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WEIGHTED QUANTILE CORRELATION TESTS FOR GUMBEL, WEIBULL AND PARETO FAMILIES*

BY

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In memoriam Professor Béla Gyires (1909–2001)

Abstract. Weighted quantile correlation tests are worked out for the Gumbel location and location-scale families. Our theoretical emphasis is on the determination of computable forms of the asymptotic distributions under the null hypotheses, which forms are based on the solution of an associated eigenvalue-eigenfunction problem. Suitable transformations then yield corresponding composite goodness-of-fit tests for the Weibull family with unknown shape and scale parameters and for the Pareto family with an unknown shape parameter. Simulations demonstrate slow convergence under the null hypotheses, and hence the inadequacy of the asymptotic critical points. Other rounds of extensive simulations illustrate the power of all three tests: Gumbel against the other extreme-value distributions, Weibull against gamma distributions, and Pareto against generalized Pareto distributions with logarithmic slow variation.

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1. INTRODUCTION

Gumbel distributions are at the heart of extreme value theory, and in many applied situations there is a need to test for them. The problem is old, as seen through [17], and for the study of various test procedures we refer to [12], [14], [15], [17], to the more recent article [1], and the references in all these papers; in particular, Marohn [15] discusses the Gumbel testing literature divided into three

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classes. The main goal of the present paper is to introduce what may be called weighted quantile correlation tests for the location and the location-scale Gumbel families, and to make these tests, by suitable transformations, applicable also to Weibull and Pareto families.

Quantile correlation test statistics for goodness of fit to location and scale families, minimizing an empirical L_2-Wasserstein distance, were introduced in [7] and [6]. In these papers special attention was paid to the particularly important test for normality, the strong power properties of which – demonstrated in extensive simulation studies – are reported in [13]. In general, the limitations of quantile correlation tests in their demand of very light underlying tails were delineated in [2], and a remedial involvement of a weight function was proposed in [3] and [4]. Weight functions were independently suggested by de Wet [9], [10] with a different motivation, namely to achieve a desirable asymptotic loss of degree of freedom. Written at about the same time and competing with [4], minimal regularity conditions for asymptotic distributions under the null hypothesis are achieved in [8].

In general, for a *known* univariate distribution function $G(x) = \mathcal{P}\{X \leqslant x\}$, let $G^{\theta,\sigma}(x) = G\big((x-\theta)/\sigma\big)$, $x \in \mathbb{R}$, where $\mathbb{R}$ is the real line, and consider the location-scale family $\mathcal{G}_{\text{l-s}} = \{G^{\theta,\sigma} : \theta \in \mathbb{R}, \sigma > 0\}$ and the location family $\mathcal{G}_{\text{l}} = \{G^{\theta,1} : \theta \in \mathbb{R}\}$ generated by G. Denote by $Q_G(t)$,

$$Q_G(t) = G^{-1}(t) = \inf\{x \in \mathbb{R} : G(x) \geqslant t\}, \quad 0 < t < 1,$$

the quantile function pertaining to G, so that $Q_{G^{\theta,\sigma}}(t) = \theta + \sigma Q_G(t)$ for all $t \in (0,1)$, and for an integrable weight function $w : (0,1) \to [0,\infty)$ satisfying

$$\int_0^1 w(t)\,dt = 1$$

suppose that the generalized second moment

$$0 < \mu_2(G,w) = \int_0^1 Q_G^2(t)w(t)\,dt = \int_{-\infty}^{\infty} x^2 w\big(G(x)\big)\,dG(x) < \infty,$$

so that the corresponding first moment

$$\mu_1(G,w) = \int_0^1 Q_G(t)w(t)\,dt = \int_{-\infty}^{\infty} x\,w\big(G(x)\big)\,dG(x)$$

is also finite. Moreover, assume that the weighted variance $v(G,w) = \mu_2(G,w) - \mu_1^2(G,w) > 0$. Let $X_1, X_2, \dots, X_n$ be a sample of size n, independent random variables on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ with common distribution function F, with pertaining order statistics $X_{1,n} \leqslant X_{2,n} \leqslant \ldots \leqslant X_{n,n}$, and let $Q_n(\cdot) = Q_{F_n}(\cdot)$ be the sample quantile function corresponding to the sample distribution function $F_n(\cdot)$ for which $Q_n(t) = X_{k,n}$ if $(k-1)/n < t \leqslant k/n$, $k = 1, 2, \dots, n$. Then the location-scale invariant weighted quantile correlation statistic for the goodness-of-fit hypothesis $F \in \mathcal{G}_{\text{l-s}}$ is

$$V_n = 1 - \frac{\big[\int_0^1 Q_n(t)Q_G(t)w(t)\,dt - \mu_1(G,w)\int_0^1 Q_n(t)w(t)\,dt\big]^2}{v(G,w)\big[\int_0^1 Q_n^2(t)w(t)\,dt - \big(\int_0^1 Q_n(t)w(t)\,dt\big)^2\big]}$$

$$= 1 - \frac{\big[\sum_{k=1}^{n} X_{k,n}\big\{\int_{(k-1)/n}^{k/n} Q_G(t)w(t)\,dt - \mu_1(G,w)\int_{(k-1)/n}^{k/n} w(t)\,dt\big\}\big]^2}{v(G,w)\big[\sum_{k=1}^{n} X_{k,n}^2 \int_{(k-1)/n}^{k/n} w(t)\,dt - \big(\sum_{k=1}^{n} X_{k,n}\int_{(k-1)/n}^{k/n} w(t)\,dt\big)^2\big]},$$

while the location-invariant statistic for the hypothesis $F \in \mathcal{G}_{\text{l}}$ is

$$W_n = \int_0^1 \{Q_n(t) - Q_G(t)\}^2 w(t)\,dt - \big[\int_0^1 \{Q_n(t) - Q_G(t)\}w(t)\,dt\big]^2$$

$$= v(G,w) + \sum_{k=1}^{n} X_{k,n}^2 \int_{(k-1)/n}^{k/n} w(t)\,dt - \big[\sum_{k=1}^{n} X_{k,n}\int_{(k-1)/n}^{k/n} w(t)\,dt\big]^2$$

$$- 2\sum_{k=1}^{n} X_{k,n}\big\{\int_{(k-1)/n}^{k/n} Q_G(t)w(t)\,dt - \mu_1(G,w)\int_{(k-1)/n}^{k/n} w(t)\,dt\big\},$$

as derived in [4]. Under suitable regularity conditions on G, the asymptotic distributions of V_n and W_n are obtained in terms of quadratic integral functionals of a Brownian bridge $B(\cdot)$ in [4] and [8], where $\{B(t)\colon 0 \leqslant t \leqslant 1\}$ is a sample continuous Gaussian process with mean zero and covariance $\mathcal{E}\big(B(s)B(t)\big) = \min(s,t) - st$, $s,t \in [0,1]$, for which $B(0) = 0 = B(1)$ almost surely.

In the next section we derive these asymptotic distributions for the Gumbel families $\mathcal{G}_{\text{l-s}}$ and $\mathcal{G}_{\text{l}}$ for a special choice of the weight function and represent these distributions in computable forms. In Section 3 the asymptotic distribution functions are computed and the speed in the limit theorems under the null hypotheses is investigated by simulation as well as the power of the location-scale test under the other extreme value distributions as alternatives. In Section 4 we transform the results to the Weibull family on $[0,\infty)$ and investigate the power against gamma distributions. Finally, in Section 5 we do the same for the Pareto family on $[1,\infty)$, for which the power is looked at against the presence of logarithmic nuisance functions.

2. TESTS FOR GUMBEL FAMILIES

Setting $G(x) = e^{-e^{-x}}$ with density $g(x) = G'(x) = e^{-x}e^{-e^{-x}}$, $x \in \mathbb{R}$, the collection $\mathcal{G}_{\text{l-s}}$ becomes the location-scale Gumbel family, while $\mathcal{G}_{\text{l}}$ becomes the location Gumbel family of extreme value distributions. Here the quantile function is $Q(t) = Q_G(t) = -\log\log t^{-1}$, so that the density-quantile function is $g\big(Q(t)\big) = t\log t^{-1}$, $0 < t < 1$, where $\log = \log_e$ stands for the natural logarithm.

The crucial point is to choose the weight function w. The basic well-motivated idea of de Wet [9], [10] is the choice that makes the underlying implicit estimation of the location and scale parameters asymptotically efficient, which, as he proves separately for the two cases in [10], will then result in what he calls an asymptotic loss of a degree of freedom; see below for the appearance of the latter phenomenon. As de Wet points out in [10], in general this cannot be done with a single w jointly for both parameters. The constant weight function 1 achieves joint efficiency for the normal distribution and for no other distribution, as proved in [6], and presumably no weight function can do this for any other distribution. Hence for a given location-scale family, as for the present Gumbel family $\mathcal{G}_{\text{l-s}}$, the expedient choice of one of de Wet's optimal weight functions, either for scale or for location, appears reasonable. De Wet's [10] weight function for the scale case is $h_{\rm s}'\big(Q(t)\big)/\big(cQ(t)\big)$, $0 < t < 1$, where $h_{\rm s}(x) = -xg'(x)/g(x)$, $x \in \mathbb{R}$, and $c = \int_0^1 h_{\rm s}'\big(Q(t)\big)Q(t)\,dt$. This function is so prohibitively complicated that we did not check for it the validity of a possible counterpart of the corresponding first statements of Theorems 2.1 and 2.2 below, exactly because there seemed to be no hope for a counterpart of the second, distributional statements.

