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ON A TAYLOR FORMULA FOR A CLASS OF ITÔ PROCESSES

BY

ECKHARD PLATEN AND WOLFGANG WAGNER (BERLIN)

Abstract. In the paper a stochastic generalization of the Taylor formula for Itô processes of diffusion type is investigated with respect to mean square, almost sure and weak convergence.

1. Introduction. In this paper we investigate a stochastic generalization of the Taylor formula for Itô processes of diffusion type. This generalization is based upon the use of multiple stochastic integrals. In [4] we developed similar ideas for the construction of time discrete approximations of Itô processes. Further, we propose Taylor approximations and give estimations of the mean square error. Also almost sure and weak convergence is investigated.

Let $(\Omega, \mathcal{F}, P)$ be the basic probability space and let $(\mathcal{F}_t)$, $t \in [t_0, T]$, be a right continuous non-decreasing family of sub-σ-algebras of $\mathcal{F}$, where $\mathcal{F}_t$ for each $t \in [t_0, T]$ contains all P-null sets of $\mathcal{F}$. We consider the process $(x_t, \mathcal{F}_t)$, $t \in [t_0, T]$, which is given by the Itô differential equation

$$dx_t = a(t, x_t)\,dt + b(t, x_t)\,dw_t, \tag{1}$$

where x_{t_0} is $\mathcal{F}_{t_0}$-measurable and $(w_t, \mathcal{F}_t)$, $t \in [t_0, T]$, is an n-dimensional standard Wiener process, $w_t = \{w_t^{(j)}\}_{j=1}^n$, while $x_t = \{x_t^{(i)}\}_{i=1}^m$ and $a(t, y) = \{a^{(i)}(t, y)\}_{i=1}^m$ are m-dimensional vectors, $b(t, y) = \{b^{(i,j)}(t, y)\}_{i,j=1}^{m,n}$ is an $(m \times n)$-matrix for $m, n \in \{1, 2, \ldots\}$, $a^{(i)}(t, y)$ and $b^{(i,j)}(t, y)$ are real-valued functions on $[t_0, T] \times R^m$. Further, we assume that a strong unique solution of (1) exists. In the following we summarize a result proved in [4] on a representation formula for Itô processes. Let

$$M := \{(j_1, \ldots, j_k) : k \in \{1, 2, \ldots\},\ j_i \in \{0, \ldots, n\} \text{ for } i \in \{1, \ldots, k\}\} \cup \{v\}$$

denote the set of row vectors $\alpha = (j_1, \ldots, j_k)$ with finite length $l(\alpha) := k$, where $l(v) := 0$. We write $-\alpha$ or $\alpha-$ if we delete the first or the last component of $\alpha \in M$ $(l(\alpha) \geqslant 1)$, respectively. Further, $\mathfrak{M}_p$, $p \in \{1, m\}$, is the set of nonanticipative (with respect to $(\mathscr{F}_t)$, $t \in [t_0, T]$) functions $g \mid [t_0, T] \to R^p$ with (1)

$$\mathrm{E} \int_{t_0}^{T} \|g(s)\|^2 \, ds < \infty .$$

For $\alpha \in M$, $s, t \in [t_0, T]$, $s \leqslant t$, and functions $g(\cdot) \in \mathfrak{M}_p$, $p \in \{1, m\}$, we define recursively multiple stochastic integrals

$$I_\alpha(g(\cdot), s, t) := \begin{cases} g(t) & \text{for } \alpha = v, \\ \int_s^t I_{\alpha-}(g(\cdot), s, u)\, dw_u^{(j_k)} & \text{for } \alpha = (j_1, \ldots, j_k), \end{cases} \tag{2}$$

where $dw_u^{(0)} := du$. Obviously, for all $\alpha = (j_1, \ldots, j_k) \in M \setminus \{v\}$ we have

$$I_\alpha(g(\cdot), s, t) = \int_s^t \int_s^{s_k} \ldots \int_s^{s_2} g(s_1)\, dw_{s_1}^{(j_1)} \ldots dw_{s_{k-1}}^{(j_{k-1})}\, dw_{s_k}^{(j_k)}.$$

In the case $g(t) \equiv 1$ we write $I_\alpha(s, t) := I_\alpha(g(\cdot), s, t)$.

