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A LIMIT THEOREM FOR SUMS OF BOUNDED FUNCTIONALS OF LINEAR PROCESSES WITHOUT FINITE MEAN

BY

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Abstract. We consider the partial sum process of a bounded functional of a linear process and the linear process has no finite mean. We assume the innovations of the linear process are independent and identically distributed and that the distribution of the innovations belongs to the domain of attraction of an α-stable law and satisfies some additional assumptions. Then we establish the finite-dimensional convergence in distribution of the partial sum process to a stable Lévy motion.

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1. INTRODUCTION

We consider a linear process defined in (1.1) below and establish the finite-dimensional convergence in distribution of the partial sum process of a bounded functional of the linear process. The linear process is given by

$$X_i = \sum_{j=1}^{\infty} b_j \epsilon_{i-j}, \quad i = 1, 2, \ldots, \tag{1.1}$$

where $b_j \sim c_0 j^{-\beta}$ $(j \geqslant 1)$, $c_0 > 0$ and $\{\epsilon_i\}$ are independent and identically distributed.

We assume that ϵ_1 belongs to the domain of attraction of an α-stable law $(0 < \alpha < 2)$. Let $\mathrm{E}\{\epsilon_1\} = 0$ when $\alpha > 1$. In this paper $a_j \sim a'_j$ means $a_j/a'_j \to 1$ as $j \to \infty$. Then a sufficient condition for the existence of X_i is that $\alpha\beta > 1$.

We study the partial sum process defined by

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{\lfloor nt \rfloor} \big(K(X_i) - \mathrm{E}\{K(X_i)\}\big), \quad 0 \leqslant t \leqslant 1, \tag{1.2}$$

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where $K(x)$ is any bounded function on R and $\lfloor a \rfloor$ stands for the largest integer less than or equal to a. When $0 < \alpha < 1$ and $1 < \alpha\beta < 2$, we establish the convergence in distribution of finite-dimensional distributions of (1.2) to those of an $\alpha\beta$-stable Lévy motion under a set of mild assumptions on ϵ_1 in Theorem 2.1 below. See Samorodnitsky and Taqqu [12] for details on stable laws and stable Lévy motions. We can include the case of $\alpha = 1$ if we deal with slowly varying functions in Lemmas 4.1 and 4.2 below. However, we do not include the case of $\alpha = 1$ to make this paper more readable and easier to understand.

A lot of researchers have been studying the asymptotic properties of partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ when $\{X_i\}$ is a linear process with i.i.d. innovations and have derived the asymptotic distributions in various cases. Concentrating on the cases where ϵ_1 belongs to the domain of attraction of an α-stable law $(0 < \alpha < 2)$, we refer to the relevant results here. There are four cases of (a) to (d) defined in Table 1 below.

TABLE 1

(a)	(b)	(c)	(d)
$0 < \alpha < 2$	$1 < \alpha < 2$	$0 < \alpha < 1$	$1 < \alpha < 2$
$\beta > 2/\alpha$	$1 < \beta < 2/\alpha$	$1 < \beta < 2/\alpha$	$1/\alpha < \beta < 1$
Hsing [8]	Surgailis [13]	none	Koul and Surgailis [9]
Pipiras and Taqqu [11]			

In the case of (a), the normalization constant is $n^{1/2}$ and the limiting distribution is the Brownian motion. Our main result, Theorem 2.1, shows that the asymptotic distribution in the case of (b) carries over to the case of (c). We need some new techniques to tackle the case of (c), for example, equation (2.5), Proposition 2.3 and Lemma 4.2 below. The normalization constant is $n^{1/(\alpha\beta)}$ and the limiting distribution is an $\alpha\beta$-stable Lévy motion in both cases. In the case of (d), the normalization constant is $n^{1-\beta+1/\alpha}$ and the limiting distribution is an α-stable fractional motion. We owe the above exposition to Table 1 of Surgailis [13]. Theorem 2 of [8] is about the cases of (b), (c), and (d). However, the result contradicts the above results and is known to be wrong.

There are some other relevant papers. Among them, Surgailis [14] investigate the asymptotics of the partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ in some other setups where $\mathrm{E}\{|\epsilon_1|^2\} < \infty$ and Wu [16] deals with the partial sum processes of unbounded $K(x)$ in the cases of (a) and (d). For the partial sum process (1.2), only convergence in distribution of finite-dimensional distributions is established in Koul and Surgailis [9], Surgailis [13], [14], Pipiras and Taqqu [11], and Wu [16] as in the present paper. Note that the functional central limit theorem is obtained in Hsing [8] with an additional assumption on β and that the uniform invariance principle of the empirical process of $\{X_i\}$ is established in Koul and Surgailis [9].

All the above papers crucially depend on Ho and Hsing [5], [6]. The authors of the two papers applied the martingale decomposition approach and successfully studied the asymptotic properties of the partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ and empirical processes of $\{X_i\}$ when $\{X_i\}$ is a long-range dependent linear process with $\mathrm{E}\{|\epsilon_1|^4\} < \infty$. Our result crucially depends on Ho and Hsing [5], [6] through Surgailis [13], too. Koul and Surgailis [10] is an excellent expository paper on the martingale decomposition approach. Hannan [4] is an earlier paper of martingale decomposition approaches. Some results in Pipiras and Taqqu [11] are also important to the proofs of Lemmas 4.1 and 4.2 below.

