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CHARLIER AND EDGEWORTH EXPANSIONS FOR DISTRIBUTIONS AND DENSITIES IN TERMS OF BELL POLYNOMIALS

BY

CHRISTOPHER S. WITHERS (LOWER HUTT, NEW ZEALAND)
AND SARALEES NADARAJAH (MANCHESTER, UNITED KINGDOM)

Abstract. We show that the coefficients of the Charlier differential series for distributions and densities are simply Bell polynomials in the cumulants. The same is true for the Edgeworth expansions of distributions and densities of sample means. We use this to obtain higher order extensions of these well-known series.

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1. INTRODUCTION AND SUMMARY

One of the great achievements in the theory of statistics have been the expansions for quantiles of the distribution of an approximately normal estimate given by Cornish and Fisher [4] and later extended by Fisher and Cornish [5]. They built on the expansions of Edgeworth for the distribution and density of a sample mean. These are derived from the 'Type A' differential series of Charlier [2]. (A more accessible reference is Stuart and Ord [7].) Later Hill and Davis [6] extended the results of Cornish and Fisher to estimates with non-normal limits. Other extensions were given by Withers [8].

None of these authors realised the central role that Bell polynomials play in these expansions. In Section 2 we show that the coefficients of the Charlier differential series are Bell polynomials in the cumulants. We give these explicitly up to order 14. In Section 3 we show that the coefficients of the Edgeworth expansions for the distribution of a standardised sample mean are also Bell polynomials. We give these up to order 16. The reformulation of these expansions in terms of Bell polynomials does not appear to have been noted by others (at least to the best knowledge of these authors).

Suppose that

$$S(t) = \sum_{r=1}^{\infty} k_r t^r / r!$$

converges in a neighbourhood of $t = 0$ for some sequence of complex numbers $\mathbf{k} = (k_1, k_2, \ldots)$. Then

$$S(t)^k / k! = \sum_{r=k}^{\infty} B_{rk}(\mathbf{k}) t^r / r!$$

for $k = 0, 1, \ldots$, where, by definition, $B_{rk}(\mathbf{k})$ is the *partial exponential Bell polynomial*. So, $B_{r0}(\mathbf{k}) = \delta_{r0}$, $B_{r1}(\mathbf{k}) = k_r$ and $B_{rr}(\mathbf{k}) = k_1^r$. Note $B_{rk}(\mathbf{k})$ are tabled on pages 307–308 of Comtet [3] for $1 \leqslant k \leqslant r \leqslant 12$. Recurrence formulas for them are given on page 136. Note also that $B_{rk}(\mathbf{k})$ are a linear combination of terms of the form $k_1^{n_1} k_1^{n_2} \ldots$, where $\sum_j n_j = k$, $\sum_j j n_j = r$. So,

$$\kappa_r \equiv \alpha_r \delta^{a+br} \Rightarrow B_{rk}(\mathbf{k}) = B_{rk}(\boldsymbol{\alpha}) \delta^{ak+br}. \tag{1.1}$$

The *complete exponential Bell polynomial* $B_r(\mathbf{k})$ is defined by

$$B_r(\mathbf{k}) = \sum_{k=0}^{r} B_{rk}(\mathbf{k}) \tag{1.2}$$

for $r \geqslant 0$. Consequently,

$$\exp\big(S(t)\big) = \sum_{r=0}^{\infty} t^r B_r(\mathbf{k}) / r!. \tag{1.3}$$

The results of this note could be useful technical tools for obtaining expansions for asymptotically normal estimates; see Withers and Nadarajah [10], [11]. Withers and Nadarajah [10] consider expansions for the log density of an asymptotically normal random variable. Withers and Nadarajah [11] consider expansions for the distribution function of an asymptotically normal random variable.

2. THE CHARLIER DIFFERENTIAL SERIES

Let X be a real absolutely continuous random variable with distribution F, density f, and finite moments and cumulants $m_r = E\,X^r$, $\kappa_r = \kappa_r(X)$.