On the other hand, de Wet's [10] weight function $w(\cdot)$ for the location case is given by

$$w(t) = h_{\rm l}\big(Q(t)\big)/C, \qquad 0 < t < 1,$$

where $h_{\rm l}(x) = \big[\big(g'(x)\big)^2 - g'(x)g(x)\big]/g^2(x)$, $x \in \mathbb{R}$, and $C = \int_0^1 h_{\rm l}\big(Q(t)\big)\,dt$. Elementary calculation yields the remarkable simplification $h_{\rm l}(x) = e^{-x}$, $x \in \mathbb{R}$, so that $w(t) = \log t^{-1} = -\log t$, $0 < t < 1$. This $w(\cdot)$ is optimal in de Wet's sense for the Gumbel location class $\mathcal{G}_{\rm l}$, and this is also our pragmatic choice of the weight function for the Gumbel location-scale class $\mathcal{G}_{\text{l-s}}$.

We have of course $\int_0^1 w(t)\,dt = -\int_0^1 \log t\,dt = 1$, and

$$\begin{aligned}\mu_1 = \mu_1(G, w) &= -\int_0^1 \Big(\log\log\frac{1}{t}\Big)\log\frac{1}{t}\,dt\\ &= -\int_0^\infty y(\log y)e^{-y}\,dy = -\Gamma'(2) = \gamma - 1,\end{aligned}$$

$$\mu_2 = \mu_2(G, w) = \int_0^1 \Big(\log\log\frac{1}{t}\Big)^2 \log\frac{1}{t}\,dt = -\Gamma''(2) = \gamma^2 - 2\gamma + \frac{\pi^2}{6},$$

where $\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}\,dx$, $u > 0$, is the usual gamma function and $\gamma = -\Gamma'(1) = 0.577215664\ldots$ is Euler's constant, so that $v = v(G, w) = \mu_2 - \mu_1^2 =$

$(\pi^2-6)/6$. The two statistics introduced above take the concrete forms

$$V_n =$$
$$= 1 - \frac{6\big[\sum_{k=1}^{n} X_{k,n}\big\{\int_{(k-1)/t}^{k/n} (-\log\log t^{-1})\log t^{-1}\,dt - (\gamma-1)\int_{(k-1)/t}^{k/n} \log t^{-1}\,dt\big\}\big]^2}{(\pi^2-6)\big[\sum_{k=1}^{n} X_{k,n}^2 \int_{(k-1)/t}^{k/n} \log t^{-1}\,dt - \big(\sum_{k=1}^{n} X_{k,n}\int_{(k-1)/t}^{k/n} \log t^{-1}\,dt\big)^2\big]}$$

and

$$\begin{aligned} W_n = \frac{\pi^2-6}{6} &+ \sum_{k=1}^{n} X_{k,n}^2 \int_{(k-1)/t}^{k/n} \log\frac{1}{t}\,dt \\ &- \left[\sum_{k=1}^{n} X_{k,n}\int_{(k-1)/t}^{k/n} \log\frac{1}{t}\,dt\right]^2 \\ &- 2\sum_{k=1}^{n} X_{k,n}\left\{\int_{(k-1)/t}^{k/n}\left(-\log\log\frac{1}{t}\right)\log\frac{1}{t}\,dt - (\gamma-1)\int_{(k-1)/t}^{k/n}\log\frac{1}{t}\,dt\right\}. \end{aligned}$$

Denoting by $\xrightarrow{\mathcal{D}}$ convergence in distribution and understanding all asymptotic relations as $n\to\infty$ unless otherwise specified, the main asymptotic results are contained in the following two theorems.

THEOREM 2.1. *If $F \in \mathcal{G}_{\text{l-s}}$, then*

$$V_n^\star = nV_n - c_n^\star \xrightarrow{\mathcal{D}} V, \tag{2.1}$$

where

(2.2)
$$\begin{aligned} c_n^\star &= \frac{\int_{1/(n+1)}^{n/(n+1)} \big[t(1-t)/g^2\big(Q_G(t)\big)\big]\,w(t)\,dt}{\int_{1/(n+1)}^{n/(n+1)} Q_G^2(t)w(t)\,dt - \big[\int_{1/(n+1)}^{n/(n+1)} Q_G(t)w(t)\,dt\big]^2} \\ &= \frac{\int_{1/(n+1)}^{n/(n+1)}[(1-t)/(t\log t^{-1})]dt}{\int_{1/(n+1)}^{n/(n+1)}(\log\log t^{-1})^2\log t^{-1}\,dt - \big[\int_{1/(n+1)}^{n/(n+1)}(-\log\log t^{-1})\log t^{-1}\,dt\big]^2} \\ &= \frac{6}{\pi^2-6}\,\log\log n + \frac{6\gamma}{\pi^2-6} + o(1) \end{aligned}$$

and

$$V=\frac{1}{v(G,w)}\left\{\int_0^1\frac{B^2(t)-t(1-t)}{g^2\big(Q_G(t)\big)}\,w(t)\,dt-\left[\int_0^1\frac{B(t)}{g\big(Q_G(t)\big)}\,w(t)\,dt\right]^2\right\}$$

$$-\left[\frac{1}{v(G,w)}\int_0^1\frac{B(t)Q_G(t)}{g\big(Q_G(t)\big)}\,w(t)\,dt-\frac{\mu_1(G,w)}{v(G,w)}\int_0^1\frac{B(t)}{g\big(Q_G(t)\big)}\,w(t)\,dt\right]^2$$

$$=\frac{6}{\pi^2-6}\left\{\int_0^1\frac{B^2(t)-t(1-t)}{t^2\log t^{-1}}\,dt-\left[\int_0^1\frac{B(t)}{t}\,dt\right]^2\right\}$$

$$-\left[\frac{6}{\pi^2-6}\int_0^1\frac{B(t)(-\log\log t^{-1})}{t}\,dt-\frac{6(\gamma-1)}{\pi^2-6}\int_0^1\frac{B(t)}{t}\,dt\right]^2,$$

in which the integrals are meaningful in the space $L_2(\Omega,\mathcal{A},\mathcal{P})$. *Furthermore, the equality in distribution*

$$V\overset{\mathcal{D}}{=}\frac{6}{\pi^2-6}\left(-1+\sum_{k=2}^{\infty}\frac{Z_k^2-1}{k}\right)-\frac{36}{(\pi^2-6)^2}\left(\sum_{k=2}^{\infty}\frac{Z_k}{(k-1)k}\right)^2 \tag{2.3}$$

holds, where $Z_1,Z_2,Z_3,\dots$ *are independent standard normal random variables.*

THEOREM 2.2. *If* $F\in\mathcal{G}_1$, *then*

$$W_n^{\diamond}=nW_n-c_n^{\diamond}\xrightarrow{\mathcal{D}}W, \tag{2.4}$$

where

(2.5)
$$c_n^{\diamond}=\int_{1/(n+1)}^{n/(n+1)}\frac{t(1-t)}{g^2\big(Q_G(t)\big)}\,w(t)\,dt=\int_{1/(n+1)}^{n/(n+1)}\frac{1-t}{t\log t^{-1}}\,dt=\log\log n+\gamma+o(1)$$

and

$$W=\int_0^1\frac{B^2(t)-t(1-t)}{g^2\big(Q_G(t)\big)}\,w(t)\,dt-\left[\int_0^1\frac{B(t)}{g\big(Q_G(t)\big)}\,w(t)\,dt\right]^2$$

$$=\int_0^1\frac{B^2(t)-t(1-t)}{t^2\log t^{-1}}\,dt-\left[\int_0^1\frac{B(t)}{t}\,dt\right]^2.$$

Furthermore,

$$W\overset{\mathcal{D}}{=}-1+\sum_{k=2}^{\infty}\frac{Z_k^2-1}{k}, \tag{2.6}$$

where $Z_1,Z_2,Z_3,\dots$ *are as in Theorem* 2.1.