The paper is organized as follows. We state the assumptions and the main theorem in Section 2. The theorem is proved and the propositions for the proof are also presented in the section. Those propositions are verified in Section 3. All the technical lemmas and the proofs are confined to Section 4.

2. LIMIT THEOREM

We begin with the assumptions and the notation. Next we state Theorem 2.1. Then we describe the propositions for the proof of Theorem 2.1 and present the proof at the end of this section. Hereafter we assume $0 < \alpha < 1$ and $1 < \alpha\beta < 2$. Recall that $b_j \sim c_0 j^{-\beta}$ and let b_j be nonnegative for simplicity of presentation.

In this paper, C, C_1, and C_2 stand for generic positive constants and their values change from place to place and $a \vee b$ and $a \wedge b$ are defined by max$\{a, b\}$ and min$\{a, b\}$, respectively. The range of integration is the whole real line when it is omitted.

We denote the distribution function and the characteristic function of ϵ_1 by $G(x)$ and $\phi(\theta)$, respectively. Assumptions A1 and A2 below are on $G(x)$ and $\phi(\theta)$, respectively. Assumption A1 means that ϵ_1 belongs to the domain of an α-stable law. Assumptions A1 and A2 ensure desirable properties of density functions.

A1. There is an $\alpha \in (0, 1)$ satisfying

$$\lim_{x\to-\infty} G(x)|x|^\alpha = c_1, \quad \lim_{x\to\infty} \big(1 - G(x)\big)x^\alpha = c_2, \quad \text{and} \quad c_1 + c_2 > 0.$$

A2. $|\phi(\theta)| < C(1 + |\theta|)^{-\delta}$ for some positive δ.

We always assume that Assumptions A1 and A2 hold.

We introduce some more notation to define another assumption. We decompose X_i into

$$X_i = X_{i,j} + \tilde{X}_{i,j}, \tag{2.1}$$

where

$$X_{i,j} = \sum_{l=1}^{j-1} b_l \epsilon_{i-l} \quad \text{and} \quad \tilde{X}_{i,j} = \sum_{l=j}^{\infty} b_l \epsilon_{i-l}.$$

Let $F_j(x)$ and $\tilde{F}_j(x)$ stand for the distribution functions of $X_{i,j}$ and $\tilde{X}_{i,j}$, respectively. The proof of Lemma 1 in [3] and Assumption A2 imply that $\tilde{F}_j(x)$, $j = 1, 2, \ldots$, are at least three times continuously differentiable and that $F_j(x)$, $j = s_0, s_0 + 1, \ldots$, are at least three times continuously differentiable for a sufficiently large positive integer s_0. Besides all the derivatives of $F_j(x)$ are bounded up to the third order uniformly in x and j. We write $f(x)$ and $F(x)$ for $f_\infty(x)$ and $F_\infty(x)$, respectively. Then $f(x)$ is the density function of X_1.

We state Assumption A3.

A3. We can choose a positive $\gamma \in (0, \alpha)$ such that

$$|F''(x)| + |F_j''(x)| \leqslant C(1 + |x|)^{-(1+\gamma)}$$

and

$$|F''(y) - F''(x)| + |F_j''(y) - F_j''(x)| \leqslant C|x - y|(1 + |x|)^{-(1+\gamma)}$$

for $|x - y| \leqslant 1$, uniformly in x and $j \geqslant s_0$.

When ϵ_1 follows an α-stable law $(0 < \alpha < 1)$, Assumptions A1–A3 hold. See Remark 2.1 below. We cannot apply the arguments in the proof of Lemma 4.2 of [9] and need the part of $j \geqslant s_0$ in Assumption A3 since $0 < \alpha < 1$ here.

REMARK 2.1. Let $S_\alpha(\sigma, \eta, \mu)$ stand for α-stable law. Then the characteristic function of $S_\alpha(\sigma, \eta, \mu)$ has the form

$$S_\alpha(\sigma, \eta, \mu) = \begin{cases} \exp\big\{ -\sigma^\alpha|\theta|^\alpha\big(1 - \mathrm{i}\eta \mathrm{sign}(\theta)\tan(\pi\alpha/2)\big) + \mathrm{i}\mu\theta\big\} & \text{for } \alpha \neq 1, \\ \exp\big\{ -\sigma|\theta|\big(1 + (2/\pi)\mathrm{i}\eta \mathrm{sign}(\theta)\log|\theta|\big) + \mathrm{i}\mu\theta\big\} & \text{for } \alpha = 1, \end{cases}$$

where $0 < \alpha \leqslant 2$, $0 < \sigma$, $-1 \leqslant \eta \leqslant 1$, $-\infty < \mu < \infty$, and i stands for the imaginary unit. When ϵ_1 follows an α-stable law $(\alpha \neq 1)$, the characteristic function $\phi_l(\theta)$ of $X_{1,l}$ is given by

$$\phi_l(\theta) = \exp\Big\{ -\Big(\sum_{j=1}^{l-1} b_j^\alpha\Big)|\theta|^\alpha\big(1 - \mathrm{i}\eta \mathrm{sign}(\theta)\tan(\pi\alpha/2)\big) + \mathrm{i}\Big(\sum_{j=1}^{l-1} b_j\Big)\mu\theta\Big\} \tag{2.2}$$

and $f_l^{(k)}(x)$, $k = 1, 2, \ldots$, is represented as

$$f_l^{(k)}(x) = \frac{(-\mathrm{i})^k}{2\pi}\int_0^\infty \theta^k\phi_l(\theta)e^{-\mathrm{i}\theta x}d\theta + \frac{(-\mathrm{i})^k}{2\pi}\int_{-\infty}^0 \theta^k\phi_l(\theta)e^{-\mathrm{i}\theta x}d\theta. \tag{2.3}$$

By appealing to integration by parts as in the proof in Lemma 3 in [8], we can show that Assumption A3 holds.