We introduce now the differential operators

$$L^{(0)} f := \frac{\partial}{\partial t} f + \sum_{i=1}^{m} \left(\frac{\partial}{\partial x^{(i)}} f \right) a^{(i)} + \frac{1}{2} \sum_{i,r=1}^{m} \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(r)}} f \right) \sum_{j=1}^{n} b^{(i,j)} b^{(r,j)}$$

and

$$L^{(k)} f := \sum_{i=1}^{m} \left(\frac{\partial}{\partial x^{(i)}} f \right) b^{(i,k)} \quad \text{for } k \in \{1, \ldots, n\}$$

which are defined on the corresponding sets of functions $f \mid [t_0, T] \times R^m \to R^m$ having the necessary partial derivatives.

Further, we use functions $f_\alpha = \{f_\alpha^{(i)}\}_{i=1}^m$, $\alpha \in M$, where

$$f_\alpha^{(i)} := \begin{cases} 0 & \text{for } \alpha = v, \\ a^{(i)} & \text{for } \alpha = (0), \\ b^{(i,j)} & \text{for } \alpha = (j), j \in \{1, \ldots, n\}, \\ L^{(j_1)} f_{-\alpha}^{(i)} & \text{for } \alpha = (j_1, \ldots, j_k), k \geqslant 2. \end{cases}$$

Now for $A \subset M$ we set

$$B(A) := \{\alpha \in M \setminus A \colon -\alpha \in A\}$$

and formulate the theorem which is proved in [4].

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(1) $\|\cdot\|$ denotes the Euclidean norm.

THEOREM 1. *Assume that for $A \subset M$, $A \neq \emptyset$, and $s, t \in [t_0, T]$, $s \leqslant t$, the following conditions are satisfied:*

(i) $\sup_{\alpha \in A} l(\alpha) < \infty$;

(ii) $-\alpha \in A$ *for all* $\alpha \in A$;

(iii) f_α *exists for all* $\alpha \in A \cup B(A)$;

(iv) $f_\alpha(s, x_s) \in \mathfrak{M}_m$ *for all* $\alpha \in A$ *and* $f_\alpha(\cdot, x_\cdot) \in \mathfrak{M}_m$ *for all* $\alpha \in B(A)$.

Then

$$x_t = x_s + \sum_{\alpha \in A} I_\alpha(f_\alpha(s, x_s), s, t) + \sum_{\alpha \in B(A)} I_\alpha(f_\alpha(\cdot, x_\cdot), s, t).$$

We can interpret this assertion as a stochastic generalization of the Taylor formula. For this purpose we write this formula in the deterministic case. That means, we have to develop the function

$$x_t = x_{t_0} + \int_{t_0}^{t} a(u, x_u)\,du, \qquad t \in [t_0, T].$$

Obviously, for $s \in [t_0, T]$ we have in that case

$$f_{(0)}(s, x_s) = a(s, x_s) = \left(\frac{d}{dt} x_t\right)(s),$$

$$f_\alpha(s, x_s) = L^{(0)}\left(\frac{d^{k-1}}{dt^{k-1}} x_t\right)(s) = \left(\frac{d^k}{dt^k} x_t\right)(s)$$

for all $\alpha \in M$ with $l(\alpha) = n(\alpha) = k$, $k \in \{2, 3, \ldots\}$,

and

$$f_\alpha(s, x_s) = 0 \quad \text{otherwise,}$$

where $n(\alpha)$ denotes the number of components of $\alpha \in M$ which are equal to 0.