The loss of a degree of freedom is the phenomenon seen in (2.6) and in the first term of (2.3), which results from the representation

$$W \stackrel{\mathcal{D}}{=} \sum_{k=1}^{\infty} (Z_k^2 - 1)k^{-1} - Z_1^2,$$

so that Z_1^2 cancels, and essentially the same phenomenon occurs jointly in the second term of V yielding the full formula in (2.3). We emphasize that while the results of de Wet [10] predict this to happen, they do not guarantee the distributional representations in (2.3) and (2.6). Such representations should be individually obtained, if possible at all, for each problem of this sort through the determination of the eigenvalues and eigenfunctions of an $L_2(0,1)$ Hilbert–Schmidt integral operator associated with the covariance of the stochastic process

$$\big\{B(t)\sqrt{w(t)}/g\big(Q_G(t)\big) \colon 0 < t < 1\big\}.$$

P r o o f o f T h e o r e m s 2.1 a n d 2.2. In order to prove (2.1) and (2.4), we check the conditions of the respective parts (ii) of Theorems 3 and 2 in [4]. Since for

$$A(t) := \frac{t(1-t)\big|g'\big(Q(t)\big)\big|}{g^2\big(Q(t)\big)} = (1-t)\left|1 + \frac{1}{\log t}\right|$$

we have $\lim_{t\downarrow 0} A(t) = 1 = \lim_{t\uparrow 1} A(t)$, and hence $\sup_{0<t<1} A(t) < \infty$, condition (1) in [4] is trivially satisfied.

It is indeed part (ii) in both theorems that may work because

$$\int_0^1 \frac{t(1-t)}{g^2\big(Q(t)\big)}\, w(t)\, dt = \int_0^1 \frac{t-1}{t\log t}\, dt = \int_0^\infty \frac{1-e^{-y}}{y}\, dy \geqslant \int_{\log 2}^\infty \frac{1}{2y}\, dy = \infty,$$

that is, condition (2) in [4], for parts (i) of the theorems there, is violated. But, for condition (3) there, writing $\log^u x = (\log x)^u$ for $x, u > 0$, we have

$$\begin{aligned} I &:= \int_0^1\int_0^1 \frac{[\min(s,t) - st]^2}{g^2\big(Q(s)\big)g^2\big(Q(t)\big)}\, w(s)w(t)\, ds\, dt = \int_0^1\int_0^1 \frac{[\min(s,t) - st]^2}{s^2t^2(\log s)(\log t)}\, ds\, dt \\ &= 2\bigg\{\int_0^1 \bigg[\int_y^\infty \frac{(1-e^{-y})^2}{e^{-y}xy}\, e^{-x}\, dx\bigg] dy + \int_1^\infty \bigg[\int_y^\infty \frac{(1-e^{-y})^2}{e^{-y}xy}\, e^{-x}\, dx\bigg] dy\bigg\} \\ &=: 2\big\{I_1 + I_2\big\}, \end{aligned}$$

and, since $\lim_{y\downarrow 0}(1-e^{-y})^2/y^2 = 1$ and $\lim_{y\downarrow 0} y\log y = 0$, we see that

$$\begin{aligned} I_1 &\leqslant e\int_0^1 \frac{(1-e^{-y})^2}{y}\bigg[\int_y^\infty \frac{e^{-x}}{x}\, dx\bigg] dy \leqslant e\int_0^1 \frac{(1-e^{-y})^2}{y}\bigg[\int_y^1 \frac{1}{x}\, dx + \int_1^\infty e^{-x}\, dx\bigg] dy \\ &= \int_0^1 \frac{(1-e^{-y})^2}{y}\,(1 - e\log y) dy = \int_0^1 \frac{(1-e^{-y})^2}{y^2}\,(y - e\, y\log y) dy < \infty, \end{aligned}$$

and

$$I_2 \leqslant \int_1^\infty \frac{(1-e^{-y})^2}{e^{-y}y} \left[\int_y^\infty \frac{e^{-x}}{y}\, dx \right] dy = \int_1^\infty \frac{(1-e^{-y})^2}{y^2}\, dy \leqslant \int_1^\infty \frac{1}{y^2}\, dy = 1.$$

Thus $I < \infty$, validating condition (3) in [4].

We now consider condition (8) in [4], which is sufficient for both (4a) and (5a) there. It suffices to show that $C_n(\delta) \to 0$ for any fixed $\delta \in (1/2, 1)$, where

$$\begin{aligned} C_n(\delta) &= \frac{1}{n^{1-\delta}} \int_{1/(n+1)}^{n/(n+1)} \frac{[t(1-t)]^\delta}{g^2\big(Q(t)\big)}\, w(t)\, dt = \frac{1}{n^{1-\delta}} \int_{1/(n+1)}^{n/(n+1)} \frac{[t(1-t)]^\delta}{t^2 \log t^{-1}}\, dt \\ &= \frac{1}{n^{1-\delta}} \int_{\log[(n+1)/n]}^{\log(n+1)} \frac{[e^{-x}(1-e^{-x})]^\delta}{x e^{-x}}\, dx \\ &= \frac{1}{n^{1-\delta}} \int_{\log[(n+1)/n]}^{1} \frac{e^{x(1-\delta)}(1-e^{-x})^\delta}{x}\, dx + \frac{1}{n^{1-\delta}} \int_1^{\log(n+1)} \frac{e^{x(1-\delta)}(1-e^{-x})^\delta}{x}\, dx \\ &=: C_n^{\delta,1} + C_n^{\delta,2}. \end{aligned}$$

Since $1 - e^{-x} < x$ for $x > 0$, we have

$$C_n^{\delta,1} = \frac{1}{n^{1-\delta}} \int_{\log[(n+1)/n]}^{1} \frac{e^{x(1-\delta)}}{x^{1-\delta}} \left(\frac{1-e^{-x}}{x}\right)^\delta dx \leqslant \frac{e}{n^{1-\delta}} \int_0^1 \frac{1}{x^{1-\delta}}\, dx = \frac{e}{\delta n^{1-\delta}} \to 0$$

and

$$\begin{aligned} C_n^{\delta,2} &\leqslant \frac{1}{n^{1-\delta}} \int_1^{\log(n+1)} \frac{e^{x(1-\delta)}}{x}\, dx \\ &= \frac{1}{n^{1-\delta}} \int_1^{\log\log n} \frac{e^{x(1-\delta)}}{x}\, dx + \frac{1}{n^{1-\delta}} \int_{\log\log n}^{\log(n+1)} \frac{e^{x(1-\delta)}}{x}\, dx \\ &\leqslant \frac{1}{n^{1-\delta}} \int_1^{\log\log n} e^{x(1-\delta)}\, dx + \frac{1}{n^{1-\delta}} \int_{\log\log n}^{\log(n+1)} \frac{e^{x(1-\delta)}}{\log\log n}\, dx \\ &= \frac{1}{n^{1-\delta}} \left[\frac{e^{x(1-\delta)}}{1-\delta}\right]_{x=1}^{\log\log n} + \frac{1}{n^{1-\delta}} \left[\frac{e^{x(1-\delta)}}{(1-\delta)\log\log n}\right]_{x=\log\log n}^{\log(n+1)} \\ &\leqslant \frac{1}{n^{1-\delta}} \frac{(\log n)^{1-\delta}}{1-\delta} + \frac{1}{n^{1-\delta}} \frac{(n+1)^{1-\delta}}{(1-\delta)\log\log n} \to 0, \end{aligned}$$

so that $C_n(\delta) \to 0$ indeed.

Next, aiming finally at conditions (6) and (7) in [4], let $Y_{1,n} \leqslant \ldots \leqslant Y_{n,n}$ be the order statistics of a sample $Y_1, \ldots, Y_n$ from the generating Gumbel distribution given by $G(x) = \mathcal{P}\{Y \leqslant x\} = \exp(-e^{-x})$, $x \in \mathbb{R}$. What we have to show is that

$$\text{(2.7)} \qquad D_{1,n} := n \int_0^{1/(n+1)} [Y_{1,n} - Q(t)]^2 dt = n \int_0^{1/(n+1)} \left[Y_{1,n} + \log\log\frac{1}{t}\right]^2 \log\frac{1}{t}\, dt \xrightarrow{\mathcal{P}} 0$$

and

$$\text{(2.8)} \qquad D_{n,n} := n \int_{n/(n+1)}^{1} [Y_{n,n} - Q(t)]^2 dt = n \int_{n/(n+1)}^{1} \left[Y_{n,n} + \log\log\frac{1}{t}\right]^2 \log\frac{1}{t}\, dt \xrightarrow{\mathcal{P}} 0,$$

where $\xrightarrow{\mathcal{P}}$ denotes convergence in probability.