Let N be a standard normal random variable with density

$$\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2).$$

Let $H_k = H_k(x)$ be the kth Hermite polynomial defined by

$$H_k(x) = \phi(x)^{-1} (-d/dx)^k \phi(x) = E(x + iN)^k$$

for $i = \sqrt{-1}$. See Withers [9]. So, $\{H_k/\sqrt{k!}\}$ form a complete orthonormal set of real functions on R with respect to $\phi(x)$:

$$\int H_j H_k \phi = k! \delta_{jk}, \tag{2.1}$$

where $\int g = \int g(x)dx$ and δ_{jk} is the Kronecker delta function, equal to 1 or 0 for $j = k$ or not.

Suppose that

$$\int f^2/\phi < \infty. \tag{2.2}$$

Then f/ϕ lies in $L_2(\phi)$ and has the Fourier expansion

$$f(x)/\phi(x) = \sum_{r=0}^{\infty} B_r H_r(x)/r!, \tag{2.3}$$

where

$$\begin{aligned} B_r = \int H_r f = E\,H_r(X) &= E(X + iN)^r \\ &= \sum_{0 \leqslant j \leqslant r/2} \binom{r}{2j} m_{r-2j} \nu_{2j}, \end{aligned} \tag{2.4}$$

where

$$\nu_{2j} = E\,N^{2j} = 1 \cdot 3 \ldots (2j-1) = (2j)!/(2^j\, j!),$$

and N is independent of X.

Note that (2.3) holds in the sense of convergence in $L_2(\phi)$:

$$\int \Big[f(x)/\phi(x) - \sum_{r=0}^{K} B_r H_r(x)/r!\Big]^2 \phi(x)dx \to 0$$

as $K \to \infty$. Observe that (2.4) gives the Fourier coefficient B_k in terms of the moments. For example, since $H_0 = 1$, $H_1 = x$, $H_2 = x^2 - 1$, $H_3 = x^3 - 3x$, $H_4 = x^4 - 6x^2 + 3$, $H_5 = x^5 - 10x^3 + 15x$, …, we have $B_0 = 1$, $B_1 = m_1$, $B_2 = m_2 - 1$, $B_3 = m_3 - 3m_1$, $B_4 = m_4 - 6m_2 + 3$, $B_5 = m_5 - 10m_3 + 15m_1$, $B_6 = m_6 - 15m_4 + 45m_2 - 15, \ldots$ Section 6.31 of Stuart and Ord [7] gives these up to B_8 for the case $m_1 = 0$.

Note that integrals like $\int f^2/\phi$ and $\int f \ln \phi$ are only meaningful if X is dimension-free. Suppose in fact that X is standardised so that $E\,X = 0$ and $\mathrm{var}(X) = 1$, that is, $m_1 = 0$ and $m_2 = 1$. Then $B_0 = 1$, $B_1 = B_2 = 0$ and (2.3) can be written as

$$f(x)/\phi(x) - 1 = \sum_{r=3}^{\infty} B_r H_r(x)/r!.$$

Note that (2.3) is known as the Gram–Charlier series. The expressions for the B_k look simpler if we convert from moments to cumulants: $B_3 = \kappa_3$, $B_4 = \kappa_4$, $B_5 = \kappa_5$, $B_6 = \kappa_6 + 10\kappa_3^2$, $B_7 = \kappa_7 + 35\kappa_3\kappa_4$, $B_8 = \kappa_8 + 56\kappa_3\kappa_5 + 35\kappa_4^2$, as noted in (6.41) of Stuart and Ord [7]. However, this conversion becomes laborious. An alternative derivation that avoids this labour is to use the fact that, for $D = d/dx$, the operator $\exp\big(h(-D)^r/r!\big)$ acting on a density f increases its rth cumulant by h but does not change the other cumulants. (This result goes back to Edgeworth. For $r = 1$ this gives Taylor's expansion. We assume that derivatives of all orders exist.) So, if

$$m_1 = 0,\ m_2 = 1, \quad k_r = \kappa_r - \delta_{r2}, \quad S(t) = \sum_{r=1}^{\infty} k_r t^r/r!,$$

then $f = \exp\big(S(-D)\big)\phi$. Consequently, by (1.3), we have the simple formula

$$f(x) = \phi(x) \sum_{r=0}^{\infty} B_r H_r(x)/r!, \tag{2.5}$$

where $B_r = B_r(\mathbf{k}) = B_r(\kappa)_{\kappa_1=\kappa_2=0}$, that is, (2.3) with $B_r = B_r(\mathbf{k})$. Using Comtet's table, (1.2) immediately gives B_k to $k = 12$:

$$\begin{aligned} B_9 &= \kappa_9 + 84\kappa_3\kappa_6 + 126\kappa_4\kappa_5 + 280\kappa_3^3, \\ B_{10} &= \kappa_{10} + 120\kappa_3\kappa_7 + 210\kappa_4\kappa_6 + 126\kappa_5^2 + 2100\kappa_3^2\kappa_4, \\ B_{11} &= \kappa_{11} + 165\kappa_3\kappa_8 + 330\kappa_4\kappa_7 + 462\kappa_5\kappa_6 + 4620\kappa_3^2\kappa_5 + 5775\kappa_3\kappa_4^2, \\ B_{12} &= \kappa_{12} + 220\kappa_3\kappa_9 + 495\kappa_4\kappa_8 + 792\kappa_5\kappa_7 + 462\kappa_6^2 + 9240\kappa_3^2\kappa_6 \\ &\quad + 27\,720\kappa_3\kappa_4\kappa_5 + 5775\kappa_4^3 + 15\,400\kappa_3^4. \end{aligned}$$

Since $k_1 = k_2 = 0$, about two thirds of the terms in (1.2) are zero. An alternative is to calculate B_k as follows:

$$B_r = \sum_{1 \leqslant k \leqslant r/3} [r]_{2k} B_{r-2k,k}(\boldsymbol{\eta}) \tag{2.6}$$

for $r \geqslant 1$, where $\eta_{r-2} = \kappa_r/r(r-1)$ and $[r]_{2k} = r!/(r-2k)!$. Observe that (2.6) follows by noting that $S(t) = t^2 S_1(t)$, where $S_1(t) = \sum_{j=1}^{\infty} \eta_j t^j/j!$. So, $\exp\big(S(t)\big) = \sum_{k=0}^{\infty} t^{2k} S_1(t)^k/k!$. Substitute $S_1(t)^k/k! = \sum_{r=k}^{\infty} B_{rk}(\boldsymbol{\eta}) t^r/r!$ and take the coefficient of $t^r/r!$ to obtain

$$B_{rk}(\boldsymbol{k})/r! = B_{r-2k,k}(\boldsymbol{\eta})/(r-2k)!. \tag{2.7}$$

Note that (2.6) now follows from (1.2). For example, Comtet's table gives B_{rk} for $r \leqslant 12$, and so $B_{13} = \kappa_{13} + 13\alpha$ at

$$\begin{aligned} \alpha &= 22\kappa_3\kappa_{10} + 55\kappa_4\kappa_9 + 99\kappa_5\kappa_8 + 132\kappa_6\kappa_7 + 1320\kappa_3^2\kappa_7 + 4620\kappa_3\kappa_4\kappa_6 \\ &\quad + 44\,352\kappa_3\kappa_5^2 + 3465\kappa_4^2\kappa_5 + 15\,400\kappa_3^3\kappa_4, \end{aligned}$$

and $B_{14} = \kappa_{14} + 13\beta$ at

$$\beta = 77\kappa_4\kappa_{10} + 154\kappa_5\kappa_9 + 231\kappa_6\kappa_8 + 132\kappa_7^2 + 14(2\kappa_3\kappa_{11} + 165\kappa_3^2\kappa_8 + 660\kappa_3\kappa_4\kappa_7 + 924\kappa_3\kappa_5\kappa_6 + 693\kappa_4\kappa_5^2 + 3080\kappa_3^3\kappa_5 + 5775\kappa_3^2\kappa_4^2) + 8085\kappa_4^2\kappa_6.$$

Charlier and Cramer showed that the Gram–Charlier series converges absolutely and uniformly under stronger conditions than the L_2 condition (2.2). For example, Cramer showed that this holds if

$$\int \dot{f}^2/\phi < \infty, \qquad f(x) \to \infty \quad \text{as } |x| \to \infty.$$

Section 6.22 of Stuart and Ord [7] gives this and another theorem of Cramer on its convergence. These theorems do not apply, for example, to the double exponential density as this does not satisfy the L_2 condition (2.2).