We define S_m by

$$S_m = \sum_{i=1}^{m} \big(K(X_i) - \mathrm{E}\{K(X_i)\}\big). \tag{2.4}$$

We are ready to state the main theorem of this paper. We omit $n \to \infty$ since it is obvious from the context.

THEOREM 2.1. *Suppose that Assumptions* A1–A3 *hold and that* $K(x)$ *is a bounded function. Then finite-dimensional distributions of* $n^{-1/(\alpha\beta)} S_{\lfloor nt \rfloor}$, $t \in [0,1]$, *converge in distribution to those of an* $\alpha\beta$*-stable Lévy motion on* $[0,1]$. *The distribution at* t *of the* $\alpha\beta$*-stable Lévy motion is given by*

$$t^{1/(\alpha\beta)}(c_2^{1/(\alpha\beta)} c_K^+ L^+ + c_1^{1/(\alpha\beta)} c_K^- L^-),$$

where

$$c_K^{\pm} = \sigma \int_0^{\infty} \big(K_\infty(\pm u) - K_\infty(0)\big) u^{-(1+1/\beta)} du,$$

$$K_\infty(x) = \mathrm{E}\{K(X_1 + x)\}, \qquad \sigma = \left\{ \frac{c_0^\alpha(\alpha\beta - 1)}{\Gamma(2-\alpha\beta)|\cos(\pi\alpha\beta/2)|\beta^{\alpha\beta}} \right\}^{1/(\alpha\beta)},$$

and L^- *and* L^+ *are mutually independent random variables whose distribution are* $S_{\alpha\beta}(1,1,0)$, *respectively. See* b_j *and Assumption* A1 *for the definitions of* c_0, c_1, *and* c_2.

When $K(x)$ is bounded and integrable, Assumption A3 is not necessary. See [7] for the details and an application of Theorem 2.1 to kernel density estimation.

We introduce decompositions of S_n before we state the propositions necessary to prove the theorems. Similar kinds of decomposition appear in [8] and [13]. We cannot replace $K_j(x)$ with $K_\infty(x)$ in W_n when we deal with the case of $0 < \alpha < 1$. This may have been a technical problem in this case. We put

$$S_n = (S_n - T_n) + (T_n - W_n) + W_n, \tag{2.5}$$

where

$$T_n = \sum_{i=1}^{n} \sum_{j=s_0}^{\infty} \big(K_j(b_j\epsilon_{i-j}) - \mathrm{E}\{K_j(b_j\epsilon_{i-j})\}\big), \tag{2.6}$$

$$K_j(x) = \mathrm{E}\{K(X_{1,j} + x)\}, \quad j \geqslant s_0, \tag{2.7}$$

$$W_n = \sum_{i=1}^{n} \sum_{j=s_0}^{\infty} \big(K_j(b_j\epsilon_i) - \mathrm{E}\{K_j(b_j\epsilon_i)\}\big), \tag{2.8}$$

Since

$$K_j(x) = \int K(\xi) f_j(\xi - x) d\xi,$$

it is easy to see from Assumption A3 that $K_j(x)$, $j \geqslant s_0$, and $K_\infty(x)$ are continuously differentiable and that all the derivatives are uniformly bounded in j and x.

We give some comments on the propositions before we state them. Propositions 2.1 and 2.2 are used to show that

$$n^{-1/(\alpha\beta)}(S_{\lfloor nt\rfloor} - T_{\lfloor nt\rfloor}) \quad \text{and} \quad n^{-1/(\alpha\beta)}(T_{\lfloor nt\rfloor} - W_{\lfloor nt\rfloor})$$

are asymptotically negligible for any fixed t. Propositions 2.1 and 2.2 correspond to Lemmas 5.1 and 5.2 in [13], respectively. Proposition 2.3 implies the weak convergence of $n^{-1/(\alpha\beta)}W_{\lfloor nt\rfloor}$ in $D[0,1]$ and it is an adapted version of Lemma 3.1 in [13]. See A2 of [2] for an exposition on $D[0,1]$. We are not able to show that $n^{-1/(\alpha\beta)}(T_{\lfloor nt\rfloor} - W_{\lfloor nt\rfloor})$ is asymptotically negligible in $D[0,1]$ at present.

All the proofs are given in Section 3.