As to (2.7), first note the elementary fact that

$$\log n\,[Y_{1,n} + \log\log n] \xrightarrow{\mathcal{D}} -Y,$$

so that

$$L_n := Y_{1,n} + \log\log n = O_{\mathcal{P}}(1/\log n).$$

Thus, introduce the function $f_n(t) = \log\log(1/t) - \log\log n$ and notice that

$$D_{1,n} \leqslant n \int_0^{1/n} \left[Y_{1,n} + \log\log n + \log\log\frac{1}{t} - \log\log n\right]^2 \log\frac{1}{t}\, dt$$
$$= L_n^2(1 + \log n) + 2L_n\, n \int_0^{1/n} f_n(t) \log\frac{1}{t}\, dt + n \int_0^{1/n} f_n^2(t) \log\frac{1}{t}\, dt,$$

where for the coefficient of $2L_n$ we have

$$d_n^{1,1} := n \int_0^{1/n} f_n(t) \log\frac{1}{t}\, dt = n \int_{\log n}^{\infty} x(\log x) e^{-x}\, dx - (\log\log n)(1 + \log n)$$
$$= n \int_{\log n}^{\infty} (1 + \log x) e^{-x}\, dx - \log\log n = 1 + n \int_{\log n}^{\infty} (\log x) e^{-x}\, dx - \log\log n$$
$$= 1 + n \int_{\log n}^{\infty} \frac{e^{-x}}{x}\, dx \leqslant 1 + \frac{n}{\log n} \int_{\log n}^{\infty} e^{-x}\, dx = 1 + \frac{1}{\log n}$$

and for the full third term $d_n^{1,2} := n\int_0^{1/n} f_n^2(t)\log t^{-1}\,dt$ we have

$$\begin{aligned}
d_n^{1,2} &= n\int_{\log n}^{\infty} x(\log x)^2 e^{-x}\,dx + (\log\log n)^2(1+\log n) \\
&\quad - 2n(\log\log n)\int_{\log n}^{\infty} x(\log x)e^{-x}\,dx \\
&= (\log\log n)^2(\log n) + n\int_{\log n}^{\infty}(\log^2 x + 2\log x)e^{-x}\,dx \\
&\quad + (\log\log n)^2(1+\log n) \\
&\quad - 2(\log\log n)\Bigg[(\log\log n)(1+\log n) + 1 + n\int_{\log n}^{\infty}\frac{e^{-x}}{x}\,dx\Bigg] \\
&= n\int_{\log n}^{\infty}\frac{2+2\log x}{x}\,e^{-x}\,dx - 2n(\log\log n)\int_{\log n}^{\infty}\frac{e^{-x}}{x}\,dx \\
&= \frac{2}{\log n} - 2n\int_{\log n}^{\infty}\frac{\log x}{x^2}\,e^{-x}\,dx + 2n(\log\log n)\int_{\log n}^{\infty}\frac{e^{-x}}{x^2}\,dx \\
&\leqslant \frac{2}{\log n} + 2n\,\frac{\log\log n}{\log^2 n}\int_{\log n}^{\infty} e^{-x}\,dx = \frac{2}{\log n} + 2\,\frac{\log\log n}{\log^2 n}.
\end{aligned}$$

Consequently, we see that (2.7) holds true.

Finally, since the distributional equality $Y_{n,n} - \log n \stackrel{\mathcal{D}}{=} Y$ holds for every $n \in \mathbb{N}$, for the proof of (2.8) we notice that $M_n := Y_{n,n} - \log n = O_{\mathcal{P}}(1)$. Considerations similar to those above imply

$$\begin{aligned}
D_{n,n} &= n\int_{n/(n+1)}^{1}\Bigg[Y_{n,n} - \log n + \log n + \log\log\frac{1}{t}\Bigg]^2 \log\frac{1}{t}\,dt \\
&= d_n^{2,1}M_n^2 + 2d_n^{2,2}M_n + d_n^{2,3},
\end{aligned}$$

where

$$\begin{aligned}
d_n^{2,1} &= n\int_{n/(n+1)}^{1}\log\frac{1}{t}\,dt \\
&= \frac{n + n^2\log\big(n/(n+1)\big)}{n+1} = \frac{n + n^2\log\big(1 - 1/(n+1)\big)}{n+1} \to 0, \\
d_n^{2,2} &= n\int_0^{\log[(n+1)/n]}[\log n + \log x]x\,e^{-x}\,dx \\
&\leqslant 2n(\log n)\int_0^{\log[(n+1)/n]} x\,dx + \frac{2n^2}{n+1}\int_0^{\log[(n+1)/n]} x\log x\,dx \\
&\leqslant n(\log n)\Bigg[\log\frac{n+1}{n}\Bigg]^2 + \frac{n^2}{n+1}\Bigg[\log\frac{n+1}{n}\Bigg]^2\Bigg[\log\log\frac{n+1}{n}\Bigg] \to 0
\end{aligned}$$

and

$$d_n^{2,3} = n \int_0^{\log[(n+1)/n]} [\log n + \log x]^2 x\, e^{-x}\, dx \leqslant n \int_0^{\log[(n+1)/n]} [\log n + \log x]^2 x\, dx$$
$$= \frac{n}{2}\left[\log \frac{n+1}{n}\right]^2 \left[\log n + \log\log \frac{n+1}{n}\right]^2$$
$$- n \int_0^{\log[(n+1)/n]} [\log n + \log x] x\, dx \to 0,$$

ensuring (2.8).

Theorems 3 and 2 in [4] now imply the basic convergence statements in (2.1) and (2.4) with the given definition of the ingredients. Also, integrating by parts,

$$\int_{1/(n+1)}^{n/(n+1)} \frac{t-1}{t\log t}\, dt = \int_{\log[(n+1)/n]}^{\log(n+1)} \frac{1-e^{-y}}{y}\, dy = \frac{n}{n+1} \log\log(n+1)$$
$$- \frac{1}{n+1} \log\log \frac{n+1}{n} - \int_{\log[(n+1)/n]}^{\log(n+1)} (\log y)\, e^{-y}\, dy$$
$$= \log\log n - \Gamma'(1) + o(1),$$

which, since the denominator in (2.2) converges to $v = \mu_2 - \mu_1^2 = (\pi^2 - 6)/6$, proves both asymptotic formulae in (2.2) and (2.5).

Now we turn to the distributional equations in (2.3) and (2.6). The condition that $I < \infty$ at the beginning of the proof ensures that the integrals in V and W exist as limits in $L_2(\Omega, \mathcal{A}, \mathcal{P})$ above the probability space $(\Omega, \mathcal{A}, \mathcal{P})$ where $B(\cdot)$ is defined. It is also the necessary and sufficient condition ([16], Sections 66, 97, 98) that the Hilbert–Schmidt covariance operator $T\colon L_2(0,1) \to L_2(0,1)$ associated with the Gaussian process

$$Z(t) = \frac{B(t)\sqrt{w(t)}}{g\big(Q(t)\big)} = \frac{B(t)}{t\sqrt{\log t^{-1}}}, \quad 0 < t < 1, \tag{2.9}$$

given by

$$Th(t) = \int_0^1 \mathrm{Cov}\big(Z(s), Z(t)\big) h(s)\, ds = \int_0^1 \frac{\min(s,t) - st}{st\sqrt{(\log s)(\log t)}}\, h(s)\, ds, \quad 0 < t < 1,$$

has a countable number of different eigenvalues λ_k with pertaining eigenfunctions h_k, so that $Th_k(t) = \lambda_k h_k(t)$, $0 < t < 1$; here k runs through a countable set to be fixed later on. Our first goal is the joint determination of λ_k and $h_k(\cdot)$.

We know from de Wet's [10] Theorem 2.4 that $\sqrt{w(\cdot)} = \sqrt{-\log(\cdot)}$ is an eigenfunction; this is what motivated the choice of $w(\cdot)$ in the first place. This fact partially motivates our trick to search for the eigenfunctions in the form of $h_k(\cdot) = f_k\big(-\log(\cdot)\big)\sqrt{-\log(\cdot)}$, where, for each k, $f_k(\cdot)$ is twice continuously differentiable on $(0,\infty)$ such that beyond $h_k \in L_2(0,1)$ we also have $\int_0^\infty |f_k(x)|\, e^{-x} dx < \infty$ and, in particular, $\lim_{x\to\infty} f_k(x)e^{-x} = 0$ and $\lim_{x\downarrow 0} f_k(x)x = 0$.

With $f(\cdot)$ standing for any of the $f_k(\cdot)$, our integral equation to solve is

$$\lambda\, f\bigg(\log\frac{1}{t}\bigg)\sqrt{\log\frac{1}{t}} = \int_0^1 \frac{\min(s,t)-st}{st\sqrt{(\log s)(\log t)}}\, f\bigg(\log\frac{1}{s}\bigg)\sqrt{\log\frac{1}{s}}\, ds, \qquad 0<t<1,$$

where λ is the corresponding eigenvalue or, what is the same for every $t\in(0,1)$,

$$\begin{aligned}\lambda\, f\bigg(\log\frac{1}{t}\bigg) t\log\frac{1}{t} &= \int_0^1 \frac{\min(s,t)-st}{s}\, f\bigg(\log\frac{1}{s}\bigg)\, ds\\ &= (1-t)\int_0^t f\bigg(\log\frac{1}{s}\bigg)\, ds + t\int_t^1 \frac{1-s}{s}\, f\bigg(\log\frac{1}{s}\bigg)\, ds.\end{aligned}$$

After substituting $t = e^{-x}$ and then $s = e^{-y}$ in the integrals, this takes the form

$$\lambda x e^{-x} f(x) = (1-e^{-x})\int_x^\infty e^{-y} f(y)\, dy + e^{-x}\int_0^x (1-e^{-y}) f(y)\, dy, \qquad x>0.$$

Differentiating and then dividing by e^{-x}, we obtain

$$\lambda x f'(x) + \lambda(1-x) f(x) = \int_x^\infty e^{-y} f(y)\, dy - \int_0^x (1-e^{-y}) f(y)\, dy, \qquad x>0.$$