By (2.1), we have a form of Parseval's identity:

$$\int f^2/\phi = \sum_{k=0}^{\infty} B_k^2/k!. \tag{2.8}$$

The integrated form of (2.3) is:

$$P(X \leqslant x) = \Phi(x) - \phi(x) \sum_{k=1}^{\infty} B_k H_{k-1}(x)/k!. \tag{2.9}$$

3. THE EDGEWORTH EXPANSION

Suppose that $X = X_n$ is a standardised sample mean of sample of size n from a population with rth cumulant l_r, say $X_n = (n/l_2)^{1/2}(\overline{Y} - l_1)$. For $r \geqslant 2$, X_n has rth cumulant $\kappa_r = \alpha_r n^{1-r/2} = \alpha_r \epsilon^{r-2}$, where $\epsilon = n^{-1/2}$ and $\alpha_r = l_r/l_2^{r/2}$: $\kappa_3 = \alpha_3\epsilon$, $\kappa_4 = \alpha_4\epsilon^2$, $\kappa_5 = \alpha_5\epsilon^3$, $\kappa_6 = \alpha_6\epsilon^4$, $\kappa_7 = \alpha_7\epsilon^5$, $\kappa_8 = \alpha_8\epsilon^6$, $\kappa_9 = \alpha_9\epsilon^7$, $\kappa_{10} = \alpha_{10}\epsilon^8, \ldots$ By (1.1), $B_{rk}(\mathbf{k}) = \epsilon^{r-2k}B_{rk}$, where $B_{rk} = B_{rk}(\boldsymbol{\alpha})$. So, by (1.2), for $r \geqslant 3$ we have

$$\begin{aligned} B_r &= \sum_{k=1}^{r} B_{rk}\epsilon^{r-2k} \\ &= \sum_{m=K_r}^{r-2} \{B_{rk}\epsilon^m :\ k = (r-m)/2,\ r-m \text{ even}\}, \end{aligned} \tag{3.1}$$

where

$$K_{3j} = j,\ K_{3j+1} = j+1,\ K_{3j+2} = j+2. \tag{3.2}$$

In particular,

$$\begin{aligned}
B_3 &= B_{31}\epsilon, \quad B_4 = B_{41}\epsilon^2, \quad B_5 = B_{51}\epsilon^3,\\
B_6 &= B_{62}\epsilon^2 + B_{61}\epsilon^4,\\
B_7 &= B_{72}\epsilon^3 + B_{71}\epsilon^5,\\
B_8 &= B_{82}\epsilon^4 + B_{81}\epsilon^6,\\
B_9 &= B_{93}\epsilon^3 + B_{92}\epsilon^5 + B_{91}\epsilon^7,\\
B_{10} &= B_{10,3}\epsilon^4 + B_{10,2}\epsilon^6 + B_{10,18}\epsilon^8,\\
B_{11} &= B_{11,3}\epsilon^5 + B_{11,2}\epsilon^7 + B_{11,1}\epsilon^9,\\
B_{12} &= B_{12,4}\epsilon^4 + B_{12,3}\epsilon^6 + B_{12,2}\epsilon^8 + B_{12,1}\epsilon^{10},\\
B_{13} &= B_{13,4}\epsilon^5 + B_{13,3}\epsilon^7 + B_{13,2}\epsilon^9 + B_{13,1}\epsilon^{11},\\
B_{14} &= B_{14,4}\epsilon^6 + B_{14,3}\epsilon^8 + B_{14,2}\epsilon^{10} + B_{14,1}\epsilon^{12}.
\end{aligned}$$