Further, for all $\alpha \in M$ with $l(\alpha) = n(\alpha) = k$ we obtain from (2) the formula

$$I_\alpha(g(t_0), t_0, t) = \int_{t_0}^{t} \ldots \int_{t_0}^{s_2} g(t_0)\,ds_1 \ldots ds_k = g(t_0)(t-t_0)^k/k!.$$

Now, for given $D_r := \{\alpha \in M \colon l(\alpha) = n(\alpha) \leqslant r\}$, $r \in \{0, 1, \ldots\}$, we get from Theorem 1

$$\begin{aligned} x_t &= x_{t_0} + \sum_{\alpha \in D_r} I_\alpha(f_\alpha(t_0, x_{t_0}), t_0, t) + \sum_{\alpha \in B(D_r)} I_\alpha(f_\alpha(\cdot, x_\cdot), t_0, t) \\ &= x_{t_0} + \sum_{k=1}^{r} \left(\frac{d^k}{dt^k} x_t\right)(t_0)\frac{(t-t_0)^k}{k!} + \int_{t_0}^{t} \ldots \int_{t_0}^{s_2} \left(\frac{d^{r+1}}{dt^{r+1}} x_t\right)(s_1)\,ds_1 \ldots ds_{r+1} \end{aligned}$$

which is the well-known Taylor formula.

We propose now a stochastic generalization of the usual Taylor approximations. For $\gamma \in \{0, 1, \ldots\}$ we set

$$A_\gamma := \{\alpha \in M: l(\alpha)+n(\alpha) \leqslant \gamma\}.$$

For $\gamma \in \{0, 1, \ldots\}$ we define the approximation $(x_t^{(\gamma)}, \mathscr{F}_t)$, $t \in [t_0, T]$, by

$$x_t^{(\gamma)} = x_{t_0} + \sum_{\alpha \in A_\gamma} f_\alpha(t_0, x_{t_0}) I_\alpha(t_0, t), \tag{3}$$

where f_α is assumed to exist for all $\alpha \in A_\gamma$.

In the following we investigate the convergence behaviour of the above-defined approximations.

2. Mean square convergence. For all $\alpha \in M$ we introduce the row vector $\alpha^* \in M$ which is obtained by deleting all components of α equal to 0, e.g. $(1, 0, 2, 1)^* = (1, 2, 1)$. For the estimation of the mean square approximation error we use also the following notation of multiple stochastic integrals:

For all $\alpha \in M$ with $\alpha^* = \alpha$ and $l(\alpha^*) = l$, $l \in \{1, 2, \ldots\}$, $k_i \in \{0, 1, \ldots\}$, $i \in \{0, \ldots, l\}$, functions $g(\cdot) \in \mathfrak{M}_p$, $p \in \{1, m\}$, and $t \in [t_0, T]$ we define recursively the multiple stochastic integrals

$$H_{\alpha^*}(k_0, \ldots, k_l, g(\cdot), t) := \begin{cases} g(t) & \text{for } l=0,\ k_0=0, \\ \int_{t_0}^{t} H_v(k_0-1, g(\cdot), u)\,du & \text{for } l=0,\ k_0 \geqslant 1, \\ \int_{t_0}^{t} \frac{(t-u)^{k_l}}{k_l!} H_{\alpha^*-}(k_0, \ldots, k_{l-1}, g(\cdot), u)\,dw_u^{(j_l)} & \text{for } l \geqslant 1, \end{cases} \tag{4}$$

where $\alpha^* = (j_1, \ldots, j_l)$ for $l \geqslant 1$.

In the sequel $k_0(\alpha)$ denotes the number of the first components of $\alpha \in M$ which are equal to 0 until the first non-zero component or until the end of α if there are only zeros. Furthermore, $k_i(\alpha)$, $i \in \{1, \ldots, l(\alpha^*)\}$, counts the number of components which are equal to 0 between the i-th non-zero and the $(i+1)$-st non-zero component or the end of $\alpha \in M$. For example, for $\alpha = (0, 1, 2, 0)$ we have $\alpha^* = (1, 2)$, $l(\alpha^*) = 2$ and $k_0(\alpha) = 1$, $k_1(\alpha) = 0$, $k_2(\alpha) = 1$.