Differentiating once more, we get

$$\lambda x f''(x) + \lambda(2-x) f'(x) - \lambda f(x) = -f(x), \qquad x>0,$$

that is,

$$x f''(x) + (2-x) f'(x) + \bigg(\frac{1}{\lambda}-1\bigg) f(x) = 0, \qquad x>0. \tag{2.10}$$

In this differential equation we recognize the one ([18], p. 100) which characterizes the classical Laguerre polynomials

$$L_k(x) = \sum_{j=0}^k \binom{k+1}{k-j}\frac{(-x)^j}{j!}, \qquad x\geqslant 0,$$

defined by

$$\int_0^\infty L_n(x) L_m(x) x e^{-x}\, dx = (n+1)\delta_{nm}, \qquad n,m = 0,1,2,\ldots, \tag{2.11}$$

where δ_{nm} is Kronecker's symbol: the equation has $L_k(\cdot)$ as the unique polynomial solution if and only if

$$\frac{1}{\lambda_k} - 1 = k, \quad \text{that is,} \quad \lambda_k = \frac{1}{k+1} \quad \text{for all } k = 0, 1, 2, \ldots,$$

and, of course, the regularity conditions imposed on $f_k(\cdot)$ are satisfied by $L_k(\cdot)$. We also note that one can also work backwards from (2.10) and see, under the regularity conditions on $f_k(\cdot)$ above, that this differential equation is in fact equivalent to $Tf(-\log t)\sqrt{-\log t} = \lambda f(-\log t)\sqrt{-\log t}$, $0 < t < 1$, the eigenvalue problem.

If we now introduce the normalized eigenfunctions

$$h_k^*(t) = \frac{1}{\sqrt{k+1}} L_k\Bigl(\log \frac{1}{t}\Bigr) \sqrt{\log \frac{1}{t}}, \quad 0 < t < 1, \ k = 0, 1, 2, \ldots,$$

then we can easily see that (2.11) is equivalent to the ordinary orthogonality relation $\int_0^1 h_n^*(t) h_m^*(t)\, dt = \delta_{nm}$, $n, m = 0, 1, 2, \ldots$ Then the zero-mean jointly normal random variables

$$Z_{k+1} = \sqrt{k+1} \int_0^1 Z(t) h_k^*(t)\, dt, \quad k = 0, 1, 2, \ldots,$$

for all $k, m = 0, 1, 2, \ldots$ have the covariance

$$\begin{aligned} E(Z_{k+1} Z_{m+1}) &= \int_0^1 \int_0^1 \frac{[\min(s,t) - st] h_k^*(s) h_m^*(t)}{st\sqrt{(\log s)(\log t)}}\, ds\, dt \\ &= \frac{\sqrt{k+1}\sqrt{m+1}}{k+1} \int_0^1 h_k^*(t) h_m^*(t)\, dt = \delta_{km}, \end{aligned}$$

and hence are independent standard normal variables. In terms of this sequence, the Karhunen–Loève expansion of $Z(t)$, holding in the space $L_2(\Omega, \mathcal{A}, \mathcal{P})$ pointwise and converging almost surely uniformly on $[\varepsilon, 1-\varepsilon]$ for every $\varepsilon \in (0,1)$, is

$$Z(t) = \sum_{k=0}^{\infty} \frac{h_k^*(t)}{\sqrt{k+1}} Z_{k+1}, \quad 0 < t < 1.$$

In this representation, by (2.9) and the fact that $h_0(t) = \sqrt{-\log t} = h_0^*(t)$, we have

$$
\begin{aligned}
W &= \int_0^1 \frac{B^2(t)-t(1-t)}{t^2\log t^{-1}}\,dt - \left[\int_0^1 \frac{B(t)}{t}\,dt\right]^2 \\
&= \int_0^1 \left[Z^2(t)-\mathcal{E}\big(Z^2(t)\big)\right]dt - \Big[\int_0^1 Z(t)h_0^*(t)\,dt\Big]^2 \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{Z_{k+1}Z_{m+1}-\mathcal{E}(Z_{k+1}Z_{m+1})}{\sqrt{k+1}\sqrt{m+1}} \int_0^1 h_k^*(t)h_m^*(t)\,dt \\
&\quad - \left[\sum_{k=0}^{\infty} \frac{Z_{k+1}}{\sqrt{k+1}} \int_0^1 h_k^*(t)h_0^*(t)\,dt\right]^2 \\
&= \sum_{k=0}^{\infty} \frac{Z_{k+1}^2-1}{k+1} - Z_1^2 = -1 + \sum_{m=2}^{\infty}\frac{Z_m^2-1}{m}.
\end{aligned}
$$

The formal manipulations may be made rigorous as in the proof of Theorem 3.6 in [6]: the random integrals and infinite series exist as $L_2(\Omega,\mathcal{A},\mathcal{P})$ limits. Of course, the final series for W converges almost surely. This proves (2.6) in Theorem 2.2.

Furthermore, aiming finally at (2.3), in the same representation,

$$
\begin{aligned}
V &= \frac{1}{v}\left[-1+\sum_{m=2}^{\infty}\frac{Z_m^2-1}{m}\right] \\
&\quad - \frac{1}{v^2}\left[\int_0^1 Z(t)\Big(-\log\log\frac{1}{t}\Big)\sqrt{\log\frac{1}{t}}\,dt\ -(\gamma-1)Z_1\right]^2 \\
&= \frac{6}{\pi^2-6}\left[-1+\sum_{m=2}^{\infty}\frac{Z_m^2-1}{m}\right] \\
&\quad - \frac{36}{(\pi^2-6)^2}\left[\sum_{k=0}^{\infty}\frac{Z_{k+1}}{\sqrt{k+1}}\,c_k-(\gamma-1)Z_1\right]^2,
\end{aligned}
$$

where $v=(\pi^2-6)/6$ as before and, by a classical formula for the derivative of the gamma function at a positive integer,

$$
\begin{aligned}
c_k &= \int_0^1 h_k^*(t)\Big(-\log\log\frac{1}{t}\Big)\sqrt{\log\frac{1}{t}}\,dt \\
&= \frac{-1}{\sqrt{k+1}}\int_0^1 L_k\Big(\log\frac{1}{t}\Big)\Big(\log\log\frac{1}{t}\Big)\log\frac{1}{t}\,dt \\
&= \frac{-1}{\sqrt{k+1}}\int_0^{\infty} L_k(x)(\log x)\,x\,e^{-x}\,dx \\
&= \frac{1}{\sqrt{k+1}}\int_0^{\infty}\left[\sum_{j=0}^{k}\binom{k+1}{k-j}\frac{(-x)^{j+1}}{j!}\,(\log x)\,e^{-x}\right]dx \\
&= \frac{1}{\sqrt{k+1}}\sum_{j=0}^{k}\frac{(k+1)k!}{(k-j)!j!}\,(-1)^{j+1}\frac{\Gamma'(j+2)}{(j+1)!}
\end{aligned}
$$

and, by the binomial theorem,

$$c_k = \sqrt{k+1}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j+1}\left[1+\frac{1}{2}+\ldots+\frac{1}{j+1}-\gamma\right] = \begin{cases}\gamma-1 & \text{if } k=0,\\ 1/(k\sqrt{k+1}) & \text{if } k>0.\end{cases}$$

Indeed, for $k \in \mathbb{N}$, writing $c_k = \sqrt{k+1}\,d_k$, rearranging the sum, using again the binomial theorem in the form

$$\sum_{l=1}^{k}(-1)^l\binom{k}{l} = -1,$$

and the formula

$$\sum_{j=m}^{k}(-1)^j\binom{k}{j} = (-1)^m\binom{k-1}{m-1},$$

which is easily proved by mathematical induction on m, we find that

$$\begin{aligned} d_k &= -\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left[1+\frac{1}{2}+\ldots+\frac{1}{j+1}\right] \\ &= \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{j+1}}{j+1} - \sum_{j=1}^{k}\binom{k}{j}(-1)^j\left[1+\frac{1}{2}+\ldots+\frac{1}{j}\right] \\ &= \frac{1}{k+1}\sum_{j=0}^{k}(-1)^{j+1}\binom{k+1}{j+1} - \sum_{m=1}^{k}\frac{1}{m}\sum_{j=m}^{k}(-1)^j\binom{k}{j} \\ &= \frac{1}{k+1}\sum_{m=1}^{k+1}(-1)^m\binom{k+1}{m} - \sum_{m=1}^{k}(-1)^m\frac{1}{m}\binom{k-1}{m-1} \\ &= -\frac{1}{k}\sum_{m=1}^{k}(-1)^m\binom{k}{m} - \frac{1}{k+1}\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}. \end{aligned}$$

Therefore, substituting $c_0 = \gamma - 1$ and $c_k = 1/(k\sqrt{k+1}\,)$ for $k \geqslant 1$ into the last formula of V above, (2.3) of Theorem 2.1 follows. This completes the proof of both theorems. ■

