big)\int_0^\infty |sx|^2(1+|sx|^2)^{-1}\eta_n(s)s^{-1}ds\\ &=: J_{31} + J_{32},\end{aligned}$$

say, and

$$J_{31} \leqslant C_z\int_{|x|\leqslant 1}|x|^2\nu(dx)\int_0^\infty s\eta_n(s)ds < \infty,$$

$$\begin{aligned}J_{32} &= C_z\int_{|x|>1}\nu(dx)\Big(\int_0^{1/|x|} + \int_{1/|x|}^\infty\Big)|sx|^2(1+|sx|^2)^{-1}\eta_n(s)s^{-1}ds\\ &=: J_{321} + J_{322},\end{aligned}$$

say. We have

$$J_{321} \leqslant 2^{-1}\int_{|x|>1}|x|\nu(dx)\int_0^{1/|x|}\eta_n(s)ds < \infty,$$

by the finiteness of I_{4211} in the proof of Theorem 3.1.

Also, we have the finiteness of J_{322} by (3.1) in Lemma 3.1.

As to J_4, note that for $a > 0$

$$\begin{aligned}|g(az,x) - g(z,ax)| &= |\langle az, x\rangle||x|^2|1-a^2|(1+|x|^2)^{-1}(1+a|x|^2)^{-1} \\ &\leqslant |z||x|^3 a(1+a^2)(1+|x|^2)^{-1}(1+a|x|^2)^{-1}.\end{aligned}$$

Then

$$\begin{aligned}&\int_0^\infty \phi(u)u^{-1}du \int_0^\infty J_4\eta_{n-1}(v)v^{-1}dv \\ &\leqslant |z| \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \phi(u)u^{-1}du \\ &\quad\times \int_0^\infty |x|^3 uv(1+u^2v^2)(1+|x|^2)^{-1}(1+u^2v^2|x|^2)^{-1}\eta_{n-1}(v)v^{-1}dv \\ &= |z| \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty |x|^3 s(1+s^2)(1+|x|^2)^{-1}(1+|sx|^2)^{-1}\eta_n(s)s^{-1}ds \\ &= |z|\Big(\int_{|x|\leqslant 1} + \int_{|x|>1} \Big)\nu(dx) \int_0^\infty |x|^3(1+s^2)(1+|x|^2)^{-1}(1+|sx|^2)^{-1}\eta_n(s)ds \\ &=: J_{41} + J_{42},\end{aligned}$$

say. Here

$$\begin{aligned}J_{41} &\leqslant |z| \int_{|x|\leqslant 1} |x|^2\nu(dx) \int_0^\infty |x|(1+s^2)(1+|x|^2)^{-1}(1+|sx|^2)^{-1}\eta_n(s)ds \\ &\leqslant 2^{-1}|z| \int_{|x|\leqslant 1} |x|^2\nu(dx) \int_0^\infty (1+s^2)(1+|sx|^2)^{-1}\eta_n(s)ds \\ &\leqslant 2^{-1}|z| \int_{|x|\leqslant 1} |x|^2\nu(dx) \int_0^\infty (1+s^2)\eta_n(s)ds < \infty,\end{aligned}$$

and

$$J_{42} = |z| \int_{|x|>1} |x|^3(1+|x|^2)^{-1}\nu(dx) \int_0^\infty (1+s^2)(1+|sx|^2)^{-1}\eta_n(s)ds < \infty.$$

The finiteness of J_{42} follows from I_{42} in the proof of Theorem 3.1.

This completes the proof of (3.7).

If we calculate the necessary cumulants, we have

$$\begin{aligned}C_{\mathcal{M}_{n+1}(\mu)}(z) &= \int_0^\infty C_\mu\big(m_n^*(t)z\big)dt \\ &= -\int_0^\infty C_\mu(uz)dm_n(u) = \int_0^\infty C_\mu(uz)\eta_n(u)u^{-1}du,\end{aligned}$$

$$\begin{aligned} C_{\mathcal{M}^{n+1}(\mu)}(z) &= \int_0^\infty C_{\mathcal{M}^n(\mu)}\big(m^*(t)z\big)dt = \int_0^\infty dt \int_0^\infty C_\mu\big(m^*(t)m^*_{n-1}(s)z\big)ds \\ &= \int_0^\infty dm(u) \int_0^\infty C_\mu(uvz)dm_{n-1}(v) \\ &= \int_0^\infty \phi(u)u^{-1}du \int_0^\infty C_\mu(uvz)\eta_{n-1}(v)v^{-1}dv \\ &= \int_0^\infty C_\mu(yz)y^{-1}dy \int_0^\infty \phi(yv^{-1})\eta_{n-1}(v)v^{-1}dv \\ &= \int_0^\infty C_\mu(yz)\eta_n(y)y^{-1}dy = C_{\mathcal{M}_{n+1}(\mu)}(z). \end{aligned}$$