PROPOSITION 2.1. *Suppose that Assumptions* A1–A3 *hold and that* $K(x)$ *is a bounded function. Then for any* r *satisfying* $\alpha\beta < r < 2 \wedge (2\alpha\beta - 1)$ *there is a positive constant* C *such that*

$$\mathrm{E}\{|S_n - T_n|^r\} < C(n^{-2\alpha\beta+2+r} + n) \quad \textit{for any positive integer } n.$$

PROPOSITION 2.2. *Suppose that Assumptions* A1–A3 *hold and that* $K(x)$ *is a bounded function. Then for any* r *satisfying* $1 \leqslant r < \alpha\beta$ *there is a positive constant* C *such that*

$$\mathrm{E}\{|T_n - W_n|^r\} < Cn^{-\alpha\beta+r+1} \quad \textit{for any positive integer } n.$$

PROPOSITION 2.3. *Suppose that Assumptions* A1–A3 *hold and that* $K(x)$ *is a bounded function. Then*

$$\sum_{j=s_0}^{\infty} \big(K_j(b_j\epsilon_1) - \mathrm{E}\{K_j(b_j\epsilon_1)\}\big)$$

belongs to the domain of attraction of an $\alpha\beta$*-stable law. As a result*,

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{n}\sum_{j=s_0}^{\infty} \big(K_j(b_j\epsilon_i) - \mathrm{E}\{K_j(b_j\epsilon_i)\}\big)$$

converges in distribution to $c_2^{1/(\alpha\beta)}c_K^+L^+ + c_1^{1/(\alpha\beta)}c_K^-L^-$. *See Theorem* 2.1 *for the definitions of* $c_K^\pm$ *and* $L^\pm$.

Now we are prepared to prove Theorem 2.1.

Proof of Theorem 2.1. Note that

$$\text{(2.9)} \quad n^{-1/(\alpha\beta)} \sum_{i=1}^{\lfloor nt\rfloor} \big(K(X_i) - \mathrm{E}\{K(X_i)\}\big)$$
$$= n^{-1/(\alpha\beta)}(S_{\lfloor nt\rfloor} - T_{\lfloor nt\rfloor}) + n^{-1/(\alpha\beta)}(T_{\lfloor nt\rfloor} - W_{\lfloor nt\rfloor}) + n^{-1/(\alpha\beta)}W_{\lfloor nt\rfloor}.$$

Since $\sum_{j=s_0}^{\infty}\big(K_j(b_j\epsilon_1)-\mathrm{E}\{K_j(b_j\epsilon_1)\}\big)$ belongs to the domain of attraction of the $\alpha\beta$-stable law in Proposition 2.3, the weak convergence of $n^{-1/(\alpha\beta)}W_{\lfloor nt\rfloor}$, $0\leqslant t\leqslant 1$, in $D[0,1]$ follows from Theorem 2.4.10 in [2].

Next we deal with the first and second terms on the right-hand side of (2.9).

Choose r_1 satisfying the condition of Proposition 2.1. Then we have

$$\mathrm{E}\{|n^{-1/(\alpha\beta)}(S_{\lfloor nt\rfloor}-T_{\lfloor nt\rfloor})|^{r_1}\} \leqslant Cn^{-r_1/(\alpha\beta)}\{(\lfloor nt\rfloor)^{-2\alpha\beta+2+r_1}+\lfloor nt\rfloor\} \tag{2.10}$$

for any t larger than $1/n$. Note that

$$0<-2\alpha\beta+2+r_1<r_1/(\alpha\beta) \quad \text{and} \quad 1<r_1/(\alpha\beta).$$

We choose r_2 satisfying the condition of Proposition 2.3 and have

$$\mathrm{E}\{|n^{-1/(\alpha\beta)}(T_{\lfloor nt\rfloor}-W_{\lfloor nt\rfloor})|^{r_2}\}\leqslant Cn^{-r_2/(\alpha\beta)}(\lfloor nt\rfloor)^{-\alpha\beta+1+r_2} \tag{2.11}$$

for any t larger than $1/n$. Note that

$$0<-\alpha\beta+1+r_2<r_2/(\alpha\beta)<1.$$

The desired result follows from the weak convergence of $n^{-1/(\alpha\beta)}W_{\lfloor nt\rfloor}$, (2.10), and (2.11). Hence the proof of the theorem is complete. ■

3. PROOFS OF PROPOSITIONS

In this section we prove Propositions 2.1–2.3. An argument similar to the proof of Proposition 2.2 can be found in [14], p. 337.

We write $\mathcal{F}_i$ for the σ-field generated by $\{\epsilon_j \mid j\leqslant i\}$.

P r o o f o f P r o p o s i t i o n 2.1. Write S_n and T_n as

$$S_n=\sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int K(\xi)\big(f_j(\xi-b_j\epsilon_{i-j}-\tilde{X}_{i,j+1})-f_{j+1}(\xi-\tilde{X}_{i,j+1})\big)d\xi + \sum_{i=1}^{n}\sum_{j=0}^{s_0-1}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\}-\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}], \tag{3.1}$$

$$T_n=\sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int K(\xi)\big(f_j(\xi-b_j\epsilon_{i-j})-\mathrm{E}\{f_j(\xi-b_j\epsilon_{i-j})\}\big)d\xi. \tag{3.2}$$

The right-hand side of (3.1) is typical of the martingale decomposition approach. By using the von Bahr and Esseen inequality (see [15]) and the boundedness of

$K(x)$, we obtain

$$(3.3)\quad \mathrm{E}\Big\{\Big|\sum_{i=1}^{n}\sum_{j=1}^{s_0-1}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\}-\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}]\Big|^r\Big\}$$
$$\leqslant C\sum_{j=1}^{s_0-1}\mathrm{E}\Big\{\Big|\sum_{i=1}^{n}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\}-\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}]\Big|^r\Big\}$$
$$\leqslant C\sum_{j=1}^{s_0-1}\sum_{i=1}^{n}\mathrm{E}\Big\{\Big|\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\}-\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}\Big|^r\Big\}$$
$$\leqslant Cn.$$