3. PERFORMANCE OF THE GUMBEL TESTS

3.1. Computation of the limiting distributions. First we aim at a precise numerical determination of the limiting distribution functions $H_{\mathrm{l}}(x) = \mathcal{P}\{W \leqslant x\}$ and $H_{\text{l-s}}(x) = \mathcal{P}\{V \leqslant x\}$ that arose in Theorems 2.2 and 2.1, along with their densities $h_{\mathrm{l}}(x) = H_{\mathrm{l}}'(x)$ and $h_{\text{l-s}}(x) = H_{\text{l-s}}'(x)$, $x \in \mathbb{R}$. While the subscript l in

H_1 for location is adequate in the present paper, we must point out that the same limiting distribution was obtained in [5] for the weighted quantile correlation test statistic for all Gamma *scale* families with a known shape parameter and the corresponding optimal de Wet weight function. In fact, $H_1(x-1) = \mathcal{P}\{W+1 \leqslant x\}$, $x \in \mathbb{R}$, was obtained by de Wet and Venter [11] a long time ago for a similar test statistic for the same Gamma scale problem. It follows from their work that if we write the characteristic function $\phi(t) = \mathcal{E}(e^{iWt}) = \int_{-\infty}^{\infty} e^{ixt}\, dH_1(x)$ in the form $\phi(t) = r(t)e^{i\vartheta(t)}$, $t \in \mathbb{R}$, then for the modulus and angle functions we have

$$r(t) = |\phi(t)| = \left(\frac{2\pi t(1+4t^2)}{\sinh(2\pi t)}\right)^{1/4} \quad \text{and} \quad \vartheta(t) = -t + \frac{1}{2}\sum_{k=2}^{\infty}\left[\arctan\frac{2t}{k} - \frac{2t}{k}\right].$$

In particular, since $\int_{-\infty}^{\infty} |t|^k r(t)\, dt < \infty$ for every $k \in \mathbb{N}$, it follows that H_1 is infinitely many times differentiable and, by the usual inversion formula,

$$\begin{aligned} &H_1(x) - H_1(0) \\ &\quad = \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{(1-e^{-ixt})\phi(t)}{it}\, dt = \frac{1}{\pi}\int_0^{\infty} \frac{r(t)}{t}\big[\sin\vartheta(t) + \sin\big(tx - \vartheta(t)\big)\big]\, dt, \end{aligned}$$

and hence

$$h_1(x) = \frac{1}{\pi}\int_0^{\infty} r(t)\cos\big(tx - \vartheta(t)\big)\, dt, \qquad x \in \mathbb{R}.$$

Taking x large, we obtain $1 - H_1(0)$, and thus all the values of $H_1(x)$ and $h_1(x)$ for $-6 \leqslant x \leqslant 5$, where $H_1(-6) \approx 0$ and $H_1(5) \approx 1$. This was done by integrating numerically from 0 to 100 after truncating the infinite series for $\vartheta(\cdot)$ at 10,000 terms.

We also determined $H_1(\cdot)$ and $h_1(\cdot)$ by simulation. We generated 1,000,000 copies of the random variable $-1 + \sum_{j=2}^{5000} (Z_j^2 - 1)/j$, and computed their empirical distribution function and a corresponding density function. The parameters in

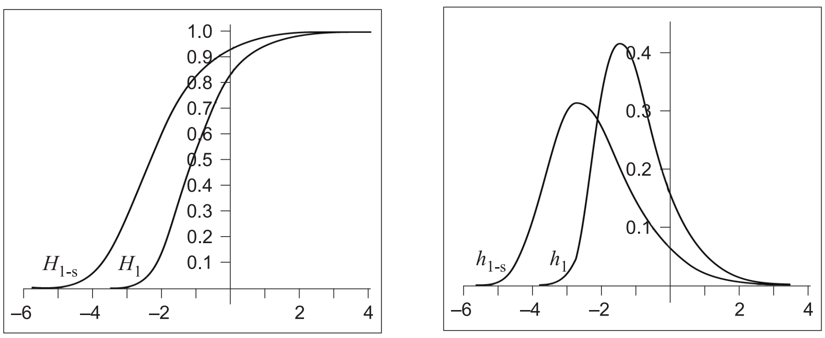

FIGURE 1. The distribution functions H_1 and $H_{1\text{-s}}$ and their densities

the numerical inversion and in the simulation were chosen such that the values of $H_1(\cdot)$ and $h_1(\cdot)$ be the same to two decimal places; the values of $H_1(\cdot)$ also agreed with the suitably shifted values in Table 1 of [11] reasonably well, the three decimals of which were obtained there by a similar numerical integration at the dawn of the computer age. Thus our two procedures corroborate each other in a clean fashion. Having thus checked the precision of the simulation, in the absence of an invertible form of the characteristic function, the functions $H_{\text{l-c}}(\cdot)$ and $h_{\text{l-c}}(\cdot)$ were computed only by the simulation method, achieving full numerical stability in the third decimal.

3.2. Speed of convergence. Using the sample sizes $n = 50, 100, 1000, 10{,}000$, $100{,}000$ and $1{,}000{,}000$ we simulated the distribution function of the location and location-scale test statistics $W_n^\diamond = nW_n - c_n^\diamond$ and $V_n^\star = nV_n - c_n^\star$ under the standard Gumbel distribution. Working with the exact forms of the centering sequences throughout, this was done using $1{,}000{,}000$ repetitions for $n \leqslant 100$; then for higher n's we gradually decreased the number of repetitions, using finally 5000 for $n = 1{,}000{,}000$. As shown in Figures 2 and 3, skipping for a clearer view the curves pertaining to $n = 10{,}000$ and $100{,}000$, we find in both cases that the convergence is very slow overall. This is particularly true for the otherwise irrelevant small quantiles, where the true distribution functions of both statistics appear far to the right from the asymptotic ones, even for astronomically large n.

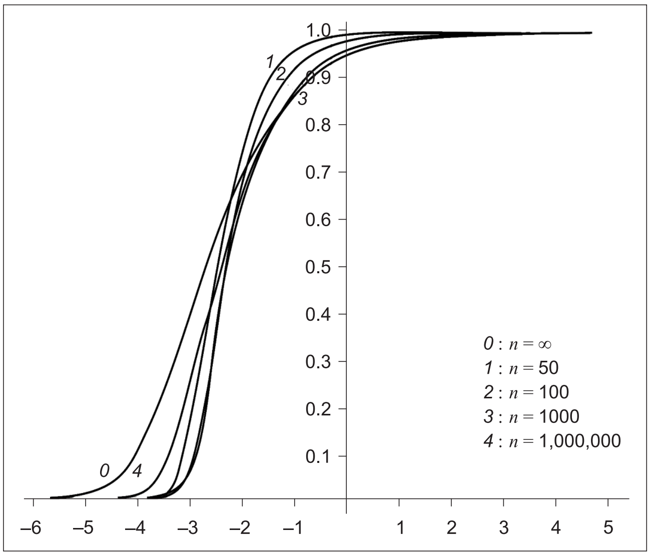

FIGURE 2. The distribution function of $V_n^\star$ for $n = 50, 100, 1000, 1{,}000{,}000, \infty$

Table 1 shows in detail corresponding and further critical values of $W_n^\diamond$ and $V_n^\star$ that belong to confidence levels 0.85, 0.90, 0.95 and 0.99.

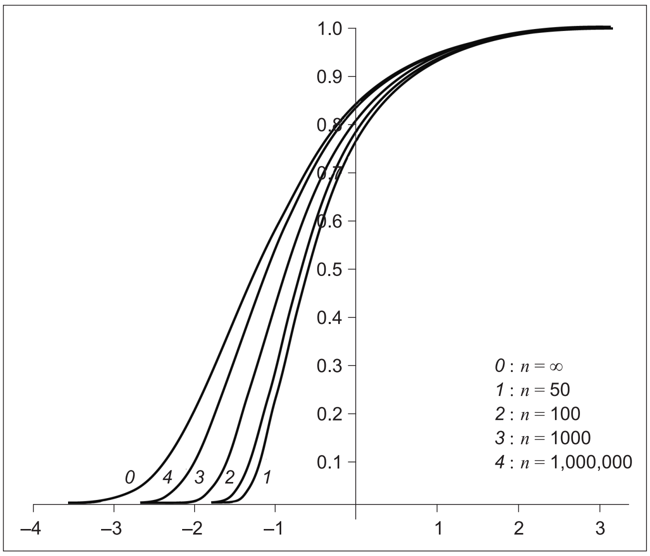

FIGURE 3. The distribution function of $W_n^\diamond$ for $n = 50, 100, 1000, 1{,}000{,}000, \infty$

Clearly, for the usual testing levels the asymptotic critical points cannot be used reliably for most sample sizes in practice. The usage of the asymptotic critical points consistently results in conservative location tests with the statistic $W_n^\diamond$, while that for the location-scale test with the statistic $V_n^\star$ yields exactly the opposite, anticonservative tests. Thus there will be consistently a greater number of rejection than desirable for $W_n^\diamond$, and a lesser number for $V_n^\star$. Therefore, the general conclusion is that for any given sample size n occurring in an everyday practical problem simulated critical points are better.