This completes the proof of Theorem 3.3. ■

The following is a goal of this section and an M_n-version of Proposition 1.1. Namely, any $\mu \in M_n(\mathbb{R}^d)$ has the stochastic integral representation defined in Definition 3.1.

THEOREM 3.4. *We have*

$$M_n(\mathbb{R}^d) = \mathcal{M}_{n+1}\big(I_{\log^{n+1}}(\mathbb{R}^d)\big).$$

P r o o f. The statement is an immediate consequence of Corollary 3.1 and Theorem 3.3. ■

4. THE CLASS $M_\infty(\mathbb{R}^d)$

THEOREM 4.1. *We have*

$$M_\infty(\mathbb{R}^d) \supset S_{\mathrm{sym}}(\mathbb{R}^d),$$

where $S_{\mathrm{sym}}(\mathbb{R}^d)$ *is the class of all symmetric stable distributions on* $\mathbb{R}^d$.

P r o o f. Let $n \geqslant 1$. When μ_A is Gaussian with zero mean and covariance matrix A, suppose $\{X_t\}$ is a Gaussian Lévy process such that the covariance matrix of X_1 is $c_n^{-1}A$, where $c_n = \int_0^\infty m_n^*(t)^2dt$. Then we have

$$\mu_A = \mathcal{L}\big(\int_0^\infty m_n^*(t)dX_t\big) \in M_n(\mathbb{R}^d)$$

for any $n \geqslant 1$. Hence $\mu \in M_\infty(\mathbb{R}^d)$.

When μ is non-Gaussian α-stable with the Lévy measure ν, we have

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)r^{-(1+\alpha)}dr = \int_S \lambda_n(d\xi) \int_0^\infty 1_B(r\xi)c_n r^{-(1+\alpha)}dr,$$

where

$$c_n = \int_0^\infty m_{n-1}^*(t)^\alpha dt \quad \text{and} \quad \lambda_n(d\xi) = c_n^{-1}\lambda(d\xi).$$

We also have

$$\begin{aligned} c_n r^{-(1+\alpha)} &= -r^{-(1+\alpha)} \int_0^\infty u^\alpha dm_{n-1}(u) = r^{-1} \int_0^\infty (ur^{-1})^\alpha \eta_{n-1}(u) u^{-1} dt \\ &= r^{-1} \int_0^\infty \eta_{n-1}(ry^{-1}) y^{-(1+\alpha)} dy = r^{-1} \int_0^\infty \eta_{n-1}(ry^{-1}) g(y^2) y^{-1} dy, \end{aligned}$$

where

$$g(s) = s^{-\alpha/2},$$

which is completely monotone. Thus, by Theorem 2.2, $c_n r^{-(1+\alpha)}$ can be regarded as $g_{n,\xi}(r)r^{-1}$, implying that ν is the Lévy measure of a distribution in $M_n(\mathbb{R}^d)$. This is true for all n, and thus $\mu \in M_\infty(\mathbb{R}^d)$. ■

5. MORE ABOUT THE CLASSES $M_n(\mathbb{R}^d)$ WHEN $d = 1$

When $d = 1$, it is known that μ is of type G if and only if $\mu = \mathcal{L}(V^{1/2}Z)$ for some infinitely divisible nonnegative random variable V independent of the standard normal random variable Z. That is, μ is a variance mixture of normal distributions. And in [2], we showed the following

PROPOSITION 5.1. *$\mu \in M(\mathbb{R})$ if and only if*

$$\mu = \mathcal{L}(V^{1/2}Z),$$

where $\mathcal{L}(V) \in I(\mathbb{R}_+)$ has an absolutely continuous Lévy measure ν_V of the form

$$\nu_V(dr) = \ell(r)r^{-1}\,dr, \quad r > 0, \tag{5.1}$$

and the function ℓ is given by

$$\ell(r) = \int_r^\infty (x-r)^{-1/2}\,\rho(dx), \tag{5.2}$$

where ρ is a measure on $(0,\infty)$ satisfying the integrability condition

$$\int_0^1 x^{1/2}\,\rho(dx) + \int_1^\infty (1+\log x)x^{-1/2}\,\rho(dx) < \infty. \tag{5.3}$$

We characterize the distribution of the random variance V in the case of $\mu \in M_n(\mathbb{R})$.