We evaluate

$$(3.4)\quad S_n-T_n-\sum_{i=1}^{n}\sum_{j=0}^{s_0-1}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\}-\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}]$$
$$=\sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int K(\xi)U_{i,j}(\xi)d\xi,$$

where

$$U_{i,j}(\xi)=f_j(\xi-b_j\epsilon_{i-j}-\tilde{X}_{i,j+1})$$
$$-f_{j+1}(\xi-\tilde{X}_{i,j+1})-f_j(\xi-b_j\epsilon_{i-j})+\mathrm{E}\{f_j(\xi-b_j\epsilon_{i-j})\}.$$

As in [13], the following expression is useful in evaluating (3.4):

$$(3.5)\quad \int K(\xi)U_{i,j}(\xi)d\xi$$
$$=\int\Big[\int\Big\{\int_{-b_ju}^{-b_j\epsilon_{i-j}}\big(f_j'(\xi+z-\tilde{X}_{i,j+1})-f_j'(\xi+z)\big)dz\Big\}K(\xi)d\xi\Big]G(du).$$

We consider seven cases to treat (3.5) and give upper bounds of (3.5) to each case.

(i) $|b_j\epsilon_{i-j}|\geqslant 1$, $|\tilde{X}_{i,j+1}|\geqslant 1$, $|b_ju|\geqslant 1$,
(ii) $|b_j\epsilon_{i-j}|\geqslant 1$, $|\tilde{X}_{i,j+1}|\geqslant 1$, $|b_ju|< 1$,
(iii) $|b_j\epsilon_{i-j}|\geqslant 1$, $|\tilde{X}_{i,j+1}|< 1$,
(iv) $|b_j\epsilon_{i-j}|< 1$, $|\tilde{X}_{i,j+1}|\geqslant 1$, $|b_ju|\geqslant 1$,
(v) $|b_j\epsilon_{i-j}|< 1$, $|\tilde{X}_{i,j+1}|\geqslant 1$, $|b_ju|< 1$,
(vi) $|b_j\epsilon_{i-j}|< 1$, $|\tilde{X}_{i,j+1}|< 1$, $|b_ju|\geqslant 1$,
(vii) $|b_j\epsilon_{i-j}|< 1$, $|\tilde{X}_{i,j+1}|< 1$, $|b_ju|< 1$.

We present the upper bounds and the proofs.

(i) $$|(3.5)| \leqslant CI(|b_j\epsilon_{i-j}| \geqslant 1)I(|\tilde{X}_{i,j+1}| \geqslant 1)|b_j|^{\alpha}.$$

P r o o f. We should deal with

$$\int\limits_{|b_j u|\geqslant 1} \big[\int \{f_j(\xi - b_j\epsilon_{i-j} - \tilde{X}_{i,j+1})$$
$$- f_j(\xi - b_j\epsilon_{i-j}) - f_j(\xi - b_j u - \tilde{X}_{i,j+1}) + f_j(\xi - b_j u)\}K(\xi)d\xi\big]dG(u)$$
$$\times I(|b_j\epsilon_{i-j}| \geqslant 1)I(|\tilde{X}_{i,j+1}| \geqslant 1).$$

The expression inside [ ] is bounded since $\int f_j(\xi)d\xi = 1$. Therefore Lemma 4.1 and $I(|b_j u| \geqslant 1)$ yield $|b_j|^{\alpha}$.

(ii) $$|(3.5)| \leqslant CI(|b_j\epsilon_{i-j}| \geqslant 1)I(|\tilde{X}_{i,j+1}| \geqslant 1).$$

P r o o f. The proof is similar to that of (i). We have no $|b_j|^{\alpha}$ in (ii) since $I(|b_j u| \geqslant 1)$ is replaced with $I(|b_j u| < 1)$.

(iii) $$|(3.5)| \leqslant CI(|b_j\epsilon_{i-j}| \geqslant 1)|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1).$$

P r o o f. We should deal with

$$\int\Big\{\int\big(|f_j(\xi - b_j\epsilon_{i-j} - \tilde{X}_{i,j+1}) - f_j(\xi - b_j\epsilon_{i-j})|$$
$$+ |f_j(\xi - b_j u - \tilde{X}_{i,j+1}) - f_j(\xi - b_j u)|\big)d\xi\Big\}dG(u)$$
$$\times I(|b_j\epsilon_{i-j}| \geqslant 1)I(|\tilde{X}_{i,j+1}| < 1).$$

The bound follows from the expression and Assumption A3.

(iv) $$|(3.5)| \leqslant CI(|\tilde{X}_{i,j+1}| \geqslant 1)|b_j|^{\alpha}.$$

P r o o f. The proof is similar to that of (i). In (iv), $I(|b_j\epsilon_{i-j}| \geqslant 1)$ is replaced with $I(|b_j\epsilon_{i-j}| < 1)$.

(v) $$|(3.5)| \leqslant CI(|\tilde{X}_{i,j+1}| \geqslant 1)\big(|b_j\epsilon_{i-j}|I(|b_j\epsilon_{i-j}| < 1) + |b_j|^{\alpha}\big).$$

P r o o f. We should deal with

$$\int\limits_{|b_j u|<1} \Big[\int\limits_{-b_j u}^{-b_j\epsilon_{i-j}} \Big\{\int |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z)|d\xi\Big\}dz\Big]dG(u) \tag{3.6}$$
$$\times I(|\tilde{X}_{i,j+1}| \geqslant 1)I(|b_j\epsilon_{i-j}| < 1).$$

The expression inside the braces is bounded due to Assumption A3. Hence (3.6) is bounded by

$$CI(|\tilde{X}_{i,j+1}| \geqslant 1)\bigl\{|b_j\epsilon_{i-j}|I(|b_j\epsilon_{i-j}| < 1) + \int_{|b_j u|<1} |b_j u| dG(u)\bigr\}. \tag{3.7}$$

The bound follows from (3.7) and Lemma 4.2.