TABLE 1. Critical points of the test statistics $V_n^\star = nV_n - c_n^\star$ and $W_n^\diamond = nW_n - c_n^\diamond$

n	$V_n^\star$				$W_n^\diamond$			
	0.85	0.90	0.95	0.99	0.85	0.90	0.95	0.99
10	−3.19	−2.97	−2.58	−1.67	0.54	0.81	1.26	2.61
20	−2.12	−1.87	−1.44	−0.43	0.44	0.72	1.21	2.57
50	−1.48	−1.21	−0.73	0.40	0.34	0.64	1.16	2.56
100	−1.24	−0.94	−0.42	0.80	0.28	0.60	1.15	2.52
200	−1.09	−0.78	−0.22	1.09	0.25	0.57	1.14	2.48
500	−0.97	−0.62	−0.03	1.37	0.22	0.55	1.14	2.47
1000	−0.92	−0.56	0.06	1.55	0.19	0.54	1.13	2.44
10,000	−0.84	−0.45	0.22	1.74	0.18	0.53	1.12	2.42
100,000	−0.82	−0.40	0.28	1.89	0.16	0.52	1.10	2.42
1,000,000	−0.82	−0.39	0.33	1.91	0.11	0.48	1.08	2.42
∞	−0.85	−0.39	0.35	2.04	0.08	0.45	1.05	2.41

3.3. Power against other extreme value distributions. The standard Gumbel distribution function $G(x) = G_0(x) = \lim_{\alpha\to 0} G_\alpha(x) = \exp(-e^{-x})$ is perhaps best viewed as the “middle member” of the three standard types of extreme value distribution functions, given by the parameter $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$, in the summary formula $G_\alpha(x) = \exp\left(-(1+\alpha x)^{-1/\alpha}\right)$, $1 + \alpha x > 0$. To assess the power of the goodness-of-fit test for the composite location-scale null hypothesis $F \in \mathcal{G}_{\text{l-c}}$, for every $\alpha \in [-1, 0.8]$ we generated a sample of size $n = 50, 100, 200, 500, 1000$ from $G_\alpha(\cdot)$, computed the location-scale statistic $V_n^\star$ on this data and compared it with the exact simulated critical points of level 0.9 in Table 1. Having performed this simulation 1,000,000 times, each point of the power plots in Figure 4, estimating very precisely the probability of rejecting $G_\alpha \in \mathcal{G}_{\text{l-c}}$ by our test, is based on that many calculations. While of course there is no inherent symmetry in the problem, we see that the test is generally better against $\alpha > 0$ than against $\alpha < 0$, but on the whole one could not possibly expect a more exemplary power behavior than the one witnessed here.

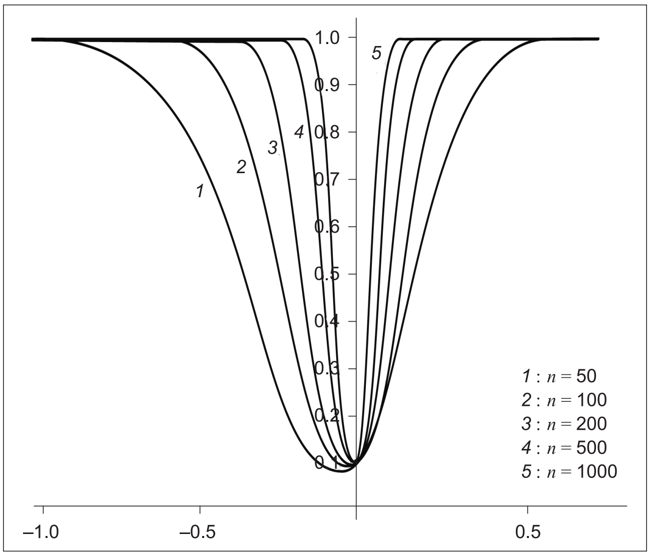

FIGURE 4. Power of the location-scale Gumbel test $V_n^\star$ against extreme value G_α

4. TEST FOR THE WEIBULL FAMILY

It belongs to statistical folklore that

$$H_{\beta,s}(x) = \mathcal{P}\{T \leqslant x\} = 1 - \exp\left(-(x/s)^\beta\right), \quad x \geqslant 0,$$

if and only if

$$\mathcal{P}\{-\log T \leqslant x\} = \exp(-e^{-\beta(x+\log s)}) = G^{-\log s, 1/\beta}(x), \quad x \in \mathbb{R},$$

that is, the positive random variable T has the Weibull distribution with some shape parameter $\beta > 0$ and some scale parameter $s > 0$ if and only if $-\log T$ has the Gumbel distribution with location parameter $\theta = -\log s$ and scale parameter $\sigma = 1/\beta$. Given a sample $X_1, \ldots, X_n$ of independent positive random variables with a common absolutely continuous distribution function $F(x) = \mathcal{P}\{X \leqslant x\}$, in almost all types of reliability studies it may be important to test the general Weibull hypothesis $F \in \mathcal{H} = \{H_{\beta,s}(\cdot)\colon \beta > 0, s > 0\}$, which by the above is equivalent to $J \in \mathcal{G}_{\text{l-c}}$, where $J(x) = \mathcal{P}\{-\log X \leqslant x\} = 1 - F(e^{-x})$, $x \in \mathbb{R}$. Thus, if we denote the order statistics of $-\log X_1, \ldots, -\log X_n$ by $L_{1,n} \leqslant \ldots \leqslant L_{n,n}$ and redefine the location-scale Gumbel statistic as

$$V_n = \\ = 1 - \frac{6\Big[\sum_{k=1}^{n} L_{k,n}\Big\{\int_{(k-1)/n}^{k/n} (-\log\log t^{-1}) \log t^{-1}\, dt - (\gamma - 1) \int_{(k-1)/n}^{k/n} \log t^{-1}\, dt\Big\}\Big]^2}{(\pi^2 - 6)\Big[\sum_{k=1}^{n} L_{k,n}^2 \int_{(k-1)/n}^{k/n} \log t^{-1}\, dt - \Big(\sum_{k=1}^{n} L_{k,n} \int_{(k-1)/n}^{k/n} \log t^{-1}\, dt\Big)^2\Big]},$$

then for this V_n we have the full form of Theorem 2.1 and the findings in Subsection 3.1 whenever $F \in \mathcal{H}$.

To see this Weibull test in work, we considered Gamma distributions as natural alternatives, given by $\mathcal{P}\{X \leqslant x\} = \Gamma_\alpha(x) = \int_0^x y^{\alpha-1} e^{-y} dy / \Gamma(\alpha)$, $x > 0$, for the shape parameter $\alpha > 0$. Of course, the exponential distribution $\Gamma_1(\cdot)$ is nothing but the Weibull distribution $H_{1,1}(\cdot)$, so it is the vicinity of $\alpha = 1$ that is of interest.

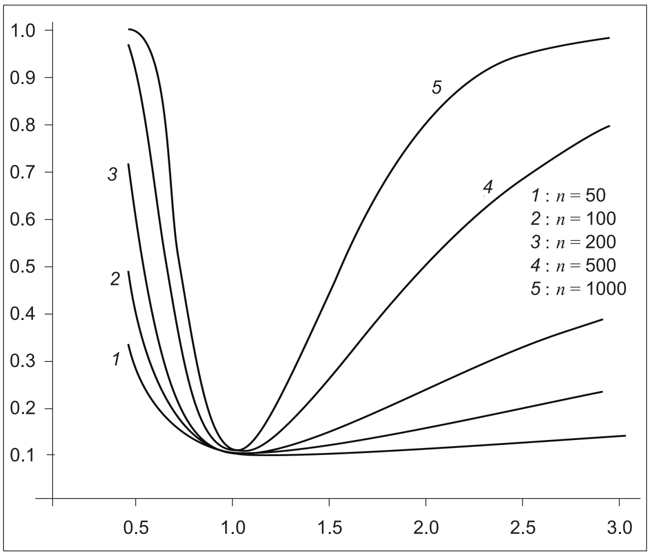

FIGURE 5. Power of the Weibull test against Γ_α, $0.45 \leqslant \alpha \leqslant 3$

Figure 5 depicts the simulated power of the Weibull test against Γ_α, again at the confidence level 0.9, using the exact critical points from Table 1. This is based on similar experiments described in the previous subsection; since the quantile function here cannot be given in a closed form, Gamma variables are not easy to simulate and it has taken days on a machine to compute the power as a function of α in the whole interval $[0.45, 3]$ for the same sample sizes as in Figure 3. The similar nature of the densities explains the large sample sizes needed to pick up appreciable power in general, but it is perhaps still surprising that the case $\alpha > 1$ is so much worse. Nevertheless, the general picture is adequate on the whole.