The following proposition shows the relation between the multiple stochastic integrals defined in (2) and (4).

PROPOSITION 1. *For all $\alpha \in M$, functions $g(\cdot) \in \mathfrak{M}_p$, $p \in \{1, m\}$, and $t \in [t_0, T]$ we have*

$$I_\alpha(g(\cdot), t_0, t) = H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)}(\alpha), g(\cdot), t) \quad P\text{-a.s.}$$

For the proof of this assertion we need two lemmas.

LEMMA 1. *For all* $t \in [t_0, t_1]$, $t_1 \in [t_0, T]$, $g(\cdot, t) \in \mathfrak{M}_p$, $p \in \{1, m\}$, *and* $j \in \{1, \ldots, n\}$ *we have*

$$\xi := \int_{t_0}^{t_1} \int_{t_0}^{t_1} g(s, t)\, dw_s^{(j)}\, dt = \int_{t_0}^{t_1} \int_{t_0}^{t_1} g(s, t)\, dt dw_s^{(j)} =: \eta \quad P\text{-a.s.}$$

Proof. By a well-known property of the Itô integral we obtain [2]

$$\mathrm{E}\,\|\xi - \eta\|^2 = \mathrm{E}\,\|\xi\|^2 - 2\mathrm{E}(\xi, \eta) + \mathrm{E}\,\|\eta\|^2 = 0,$$

which completes the proof.

LEMMA 2. *For the multiple stochastic integrals defined in* (4) *we have*

$$R := \int_{t_0}^{t} H_{\alpha^*}(k_0, \ldots, k_{l-1}, k_l, g(\cdot), z)\, dz = H_{\alpha^*}(k_0, \ldots, k_{l-1}, k_l + 1, g(\cdot), t).$$

Proof. We consider two cases.

1. $l = 0$.

We have $\alpha^* = v$ and the assertion follows from (4).

2. $l \geqslant 1$.

By (4) we have

$$R = \int_{t_0}^{t} \int_{t_0}^{t} \chi_{[t_0, z]}(u) \frac{(z-u)^{k_l}}{k_l!} H_{\alpha^*}(k_0, \ldots, k_{l-1}, g(\cdot), u)\, dw_u^{(j_l)} dz,$$

where

$$\chi_{[t_0, z]}(u) := \begin{cases} 1 & \text{for } u \in [t_0, z], \\ 0 & \text{otherwise}. \end{cases}$$

Using Lemma 1 and (4) we get

$$\begin{aligned} R &= \int_{t_0}^{t} H_{\alpha^*-}(k_0, \ldots, k_{l-1}, g(\cdot), u) \int_{t_0}^{t} \chi_{[t_0, z]}(u) \frac{(z-u)^{k_l}}{k_l!}\, dz dw_u^{(j_l)} \\ &= \int_{t_0}^{t} H_{\alpha^*-}(k_0, \ldots, k_{l-1}, g(\cdot), u) \frac{(t-u)^{k_l+1}}{(k_l+1)!}\, dw_u^{(j_l)} \\ &= H_{\alpha^*}(k_0, \ldots, k_{l-1}, k_l + 1, g(\cdot), t), \end{aligned}$$

which completes the proof.

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[2] $(\cdot, \cdot)$ denotes the usual scalar product.

Proof of Proposition 1. We prove the proposition by induction with respect to $l(\alpha)$.

1. $l(\alpha) = 0$.

We have $\alpha = \alpha^* = v$ and the assertion follows from (2) and (4).

2. $l(\alpha) = l \geqslant 1$, $\alpha = (j_1, \ldots, j_l)$.

2.1. $j_l \in \{1, \ldots, n\}$.