From Comtet's table and (2.7) we have immediately

$$\begin{aligned}
B_{r1} &= \alpha_r,\\
B_{62} &= 10\alpha_3^2,\\
B_{72} &= 35\alpha_4\alpha_3,\\
B_{82} &= 56\alpha_5\alpha_3 + 35\alpha_4^2,\\
B_{93} &= 280\alpha_3^3, \quad B_{92} = 84\alpha_6\alpha_3 + 126\alpha_5\alpha_4,\\
B_{10,3} &= 2100\alpha_4\alpha_3^2, \quad B_{10,2} = 120\alpha_7\alpha_3 + 210\alpha_6\alpha_4 + 126\alpha_5^2,\\
B_{11,3} &= 4620\alpha_3^2\alpha_5 + 5775\alpha_3\alpha_4^2,\\
B_{11,2} &= 165\alpha_3\alpha_8 + 330\alpha_4\alpha_7 + 462\alpha_5\alpha_6,\\
B_{12,4} &= 15400\alpha_3^4, \quad B_{12,3} = 9240\alpha_3^2\alpha_6 + 27720\alpha_3\alpha_4\alpha_5 + 5775\alpha_4^3,\\
B_{12,2} &= 220\alpha_3\alpha_9 + 495\alpha_4\alpha_8 + 792\alpha_5\alpha_7 + 462\alpha_6^2,\\
B_{13,4} &= 13 \cdot 15400\alpha_3^3\alpha_4,\\
B_{13,3} &= 13(1320\alpha_3^2\alpha_7 + 4620\alpha_3\alpha_4\alpha_6 + 44352\alpha_3\alpha_5^2 + 3465\alpha_4^2\alpha_5),\\
B_{13,2} &= 13(22\alpha_3\alpha_{10} + 55\alpha_4\alpha_9 + 99\alpha_5\alpha_8 + 132\alpha_6\alpha_7),\\
B_{14,4} &= 13 \cdot 14(3080\alpha_3^3\alpha_5 + 5775\alpha_3^2\alpha_4^2),\\
B_{14,3} &= 13[14(165\alpha_3^2\alpha_8 + 660\alpha_3\alpha_4\alpha_7 + 924\alpha_3\alpha_5\alpha_6 + 693\alpha_4\alpha_5^2)\\
&\qquad + 8085\alpha_4^2\alpha_6],\\
B_{14,2} &= 13(77\alpha_4\alpha_{10} + 154\alpha_5\alpha_9 + 231\alpha_6\alpha_8 + 132\alpha_7^2 + 28\alpha_3\alpha_{11}).
\end{aligned}$$

Substituting (3.1) into (2.5) and (2.9) gives the Edgeworth expansions

$$f(x)/\phi(x) = 1 + \sum_{j=1}^{\infty} h_{1j}(x)n^{-j/2} \tag{3.3}$$

and

$$F(x) = \Phi(x) - \phi(x) \sum_{j=1}^{\infty} h_{0j}(x) n^{-j/2}, \tag{3.4}$$

where

$$h_{1j}(x) = \sum_{k=1}^{j} \{B_{rk} H_r(x)/r! : \ r = j + 2k\}$$

and

$$h_{0j}(x) = \sum_{k=1}^{j} \{B_{rk} H_{r-1}(x)/r! : \ r = j + 2k\}.$$

In particular,

$$\begin{aligned}
h_{11} &= B_{31} H_3/3!, \\
h_{12} &= B_{41} H_4/4! + B_{62} H_6/6!, \\
h_{13} &= B_{51} H_5/5! + B_{72} H_7/7! + B_{93} H_9/9!, \\
h_{14} &= B_{61} H_6/6! + B_{82} H_8/8! + B_{10,3} H_{10}/10! + B_{12,4} H_{12}/12!, \\
h_{15} &= B_{71} H_7/7! + B_{92} H_9/9! + B_{11,3} H_{11}/11! + B_{13,4} H_{13}/13! \\
&\quad + B_{15,5} H_{15}/15!,
\end{aligned}$$

and so on. Note that h_{0j} is just h_{1j} with H_r replaced by H_{r-1}. Differentiating f p times gives

$$(-1)^p f^{(p)}(x)/\phi(x) = H_p(x) + \sum_{j=1}^{\infty} h_{p+1,j}(x) n^{-j/2},$$

where

$$h_{p+1,j}(x) = \sum_{k=1}^{j} \{B_{rk} H_{r+p}(x)/r! : \ r = j + 2k\}.$$

That is, $h_{p+1,j}$ is just h_{1j} with H_r replaced by H_{r+p}.