5. TEST FOR THE PARETO FAMILY

Finally, in analogous or partially redefined notation relative to the previous section, we consider the class $\mathcal{H}_C = \{H_{\beta,C}(\cdot)\colon \beta > 0\}$ of Pareto distributions with arbitrary shape parameter $\beta > 0$, given by $H_{\beta,C}(x) = \mathcal{P}\{T \leqslant x\} = 1 - (C/x)^\beta$, $x \geqslant C$, where $C > 0$ is a *known* scale parameter. Since C is also the left endpoint of the support, the difficulties associated with an unknown C are well known; in practice, with a known or very closely estimated C, one would use the random variable T/C to achieve a unit scale parameter for the new, re-scaled variable. The Pareto model for some $\beta > 0$ and a known $C > 0$ holds if and only if

$$\mathcal{P}\{-\log\log(T/C) \leqslant x\} = \exp(-e^{-(x-\log\beta)}) = G^{\log\beta,1}(x), \qquad x \in \mathbb{R},$$

that is, the positive random variable T has the Pereto distribution with some shape parameter $\beta > 0$ and a known scale parameter $C > 0$ if and only if $-\log\log(T/C)$ has the Gumbel distribution with location parameter $\theta = \log\beta$ and scale parameter $\sigma = 1$. Given a sample $X_1, \ldots, X_n$ of independent random variables taking values in $[C, \infty)$ for a known $C > 0$, specified by the statistical problem at hand, and a common absolutely continuous distribution function $F_C(x) = \mathcal{P}\{X \leqslant x\}$, practically all potentially long-tailed phenomena require testing the Pareto hypothesis $F_C \in \mathcal{H}_C$, which is equivalent to $J_C \in \mathcal{G}_1$, where

$$J_C(x) = \mathcal{P}\{-\log\log(X/C) \leqslant x\} = 1 - F_C\big(C\exp(e^{-x})\big), \qquad x \in \mathbb{R}.$$

So, this time let $R_{1,n} \leqslant \ldots \leqslant R_{n,n}$ be the order statistics of $-\log\log(X_1/C), \ldots,$ $-\log\log(X_n/C)$, and then the redefined Gumbel location statistic is

$$W_n = \frac{\pi^2 - 6}{6} + \sum_{k=1}^n R_{k,n}^2 \int_{(k-1)/n}^{k/n} \log\frac{1}{t}\,dt - \left[\sum_{k=1}^n R_{k,n} \int_{(k-1)/n}^{k/n} \log\frac{1}{t}\,dt\right]^2$$
$$- 2\sum_{k=1}^n R_{k,n}\left\{\int_{(k-1)/n}^{k/n} \left(-\log\log\frac{1}{t}\right)\log\frac{1}{t}\,dt - (\gamma - 1)\int_{(k-1)/n}^{k/n} \log\frac{1}{t}\,dt\right\}.$$

For this W_n the full form of Theorem 2.2 and the respective findings in Subsection 3.1 all hold whenever $F_C \in \mathcal{H}_C$.

To exhibit power properties we consider two versions of a type of an alternative: a generalized Pareto distribution with logarithmic slow variation, given by the distribution function $G_{\beta,\alpha}(x) = 1 - \big(\log^{\alpha}(x+e-1)\big)/x^{\beta}$, $x \geqslant 1$, where $\beta > 0$ and, for this function to be nondecreasing, the parameter α must be restricted as $\alpha \in (-\infty, \kappa\,\beta]$, where $\kappa \approx 2.4286$. Since $G_{\beta,0}(\cdot) = H_{\beta,1}(\cdot)$, it is the parameter intervals about $\alpha = 0$ that are of interest. Using again the exact critical points from Table 1 for a 0.9 test, we exhibit the power functions for $\beta = 1$ and $\beta = 1/4$ in Figures 6 and 7 on the intervals $[-1.5, 1.5]$ and $[-2.5, 0.5]$, respectively.

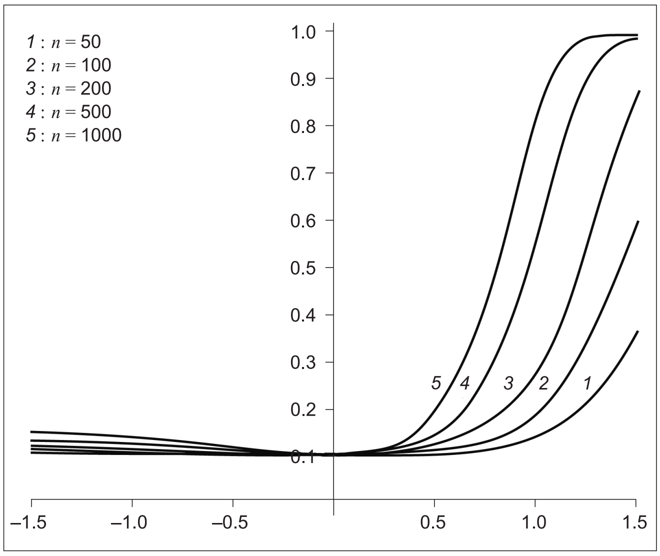

FIGURE 6. Power of the Pareto test against $G_{1,\alpha}$, $-1.5 \leqslant \alpha \leqslant 1.5$

It is intuitively clear in advance that the smaller $\beta > 0$ is, the better the power should be against every permissible $\alpha \leqslant \kappa\,\beta$, $\alpha \neq 0$. This heuristic expectation turns out to be experimentally true in general and may be seen for $\beta = 1/4$ and $\beta = 1$ in comparing the two figures; of course, we have similar figures for other β values, different from $1/4$ and 1, that are not included here. One can also argue that accepting falsely $G_{\beta,\alpha}(\cdot)$ may not be a big problem in practice if β is large and the modulus $|\alpha|$ of the exponent of the logarithmic nuisance function is not too big.

According to Figure 6, the test is still roughly satisfactory against $G_{1,\alpha}$ for $\alpha > 0$. However, for $n \leqslant 1000$ there is practically no power against $G_{1,\alpha}$ for $\alpha < 0$ when positive powers of the logarithm are in the denominator. The power is still an increasing function of n, which is to be expected in view of the fact that theoretically the test is consistent against $G_{\beta,\alpha}$ for every $\beta > 0$ and every $\alpha \in (-\infty, \kappa\,\beta\,] \setminus \{0\}$.

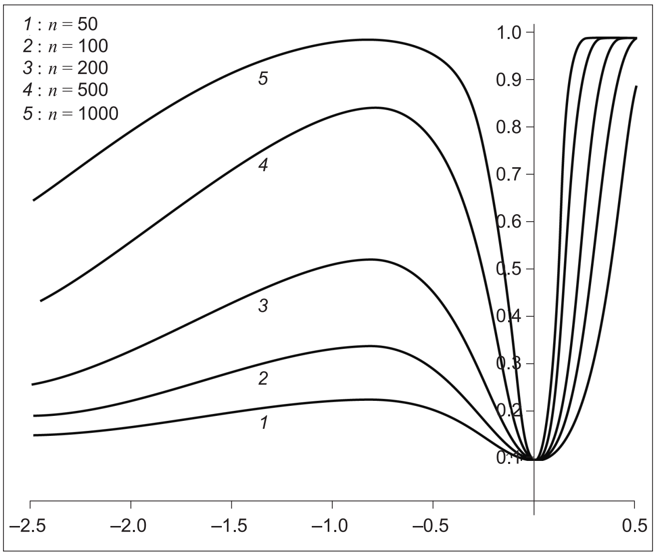

FIGURE 7. Power of the Pareto test against $G_{1/4,\alpha}$, $-2.5 \leqslant \alpha \leqslant 0.5$

The overall improvement for the small $\beta = 1/4$ in Figure 7, both in comparison with the situation for $\beta = 1$ in Figure 6 and in absolute terms, is simply dramatic. Since very large exponents of the logarithm in the denominator are like a small algebraic power, one may clearly argue in a heuristic fashion that for every $\beta > 0$ among the negative α there should exist an optimal one for which the power is maximal, but even a conjecture would be difficult to make concerning the value of such an $\alpha = \alpha_\beta < 0$. In particular, we must leave it here as an interesting puzzle or challenge why $\alpha_{1/4}$ should be about -0.8, as seen on Figure 7.

We note in closing that, similarly to the consistency statement above, the Weibull test in Section 4 is also consistent against every alternative Γ_α, $\alpha > 0$, $\alpha \neq 1$, considered there, just as the Gumbel test in Section 3 is consistent against every other extreme value distribution G_α, $\alpha \neq 0$. All these nice theoretical facts may be proved by going back to the derivation of the test statistics in [4] and using the theory of asymptotic consistency for empirical quantile functions. In fact, some general consistency statements can be proved under suitable regularity conditions on F and the G that generates the location, scale or location-scale family $\mathcal{G}$, which conditions are somewhat less demanding than those required in [4] and [8] for the existence of the asymptotic distributions of the corresponding test statistics under the null hypotheses. These are not worked out in the literature presumably because, as seen in some of the examples above, they are of limited practical importance.

Conversely, the power examples above suggest that the quantile correlation statistics proposed here for testing composite goodness of fit to Gumbel, Weibull and Pareto families may prove to be useful in a variety of practical scenarios.

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