(vi) $$|(3.5)| \leqslant C|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)|b_j|^{\alpha}.$$

P r o o f. We should deal with

$$\begin{aligned}(3.8)\quad \Bigl[\int_{|b_j u|\geqslant 1} \Bigl\{ \int_0^{-b_j\epsilon_{i-j}} \bigl(\int |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z)| d\xi\bigr) dz\Bigr\} dG(u) \\ + \int_{|b_j u|\geqslant 1} \bigl\{ \int |f_j(\xi - \tilde{X}_{i,j+1}) - f_j(\xi) - f_j(\xi - b_j u - \tilde{X}_{i,j+1}) \\ + f_j(\xi - b_j u)| d\xi \bigr\} dG(u)\Bigr] I(|\tilde{X}_{i,j+1}| < 1) I(|b_j\epsilon_{i-j}| < 1).\end{aligned}$$

Assumption A3 implies that (3.8) is bounded by

$$C|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1) \int_{|b_j u|\geqslant 1} dG(u).$$

Hence the bound follows from Lemma 4.2.

(vii) $$|(3.5)| \leqslant C|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)\bigl(|b_j\epsilon_{i-j}|I(|b_j\epsilon_{i-j}| < 1) + |b_j|^{\alpha}\bigr).$$

P r o o f. This bound follows from Assumption A3, (3.5) and Lemma 4.2. Actually, in this case, we have

$$\int \Bigl[\int \Bigl\{ \int_0^{|b_j\epsilon_{i-j}|} |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z)| dz \Bigr\} d\xi \Bigr] G(du) \leqslant C|\tilde{X}_{i,j+1}||b_j\epsilon_{i-j}|$$

and

$$\begin{aligned}\int \Bigl[\int \Bigl\{ \int_0^{|b_j u|} |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z)| dz \Bigr\} d\xi \Bigr] I(|b_j u| < 1) G(du) \\ \leqslant C|\tilde{X}_{i,j+1}||b_j|^{\alpha}.\end{aligned}$$

Here we used (4.11) in [13] to evaluate the expressions inside the brackets by taking $v = |b_j\epsilon_{i-j}|$ and $|b_j u|$, $x = -\infty$, and $y = \infty$.

The above bounds for (i)–(vii) and Lemmas 4.1 and 4.2 yield

$$\mathrm{E}\bigl\{\bigl|\int K(\xi) U_{i,j}(\xi) d\xi\bigr|^r\bigr\} \leqslant C j^{1-2\alpha\beta}. \tag{3.9}$$

We evaluate (3.4) by using (3.9). Notice that (3.4) is equal to the sum of the following expressions A_n and B_n:

$$A_n = \sum_{i=1}^{n} \sum_{l=s_0}^{n-i+1} \int K(\xi) U_{i+l-1,l}(\xi) d\xi, \tag{3.10}$$

$$B_n = \sum_{l=1}^{\infty} \sum_{j=1}^{n} \int K(\xi) U_{j,j+l}(\xi) I(j+l \geqslant s_0) d\xi. \tag{3.11}$$

We can apply the von Bahr and Esseen inequality to A_n and B_n because $U_{i,j}$ is $\mathcal{F}_{i-j}$-measurable and $\mathrm{E}\{U_{i,j}|\mathcal{F}_{i-j-1}\} = 0$ almost everywhere. We will derive the bounds for $\mathrm{E}\{|A_n|^r\}$ and $\mathrm{E}\{|B_n|^r\}$ by (3.9), the von Bahr and Esseen inequality, and Minkowski's inequality.

Noticing that $-2\alpha\beta + 1 < -r$, we have

$$\begin{aligned} \mathrm{E}\{|A_n|^r\} &\leqslant 2 \sum_{i=1}^{n} \mathrm{E}\Big\{\Big| \sum_{l=s_0}^{n-i+1} \int K(\xi) U_{i+l-1,l}(\xi) d\xi \Big|^r\Big\} \\ &\leqslant C \sum_{i=1}^{n} \Big(\sum_{l=1}^{n-i+1} (l^{-2\alpha\beta+1})^{1/r} \Big)^r \leqslant Cn. \end{aligned} \tag{3.12}$$

As for B_n, we have

$$\begin{aligned} \mathrm{E}\{|B_n|^r\} &\leqslant C \sum_{l=1}^{\infty} \Big(\sum_{j=1+l}^{n+l} (j^{-2\alpha\beta+1})^{1/r} \Big)^r \\ &\leqslant C n^{-2\alpha\beta+2+r} \sum_{l=1}^{\infty} \frac{1}{n} \bigg\{ \Big(\frac{l}{n}\Big)^{(-2\alpha\beta+1)/r+1} - \Big(1 + \frac{l}{n}\Big)^{(-2\alpha\beta+1)/r+1} \bigg\}^r \\ &\leqslant C n^{-2\alpha\beta+2+r} \int_0^{\infty} \bigg\{ u^{(-2\alpha\beta+1)/r+1} - \Big(1 + \frac{1}{n} + u\Big)^{(-2\alpha\beta+1)/r+1} \bigg\}^r du \\ &\leqslant C n^{-2\alpha\beta+2+r}. \end{aligned} \tag{3.13}$$

The integration is bounded because of the assumption on r.