THEOREM 5.1. *Let* $n = 1, 2, \dots$ *A necessary and sufficient condition for that* $\mu \in M_0(\mathbb{R})$ *belongs to a smaller class* $M_n(\mathbb{R})$ *is the following*:

$$\rho(dx) = 2^{-1}\,(2\pi x)^{-1/2}\Big\{\int_0^\infty \phi(u_1)u_1^{-1}du_1 \dots \int_0^\infty \phi(u_{n-1})u_{n-2}^{-1}du_{n-2} \times \int_0^\infty \phi(u_{n-1})u_{n-1}^{-1}g\Big(x\big(\prod_{i=1}^{n-1}u_i\big)^{-2}\Big)du_{n-1}\Big\}dx, \tag{5.4}$$

where $g(\cdot)$ *is completely monotone.*

The proof is almost the same as that of Theorem 5.2 in [2].

Proof. (i) The "only if" part. Suppose $\mu \in M_n(\mathbb{R})$. Since $M_n(\mathbb{R}) \subset G(\mathbb{R}^d)$, we have $\mu = \mathcal{L}(V^{1/2}Z)$ for some $V \in I(\mathbb{R}_+)$. Thus, we get for $z \in \mathbb{R}$

$$\begin{aligned}
&E[\exp(izV^{1/2}Z)] = E[\exp(-Vz^2/2)]\\
&= \exp\Big\{-2^{-1}Az^2 + \int_{0+}^\infty \big(\exp(-vz^2/2) - 1\big)\,\nu_V(dv)\Big\}\\
&= \exp\Big\{-2^{-1}Az^2 + \int_{0+}^\infty \nu_V(dv)\int_{-\infty}^\infty \big(\exp(izv^{1/2}u) - 1\big)\phi(u)\,du\Big\}\\
&= \exp\Big\{-2^{-1}Az^2 + \int_{-\infty}^\infty \big(\exp(izx) - 1\big)dx\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\Big\},
\end{aligned}$$

where $A \geqslant 0$. Therefore, the Lévy measure ν of μ is of the form

$$\nu(dx) = \Big(\int_{0+}^\infty \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv)\Big)\,dx. \tag{5.5}$$

By Theorem 2.2, $\mu \in M_n(\mathbb{R})$ if and only if $\nu(dx) = |x|^{-1}g_n(x^2)dx$, where g_n is given by (2.6). Since $\mu \in M_0(\mathbb{R}^d)$, g_n is completely monotone. It can be written as

$$g_n(r) = \int_0^\infty e^{-ry/2}\,Q(dy), \quad r > 0,$$

for a measure Q on $(0, \infty)$ given by

$$Q(dy) = (2\pi)^{-1/2}\,(2y)^{-1}\Big\{\int_0^\infty \phi(u_1)u_1^{-1}du_1 \dots \int_0^\infty \phi(u_{n-1})u_{n-2}^{-1}du_{n-2} \times \int_0^\infty \phi(u_{n-1})u_{n-1}^{-1}g\Big(y^{-1}\big(\prod_{i=1}^{n-1}u_i\big)^{-2}\Big)du_{n-1}\Big\}dy,$$

where $g(\cdot)$ is completely monotone.

By (5.5), we get

$$\int_{0+}^{\infty} \phi(v^{-1/2}x)v^{-1/2}\,\nu_V(dv) = |x|^{-1}g_n(x^2). \tag{5.6}$$

Since

$$r^{-1/2} = (2\pi)^{-1/2}\int_0^{\infty} e^{-rw/2}w^{-1/2}\,dw, \quad r>0,$$

we obtain

$$\begin{aligned} r^{-1/2}g(r) &= (2\pi)^{-1/2}\int_0^{\infty}\int_0^{\infty} e^{-r(w+y)/2}w^{-1/2}\,dwQ(dy) \\ &= (2\pi)^{-1/2}\int_0^{\infty} Q(dy)\int_y^{\infty} e^{-ru/2}(u-y)^{-1/2}\,du \\ &= (2\pi)^{-1/2}\int_0^{\infty} e^{-ru/2}du\int_0^{u}(u-y)^{-1/2}\,Q(dy). \end{aligned}$$