In this case we have $\alpha -{}^* = \alpha^* -$ and $k_{l(\alpha^*)} = 0$. From (2), the induction assumption and (4) we obtain

$$\begin{aligned} I_\alpha(g(\cdot), t_0, t) &= \int_{t_0}^{t} I_{\alpha-}(g(\cdot), t_0, u)\, dw_u^{(j_l)} \\ &= \int_{t_0}^{t} H_{\alpha^*-}(k_0(\alpha), \ldots, k_{l(\alpha^*)-1}(\alpha), g(\cdot), u)\, dw_u^{(j_l)} \\ &= H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)-1}(\alpha), 0, g(\cdot), t). \end{aligned}$$

2.2. $j_l = 0$.

We have $k_{l(\alpha^*)}(\alpha) \geqslant 1$ and $\alpha^* = \alpha -{}^*$. From (2), the induction assumption and Lemma 2 we get

$$\begin{aligned} I_\alpha(g(\cdot), t_0, t) &= \int_{t_0}^{t} I_{\alpha-}(g(\cdot), t_0, u)\, du \\ &= \int_{t_0}^{t} H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)}(\alpha)-1, g(\cdot), u)\, du \\ &= H_{\alpha^*}(k_0(\alpha), \ldots, k_{l(\alpha^*)}(\alpha), g(\cdot), t), \end{aligned}$$

which completes the proof.

In the sequel we use the notation

$$C_r^k := \frac{r!}{k!\,(r-k)!},$$

where $k, r \in \{0, 1, \ldots\}$, $k \leqslant r$.

The following assertions will be useful for the mean square error estimation:

Proposition 2. *If* $\alpha, \beta \in M$, $t \in [t_0, T]$ *and* $f(\cdot), g(\cdot) \in \mathfrak{M}_p$, $p \in \{1, m\}$, *then for*

$$\mathrm{E}\big(I_\alpha(f(\cdot), t_0, t), I_\beta(g(\cdot), t_0, t)\big) =: P_{f,g}^{\alpha,\beta}$$

we have

$$P_{f,g}^{\alpha,\beta} = 0 \quad \text{for } \alpha^* \neq \beta^* \tag{5}$$

and

$$P_{f,g}^{\alpha,\beta} \leqslant K_{f,g}\Big(\prod_{i=0}^{l(\alpha^*)} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)}\Big)\frac{(t-t_0)^{\sigma(\alpha,\beta)}}{\sigma(\alpha,\beta)!} \qquad \textit{for } \alpha^* = \beta^*, \tag{6}$$

where

$$K_{f,g} := \sup_{s_1,s_2\in[t_0,T]} \mathrm{E}\,|(f(s_1),\, g(s_2))|$$

and

$$\sigma(\alpha,\beta) := \sum_{i=0}^{l(\alpha^*)} (k_i(\alpha)+k_i(\beta))+l(\alpha^*).$$

In (6) *equality holds if* $f(\cdot) \equiv g(\cdot) \equiv 1$.

By partial integration we obtain easily the following

LEMMA 3. *For* $k, r\in\{0, 1, \ldots\}$ *and* $t\in[t_0, T]$ *we have*

$$\int_{t_0}^{t} (t-u)^k(u-t_0)^r\,du = \frac{k!\,r!}{(k+r+1)!}(t-t_0)^{k+r+1}.$$

Proof of Proposition 2. By Proposition 1 and (4) the assertion (5) can easily be shown by the use of well-known properties of the Itô integral.

We prove (6) by induction with respect to $l(\alpha^*)$.

1. $l(\alpha^*) = l(\beta^*) = 0$.

Obviously, we have $\alpha^* = \beta^* = v$ and from Proposition 1 and (4) in the case $l(\alpha) \geqslant 1$, $l(\beta) \geqslant 1$ we get

$$P_{f,g}^{\alpha,\beta} = \mathrm{E}\Big(\int_{t_0}^{t}\ldots\int_{t_0}^{u_2} f(u_1)\,du_1\ldots du_{l(\alpha)},\ \int_{t_0}^{t}\ldots\int_{t_0}^{u_2'} g(u_1')\,du_1'\ldots du_{l(\beta)}'\Big)$$