The assertion of the proposition follows from (3.12) and (3.13). Hence the proof is complete. ■

P r o o f o f P r o p o s i t i o n 2.2. First we consider the properties of $K_j(x)$. Let $K_j(0) = 0$ by redefining $K_j(x)$ by $K_j(x) - K_j(0)$. Then we have

$$|K_j(x)| \leqslant C(1 \wedge |x|) \tag{3.14}$$

by the Taylor expansion at 0 and the uniform boundedness of the derivatives. By using (3.14) and Lemmas 4.1 and 4.2, we get

$$\begin{aligned} \mathrm{E}\{|K_j(b_j\epsilon_1)|^r\} &\leqslant C\big(\mathrm{P}(|b_j\epsilon_1| \geqslant 1) + \mathrm{E}\{|b_j\epsilon_1|^r I(|b_j\epsilon_1| < 1)\}\big) \\ &\leqslant C|b_j|^{\alpha} \leqslant Cj^{-\alpha\beta}. \end{aligned} \tag{3.15}$$

We write $T_n - W_n$ in the following form to apply the von Bahr and Esseen inequality:

$$T_n - W_n = -\sum_{k=1}^{n} A_n(k) + \sum_{k=1}^{\infty} B_n(k), \tag{3.16}$$

where

$$A_n(k) = \sum_{j=k\vee s_0}^{\infty} \big(K_j(b_j\epsilon_{n+1-k}) - \mathrm{E}\{K_j(b_j\epsilon_{n+1-k})\}\big),$$
$$B_n(k) = \sum_{j=k\vee s_0}^{k+n-1} \big(K_j(b_j\epsilon_{1-k}) - \mathrm{E}\{K_j(b_j\epsilon_{1-k})\}\big).$$

We evaluate the two terms on the right-hand side of (3.16) by using the von Bahr and Esseen inequality, Minkowski's inequality, and (3.15). Then we have

$$\mathrm{E}\Big\{\Big|\sum_{k=1}^{n} A_n(k)\Big|^r\Big\} \leqslant C\sum_{k=1}^{n}\Big(\sum_{j=k}^{\infty} j^{-\alpha\beta/r}\Big)^r \leqslant Cn^{-\alpha\beta+r+1}, \tag{3.17}$$

$$\begin{aligned}\mathrm{E}\Big\{\Big|\sum_{k=1}^{\infty} B_n(k)\Big|^r\Big\} &\leqslant C\sum_{k=1}^{\infty}\Big(\sum_{j=k}^{k+n-1} j^{-\alpha\beta/r}\Big)^r \\ &\leqslant Cn^{-\alpha\beta+r+1}\sum_{k=1}^{\infty}\frac{1}{n}\left\{\left(\frac{k}{n}\right)^{-\alpha\beta/r+1} - \left(1+\frac{k}{n}\right)^{-\alpha\beta/r+1}\right\}^r \\ &\leqslant Cn^{-\alpha\beta+r+1}\int_0^{\infty}\left\{u^{-\alpha\beta/r+1} - \left(1+\frac{1}{n}+u\right)^{-\alpha\beta/r+1}\right\}^r du \\ &\leqslant Cn^{-\alpha\beta+r+1}.\end{aligned} \tag{3.18}$$

The relations (3.16)–(3.18) yield the assertion of the proposition. Hence the proof is complete. ■

Proof of Proposition 2.3. Set

$$\eta_K(z) = \sum_{j=s_0}^{\infty} [K_j(b_j z) - \mathrm{E}\{K_j(b_j\epsilon_1)\}].$$

We deal only with the case where $c_K^- < 0 < c_K^+$. The other cases can be treated in the same way. If we establish

$$\lim_{z\pm\infty} |z|^{-1/\beta}\eta_K(z) = \frac{c_0^{1/\beta}}{\beta}\int_0^{\infty}\big(K_\infty(\pm s) - K_\infty(0)\big)s^{-(1+1/\beta)}ds, \tag{3.19}$$

the proposition will follow from the argument of Lemma 3.1 in [13]. Note that the argument implies that

$$\lim_{x\to-\infty}|x|^{\alpha\beta}\mathrm{P}\big(\eta_K(\epsilon_1)<x\big)=c_1\frac{c_0^{\alpha}}{\beta^{\alpha\beta}}\Big(-\int_0^{\infty}\big(K_\infty(-s)-K_\infty(0)\big)s^{-(1+1/\beta)}ds\Big)^{\alpha\beta}$$

and

$$\lim_{x\to\infty}x^{\alpha\beta}\mathrm{P}\big(\eta_K(\epsilon_1)>x\big)=c_2\frac{c_0^{\alpha}}{\beta^{\alpha\beta}}\Big(\int_0^{\infty}\big(K_\infty(s)-K_\infty(0)\big)s^{-(1+1/\beta)}ds\Big)^{\alpha\beta}$$

when we have (3.19). Consequently, the assertion of the proposition follows. We will establish only (3.19) when $z\to\infty$. We can proceed in the same way when $z\to-\infty$. Let us consider $K_j(x)-K_j(0)$ as in the proof of Proposition 2.2 and recall (3.14). Then it is easy to see that $\eta_K(z)$ is well defined.