Taking $x = r^{1/2} > 0$ in (5.6), we get

$$\begin{aligned} (2\pi)^{-1/2}&\int_{0+}^{\infty} e^{-r/2v}v^{-1/2}\,\nu_V(dv) \\ &= (2\pi)^{-1/2}\int_0^{\infty} e^{-ru/2}du\int_0^{u}(u-y)^{-1/2}\,Q(dy). \end{aligned} \tag{5.7}$$

Let

$$\begin{aligned} \rho(dx) &= -x^{1/2}Q\big(d(x^{-1})\big) \\ &= -2^{-1}\,(2\pi x)^{-1/2}\Big\{\int_0^{\infty}\phi(u_1)u_1^{-1}du_1\ldots\int_0^{\infty}\phi(u_{n-1})u_{n-2}^{-1}du_{n-2} \\ &\quad\times\int_0^{\infty}\phi(u_{n-1})u_{n-1}^{-1}g\Big(x\big(\prod_{i=1}^{n-1}u_i\big)^{-2}\Big)du_{n-1}\Big\}dx. \end{aligned} \tag{5.8}$$

Then $\ell(r)$ in (5.2) becomes

$$\begin{aligned} \ell(r) &= -\int_r^{\infty}(x-r)^{-1/2}x^{1/2}Q\big(d(x^{-1})\big) = \int_0^{r^{-1}}(y^{-1}-r)^{-1/2}y^{-1/2}Q(dy) \\ &= \int_0^{r^{-1}}(1-yr)^{-1/2}Q(dy) = r^{-1/2}\int_0^{r^{-1}}(r^{-1}-y)^{-1/2}Q(dy). \end{aligned}$$

Thus, by (5.7),

$$\int_{0+}^{\infty} e^{-r/2v}v^{-1/2}\,\nu_V(dv) = \int_0^{\infty} e^{-ru/2}u^{-1/2}\ell(u^{-1})\,du$$

or

$$\int_{0+}^{\infty} e^{-r/2v} v^{-1/2}\, \nu_V(dv) = \int_0^{\infty} e^{-r/2v} v^{-3/2} \ell_n(v)\, dv, \quad r > 0.$$

Therefore,

$$v^{-1/2}\, \nu_V(dv) = v^{-3/2} \ell(v)\, dv, \quad v > 0,$$

which yields (5.1).

The integrability condition (5.3) for Q is obtained from the fact that

$$\infty > \int_{\mathbb{R}} (x^2 \wedge 1)\, \nu(dx) = \int_{\mathbb{R}} (|x| \wedge |x|^{-1}) g_n(x^2) dx.$$

For, this yields that

$$\int_0^1 x dx \int_0^{\infty} \exp(-x^2 y/2)\, Q(dy) < \infty$$

and

$$\int_1^{\infty} x^{-1} dx \int_0^{\infty} \exp(-x^2 y/2)\, Q(dy) < \infty,$$

and hence

$$\int_0^{\infty} \big[y^{-1}\big(1 - \exp(-y/2)\big) + 2^{-1} \int_y^{\infty} u^{-1} \exp(-u/2)\, du\big] Q(dy) < \infty.$$

It is obvious that the above condition is equivalent to

$$\int_0^1 (1 + \log y^{-1}) Q(dy) + \int_1^{\infty} y^{-1} Q(dy) < \infty. \tag{5.9}$$

On the other hand,

$$\int_0^1 x^{1/2} \rho(dx) = -\int_0^1 x Q\big(d(x^{-1})\big) = \int_1^{\infty} y^{-1} Q(dy)$$

and

$$\int_1^{\infty} (1 + \log x) x^{1/2} \rho(dx) = -\int_1^{\infty} (1 + \log x) Q\big(d(x^{-1})\big) = \int_0^1 (1 + \log y^{-1}) Q(dy).$$

Thus, we get (5.3) from (5.9) and (5.4) by (5.8). The "only if" part is thus proved.

(ii) The "if" part. Suppose $\mu = \mathcal{L}(V^{1/2} Z)$ and the Lévy measure ν_V of V satisfies (5.1)–(5.3).