$$= \int_{t_0}^{t}\ldots\int_{t_0}^{u_2}\int_{t_0}^{t}\ldots\int_{t_0}^{u_2'} \mathrm{E}(f(u_1),\, g(u_1'))\,du_1'\ldots du_{l(\beta)}'du_1\ldots du_{l(\alpha)}$$

$$\leqslant K_{f,g}\frac{(t-t_0)^{l(\alpha)+l(\beta)}}{l(\alpha)!\,l(\beta)!} = K_{f,g}\Big(\prod_{i=0}^{l(\alpha^*)} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)}\Big)\frac{(t-t_0)^{\sigma(\alpha,\beta)}}{\sigma(\alpha,\beta)!}.$$

Analogously we get this assertion for the case $l(\alpha) = 0$ or $l(\beta) = 0$.

2. $l(\alpha^*) = l(\beta^*) = l \geq 1$, $\alpha^* = \beta^* = (j_1, \ldots, j_l)$.
From Proposition 1, (4) and the induction assumption we obtain

$$P_{f,g}^{\alpha;\beta} = \mathrm{E}\Big(\int_{t_0}^{t} \frac{(t-u)^{k_l(\alpha)}}{k_l(\alpha)!} H_{\alpha^*-}\big(k_0(\alpha), \ldots, k_{l-1}(\alpha), f(\cdot), u\big) dw_u^{(j_l)},$$

$$\int_{t_0}^{t} \frac{(t-u)^{k_l(\beta)}}{k_l(\beta)!} H_{\beta^*-}\big(k_0(\beta), \ldots, k_{l-1}(\beta), g(\cdot), u\big) dw_u^{(j_l)}\Big)$$

$$= \int_{t_0}^{t} \frac{(t-u)^{k_l(\alpha)+k_l(\beta)}}{k_l(\alpha)!\, k_l(\beta)!} \mathrm{E}\Big(H_{\alpha^*-}\big(k_0(\alpha), \ldots, k_{l-1}(\alpha), f(\cdot), u\big),$$

$$H_{\beta^*-}\big(k_0(\beta), \ldots, k_{l-1}(\beta), g(\cdot), u\big)\Big) du$$

$$\leq K_{f,g}\Big(\prod_{i=0}^{l-1} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)}\Big) \int_{t_0}^{t} \frac{(t-u)^{k_l(\alpha)+k_l(\beta)} (u-t_0)^{\bar\sigma(\alpha,\beta)}}{k_l(\alpha)!\, k_l(\beta)!\, \bar\sigma(\alpha, \beta)!} du,$$

where

$$\bar\sigma(\alpha, \beta) := \sum_{i=0}^{l-1} \big(k_i(\alpha)+k_i(\beta)\big)+l-1.$$

Using Lemma 3 we get

$$P_{f,g}^{\alpha,\beta} \leq K_{f,g}\Big(\prod_{i=0}^{l-1} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)}\Big) \frac{(k_{l(\alpha)}+k_{l(\beta)})\, \bar\sigma(\alpha, \beta)!\, (t-t_0)^{\sigma(\alpha,\beta)}}{k_l(\alpha)!\, k_l(\beta)!\, \bar\sigma(\alpha, \beta)!\, \sigma(\alpha, \beta)!}$$

$$= K_{f,g}\Big(\prod_{i=0}^{l} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)}\Big) \frac{(t-t_0)^{\sigma(\alpha,\beta)}}{\sigma(\alpha, \beta)!}.$$

It is easy to see that, in the whole proof, equality holds if $f(\cdot) \equiv g(\cdot) \equiv 1$. Thus the proof is completed.