As in [13], we represent $\eta_K(z)$ as

$$z^{-1/\beta}\eta_K(z)=z^{-1/\beta}\int_{s_0}^{\infty}\big(K_{\lfloor u\rfloor}(b_{\lfloor u\rfloor}z)-K_{\lfloor u\rfloor}(0)\big)du+O(z^{-1/\beta}). \tag{3.20}$$

By making a change of variables $zc_0u^{-\beta}=s$, we obtain

$$\begin{aligned}z^{-1/\beta}\eta_K(z)=\frac{c_0^{1/\beta}}{\beta}\int_0^{zc_0s_0^{-\beta}}&\big(K_{\lfloor (zc_0/s)^{1/\beta}\rfloor}(b_{\lfloor (zc_0/s)^{1/\beta}\rfloor}z)\\&-K_{\lfloor (zc_0/s)^{1/\beta}\rfloor}(0)\big)s^{-(1+1/\beta)}ds+O(z^{-1/\beta}).\end{aligned} \tag{3.21}$$

The inequality (3.14) implies that

$$\begin{aligned}&|K_{\lfloor (zc_0/s)^{1/\beta}\rfloor}(b_{\lfloor (zc_0/s)^{1/\beta}\rfloor}z)-K_{\lfloor (zc_0/s)^{1/\beta}\rfloor}(0)|\\&\qquad< C_1\big(\{(zc_0/s)^{-1}z\}\wedge C_2\big).\end{aligned} \tag{3.22}$$

Now we see that (3.19) follows from (3.21), (3.22), Lemma 4.3, and the dominated convergence theorem. Hence the proof of the proposition is complete. ∎

4. TECHNICAL LEMMAS

All the technical lemmas and the proofs are given in this section. First we state the lemmas, and then we give their proofs.

LEMMA 4.1. *Suppose that Assumptions* A1 *and* A2 *hold. Then*

$$\mathrm{P}(|b_j\epsilon_1|\geqslant 1)\leqslant C|b_j|^{\alpha}\quad \textit{and}\quad \mathrm{P}(|\tilde{X}_{1,j}|\geqslant 1)\leqslant Cj^{-\alpha\beta+1}\quad \textit{for any } j\geqslant 1.$$

LEMMA 4.2. *Suppose that Assumptions* A1 *and* A2 *hold. Then for any* $\gamma \geqslant 1$ *there exists a positive constant* C_γ *such that*

$$\mathrm{E}\{|b_j\epsilon_1|^\gamma I(|b_j\epsilon_1| < 1)\} \leqslant C_\gamma |b_j|^\alpha \ \text{ and } \ \mathrm{E}\{|\tilde{X}_{1,j}|^\gamma I(|\tilde{X}_{1,j}| < 1)\} \leqslant C_\gamma j^{-\alpha\beta+1}$$

for any $j \geqslant 1$.

LEMMA 4.3. *Suppose that Assumptions* A1 *and* A2 *hold and that* $K(x)$ *is a bounded function. Then*

$$\lim_{j\to\infty} \sup_x |K_j(x) - K_\infty(x)| = 0.$$

We prove the lemmas.

Proof of Lemma 4.1. The inequalities follow from (3.35) in [11] with $b_j \sim c_0 j^{-\beta}$. ∎

Proof of Lemma 4.2. We verify only the latter inequality with $\gamma = 1$. When j is sufficiently large, $2|b_l| < 1$ for any $l \geqslant j$. Then, by using (3.41) in [11], we get

$$\begin{aligned}
\mathrm{E}\{|\tilde{X}_{1,j}| I(|\tilde{X}_{1,j}| < 1)\} &\leqslant C \sum_{l=j}^{\infty} \Big(\int_0^{2|b_l|} dx + |b_l|^\alpha \int_{2|b_l|}^{1} x^{-\alpha} dx \Big) \\
&\leqslant C \sum_{l=j}^{\infty} [|b_l| + |b_l|^\alpha \{1 - (2|b_l|)^{1-\alpha}\}] \\
&\leqslant C \sum_{l=j}^{\infty} |b_l|^\alpha \leqslant Cj^{-\alpha\beta+1}. \quad ∎
\end{aligned}$$

Proof of Lemma 4.3. By using the differentiability and boundedness of $f_j(x)$ and Lemmas 4.1 and 4.2, we have

$$\begin{aligned}
|f(x) - f_j(x)| &\leqslant \int |f_j(x-y) - f_j(x)| d\tilde{F}_j(y) \\
&\leqslant \int_{|y|<1} |f_j(x-y) - f_j(x)| d\tilde{F}_j(y) + C \int_{|y|\geqslant 1} d\tilde{F}_j(y) \\
&\leqslant C\Big(\int_{|y|<1} |y| d\tilde{F}_j(y) + \mathrm{P}(|\tilde{X}_{1,j}| \geqslant 1)\Big) \leqslant Cj^{-\alpha\beta+1}.
\end{aligned}$$

Then

$$\lim_{j\to\infty} \sup_x |f(x) - f_j(x)| = 0, \tag{4.1}$$

which and Scheffé's theorem imply

$$\lim_{j\to\infty} \int |f(x) - f_j(x)| dx = 0.$$

Hence

$$|K_j(x) - K_\infty(x)| \leqslant \int |K(x+y)||f_j(y) - f(y)| dy \to 0 \quad \text{as } j \to \infty. \quad ∎$$

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