We first claim that the integrability condition (5.3) implies that ν_V is really a Lévy measure on $(0,\infty)$ of a positive infinitely divisible random variable, namely it satisfies

$$\int_0^\infty (r\wedge 1)\nu_V(dr)<\infty. \tag{5.10}$$

We have

$$\int_0^\infty (r\wedge 1)\nu_V(dr)=\int_0^1 r\nu_V(dr)+\int_1^\infty \nu_V(dr).$$

As to the first integral, we have

$$\begin{aligned}
\int_0^1 r\nu_V(dr) &= \int_0^1 \ell(r)dr=\int_0^1 dr\int_r^\infty (x-r)^{-1/2}\rho(dx)\\
&=\int_0^1 \rho(dx)\int_0^x (x-r)^{-1/2}dr+\int_1^\infty \rho(dx)\int_0^1 (x-r)^{-1/2}dr\\
&=2\int_0^1 x^{1/2}\rho(dx)+2\int_1^\infty \big(x^{1/2}-(x-1)^{1/2}\big)\rho(dx)\\
&\leqslant 2\int_0^1 x^{1/2}\rho(dx)+\text{const}\times\int_1^\infty x^{-1/2}\rho(dx)\\
&=-2\int_0^1 xQ\big(d(x^{-1})\big)-\text{const}\times\int_1^\infty Q\big(d(x^{-1})\big)\\
&=2\int_1^\infty x^{-1}Q(dx)+\text{const}\times\int_0^1 Q(dx).
\end{aligned}$$

Next, as to the second integral, we obtain

$$\begin{aligned}
\int_1^\infty \nu_V(dr) &= \int_1^\infty r^{-1}\ell(r)dr=\int_1^\infty r^{-1}dr\int_r^\infty (x-r)^{-1/2}\rho(dx)\\
&=\int_1^\infty \rho(dx)\int_1^x r^{-1}(x-r)^{-1/2}dr=\int_1^\infty (\log x+\text{const})x^{-1/2}\rho(dx)\\
&=-\int_1^\infty (\log x+\text{const})Q\big(d(x^{-1})\big)=\int_0^1 (\log x^{-1}+\text{const})Q(dx).
\end{aligned}$$

Therefore, (5.3) implies (5.10). Furthermore, as we have already seen, ν_μ is expressed as in (5.5). So, to complete the proof, it is enough to show that when we put

$$g_n(x^2)=|x|\int_0^\infty \phi(v^{-1/2}x)v^{-1/2}\nu_V(dv),$$

then $g_n(r)$ is as (2.6) in Theorem 2.2. However, for that, it is enough to follow the proof of the “only if” part from bottom to top. This completes the proof. ■

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REFERENCES

[1] T. Aoyama and M. Maejima, *Characterizations of subclasses of type G distributions on $\mathbb{R}^d$ by stochastic integral representations*, Bernoulli 13 (2007), pp. 148–160.
[2] T. Aoyama, M. Maejima and J. Rosiński, *A subclass of type G selfdecomposable distributions*, J. Theoret. Probab. 21 (2008), pp. 14–34.
[3] O. E. Barndorff-Nielsen, M. Maejima and K. Sato, *Some classes of multivariate infinitely divisible distributions admitting stochastic integral representation*, Bernoulli 12 (2006), pp. 1–33.
[4] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd edition, Wiley, 1966.
[5] Z. J. Jurek, *Relations between the s-selfdecomposable and selfdecomposable measures*, Ann. Probab. 13 (1985), pp. 592–608.
[6] B. Rajput and J. Rosiński, *Spectral representations of infinitely divisible processes*, Probab. Theory Related Fields 82 (1989), pp. 451–487.
[7] J. Rosiński, *On series representations of infinitely divisible random vectors*, Ann. Probab. 18 (1990), pp. 405–430.
[8] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.
[9] K. Sato, *Stochastic integrals in additive processes and application to semi-Lévy processes*, Osaka J. Math. 41 (2004), pp. 211–236.
[10] K. Sato, *Additive processes and stochastic integrals*, Illinois J. Math. 50 (Doob Volume) (2006), pp. 825–851.
[11] K. Urbanik and W. A. Woyczyński, *Random integrals and Orlicz spaces*, Bull. Acad. Polon. Sci. 15 (1967), pp. 161–169.

Department of Mathematics
Faculty of Science and Technology
Tokyo University of Science
2641, Yamazaki, Noda 278-8510, Japan
E-mail: aoyama_takahiro@ma.noda.tus.ac.jp

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