For some exact conditions for the expansions (3.3) and (3.4), see Theorem 19.2 and Corollary 20.4 of Bhattacharya and Rao [1]. For adjustments to (3.4) for lattice random variables see their Theorem 23.1. Similarly, the Parseval identity (2.8) can be rewritten as

$$\int f^2/\phi = \sum_{r=0}^{\infty} b_r n^{-r},$$

where

$$\begin{aligned}
b_0 &= 1, \quad b_1 = B_{31}^2/3! = \alpha_3^2/6, \\
b_2 &= B_{41}^2/4! + B_{62}^2/6! = \alpha_4^2/24 + 5\alpha_3^4/36, \\
b_3 &= B_{51}^2/5! + 2B_{62} b_{64}/6! + B_{72}^2/7! + B_{93}^2/9! \\
&= \alpha_5^2/120 + \alpha_3^2 \alpha_6/36 + 35\alpha_3^2\alpha_4^2/144 + 35\alpha_3^6/162,
\end{aligned}$$

and

$$\begin{aligned}
b_4 &= B_{61}^2/6! + 2B_{72}B_{71}/7! + B_{82}^2/8! + 2B_{93}B_{92}/9! + B_{10,3}^2/10! \\
&\quad + B_{12,4}^2/12!, \\
b_5 &= B_{71}^2/7! + 2b_{82}B_{81}/8! + (2B_{93}B_{91} + B_{92}^2)/9! + 2B_{10,3}B_{10,2}/10! \\
&\quad + B_{11,3}^2/11! + 2B_{12,4}B_{12,3}/12! + B_{13,4}^2/13! + B_{15,5}^2/15!.
\end{aligned}$$

Note that h_{p5}, b_5 also need $B_{15,5}$ while h_{p6}, b_6 need $B_{16,5}$ and $B_{18,6}$. These are given by

$$\begin{aligned}
B_{3j,j} &= (\alpha_3/6)^j (3j)!/j!, \\
B_{3j+1,j} &= (\alpha_3/6)^{j-1}(\alpha_4/24)(3j+1)!/(j-1)!, \\
B_{3j+2,j} &= [(\alpha_3/6)^{j-1}(\alpha_5/120) + (j-1)(\alpha_3/6)^{j-2}(\alpha_4/12)^2/8] \\
&\qquad \times (3j+2)!/(j-1)!.
\end{aligned} \tag{3.5}$$

To prove this, set $\lambda_{r-2} = \alpha_r/r(r-1)$. By (2.7), we have

$$B_{3j+m,j}/(3j+m)! = B_{j+m,j}(\boldsymbol{\lambda})/(j+m)! \qquad \text{for } m = 0, 1, 2.$$

Now use

$$\begin{aligned}
B_{j,j}(\boldsymbol{\lambda}) &= \lambda_1^j, \\
B_{j+1,j}(\boldsymbol{\lambda}) &= j(j+1)\lambda_1^{j-1}\lambda_2/2, \\
B_{j+2,j}(\boldsymbol{\lambda}) &= j(j+1)(j+2)[\lambda_1^{j-1}\lambda_3/6 + (j-1)\lambda_1^{j-2}\lambda_2^2/8].
\end{aligned}$$

So, we obtain (3.5). (A modification of this argument gives $B_r \sim \epsilon^{K_r}$.)

Consequently, we have explicit formulas for h_{pj} for $j \leqslant 16$. These expressions appear to be new, as well as our expressions for b_j. (But $h_{p,17}$ needs the equality $B_{45,14}/45! = B_{17,14}(\boldsymbol{\lambda})/17!$.)

EXAMPLE 3.1. Suppose that X_0 is a gamma random variable with mean γ. Its rth cumulant is $(r-1)!\gamma$. Its standardised form $X = (X_0 - \gamma)/\sqrt{\gamma}$ has rth cumulant $(r-1)!\gamma^{1-r/2} I(r \geqslant 2)$. Set $s = t/\sqrt{\gamma}$. Then

$$\begin{aligned}
k_r &= (r-1)!\gamma^{1-r/2} I(r \geqslant 3), \\
S(t) &= \gamma \sum_{r=3}^{\infty} s^r/r = -\gamma[\ln(1-s) + s + s^2/2], \\
\exp\big(S(t)\big) &= \exp\big(-\gamma(s + s^2/2)\big)(1-s)^{-\gamma}, \\
B_r &= \gamma^{-r/2} r! \sum_{a+2b+c=r} (-\gamma)^a (-\gamma/2)^b [\gamma]_c / a!b!c!,
\end{aligned}$$

where $[\gamma]_c = \gamma(\gamma+1)\dots(\gamma+c-1)$. However, it is simpler just to apply the previous results of this section with $n = \gamma$ and l_r the rth cumulant of an exponential distribution with mean 1, that is, $l_r = \alpha_r = (r-1)!$. Substituting this we see that B_r is given by (3.1) with

$$\begin{aligned}
B_{r1} &= (r-1)!,\\
B_{62} &= 40,\\
B_{72} &= 420,\\
B_{82} &= 3948,\\
B_{93} &= 2240, \quad B_{92} = 38304,\\
B_{10,3} &= 50400, \quad B_{10,2} = 2065824,\\
B_{11,3} &= 859320, \quad B_{11,2} = 4419360,\\
B_{12,4} &= 246400, \quad B_{12,3} = 13665960, \quad B_{12,2} = 53048160,\\
B_{13,4} &= 9609600, \quad B_{13,3} = 839041632, \quad B_{13,2} = 684478080,\\
B_{14,4} &= 258978720, \quad B_{14,3} = 3060393336, \quad B_{14,2} = 9464307840.
\end{aligned}$$

Note that h_{p5}, b_5, h_{p6}, b_6 also need

$$B_{15,5} = 44844800, \quad B_{16,5} = 2690688000 \quad \text{and} \quad B_{18,6} = 12197785600.$$

Observe that (3.2) explains the behaviour of B_k noted in Example 6.3, page 229 of Stuart and Ord [7].

In this section, we have focused on the case of the sample mean. The results could be extended for: (a) smooth functions of a vector of sample means; and (b) sample quantiles. The classes of statistics, (a) and (b), have been well studied in the recent literature on Edgeworth expansions. Bell polynomials might also be useful in simplifying the explicit computation of higher order Edgeworth expansions for these more general classes of statistics. We hope to address this issue in a future work.

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REFERENCES

[1] R. N. Bhattacharya and R. R. Rao, *Normal Approximation and Asymptotic Expansions*, Wiley, New York 1976.

[2] C. V. L. Charlier, *Applications de la théorie des probabilités à l'astronomie*, Tome II. *Les applications de la théorie des probabilités aux sciences mathématiques et aux science physique*, E. Borel (Ed.), Gauthier-Villars, Paris 1931.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht 1974.

[4] E. A. Cornish and R. A. Fisher, *Moments and cumulants in the specification of distributions*, Review of the International Statistical Institute 5 (1937), pp. 307–320. Reproduced in the collected papers of R. A. Fisher, volume 4.

[5] R. A. Fisher and E. A. Cornish, *The percentile points of distributions having known cumulants*, Technometrics 2 (1960), pp. 209–225.
[6] G. W. Hill and A. W. Davis, *Generalised asymptotic expansions of Cornish–Fisher type*, Ann. Math. Statist. 39 (1968), pp. 1264–1273.
[7] A. Stuart and K. Ord, *Kendall's Advanced Theory of Statistics*, Vol. 1, 5th edition, Griffin, London 1987.
[8] C. S. Withers, *Asymptotic expansions for distributions and quantiles with power series cumulants*, J. Roy. Statist. Soc. B 46 (1984), pp. 389–396. Correction: 48 (1986), p. 258.
[9] C. S. Withers, *A simple expression for the multivariate Hermite polynomials*, Statist. Probab. Lett. 47 (2000), pp. 165–169.
[10] C. S. Withers and S. Nadarajah, *Expansions for* log *densities of asymptotically normal estimates*, Statist. Papers (2008), doi: 10.1007/s00362-008-0135-2.
[11] C. S. Withers and S. Nadarajah, *Tilted Edgeworth expansions for asymptotically normal vectors*, Ann. Inst. Statist. Math. (2008), doi: 10.1007/s10463-008-0206-0.

Applied Mathematics Group
Industrial Research Limited
Lower Hutt, New Zealand
E-mail: c.withers@cri.irl.nz

School of Mathematics
University of Manchester
Manchester M13 9PL
United Kingdom
E-mail: mbbsssn2@manchester.ac.uk

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