Now we obtain the following mean square error estimations:

THEOREM 2. *If for given $\gamma \in \{1, 2, \ldots\}$ and for all $\alpha \in A_\gamma \cup B(A_\gamma)$ there exists f_α such that $f_\alpha(t_0, x_{t_0}) \in \mathfrak{M}_m$ for $\alpha \in A_\gamma$ and $f_\alpha(\cdot, x_\cdot) \in \mathfrak{M}_m$ for $\alpha \in B(A_\gamma)$, then for $t \in [t_0, T]$*

$$\mathrm{E}\,\|x_t - x_t^{(\gamma)}\|^2 = \sum_{l=0}^{\gamma+1} \sum_{(\alpha,\beta)\in G_l^{(\gamma)}} \mathrm{E}\big(I_\alpha(f_\alpha(\cdot, x_\cdot), t_0, t), I_\beta(f_\beta(\cdot, x_\cdot), t_0, t)\big)$$

$$\leq \max\Big(1, \frac{t-t_0}{\gamma+2}\Big)\Big(\sum_{\alpha\in B(A_\gamma)} K_{f_\alpha}^{1/2} \frac{\big((2n(\alpha))!\big)^{1/2}}{n(\alpha)!}\Big)^2 \frac{(t-t_0)^{\gamma+1}}{(\gamma+1)!},$$

where (³)

$$G_l^{(\gamma)} := \Big\{(\alpha, \beta) \in B(A_\gamma) \times B(A_\gamma): \quad l(\alpha^*) = l(\beta^*) = l \text{ and } n(\alpha) = n(\beta) = \Big[\frac{\gamma+2-l}{2}\Big]\Big\}$$

and

$$K_{f_\alpha} := \sup_{t \in [t_0, T]} \mathrm{E}\,\|f_\alpha(t, x_t)\|^2.$$

Remark 1. By a straightforward application of Proposition 2 it follows from the first part of Theorem 2 that

$$\mathrm{E}\,\|x_t - x_t^\gamma\|^2 \leqslant \Big\{ \sum_{l=0}^{\gamma+1} \Big(\sum_{(\alpha,\beta) \in G_l^{(\gamma)}} K_{f_\alpha, f_\beta} \prod_{i=0}^{l} C_{k_i(\alpha)+k_i(\beta)}^{k_i(\alpha)} \Big) \varrho_l^{(\gamma)} \Big\} \frac{(t-t_0)^{\gamma+1}}{(\gamma+1)!},$$

where

$$\varrho_l^{(\gamma)} := \left(\frac{t-t_0}{\gamma+2}\right)^{2([(\gamma+2-l)/2]-(\gamma+1-l)/2)}.$$

Remark 2. If we assume additionally in Theorem 2 that for all $\alpha \in B(A_\gamma)$

$$K_{f_\alpha} \leqslant C_1\, C_2^{l(\alpha)+n(\alpha)} \left[\frac{l(\alpha)+n(\alpha)}{2}\right]!,$$

then it follows (similarly as in the proof of Theorem 2) that

$$\mathrm{E}\,\|x_t - x_t^{(\gamma)}\|^2 \leqslant C_3 \big(C_4 (t-t_0)\big)^{\gamma+1} \Big/ \left[\frac{\gamma+1}{2}\right]!. \tag{7}$$

Obviously, the mean square convergence of the sequence of approximations for $\gamma \to \infty$ follows from (7). Using Proposition 2 it is possible to show also mean square convergence in many other cases. From (7) we also obtain, by a well-known assertion (see [3], p. 20), almost sure convergence which will be considered in the next section.

Proof of Theorem 2.

1. By Theorem 1 and (3) we have

$$\begin{aligned} \mathrm{E}\,\|x_t - x_t^{(\gamma)}\|^2 &= \mathrm{E}\,\Big\| \sum_{\alpha \in B(A_\gamma)} I_\alpha\big(f_\alpha(\cdot, x_\cdot), t_0, t\big)\Big\|^2 \\ &= \sum_{(\alpha,\beta) \in B(A_\gamma) \times B(A_\gamma)} \mathrm{E}\Big(I_\alpha\big(f_\alpha(\cdot, x_\cdot), t_0, t\big), I_\beta\big(f_\beta(\cdot, x_\cdot), t_0, t\big)\Big). \end{aligned}$$

It follows from Proposition 2 and the definition of A_γ and $B(A_\gamma)$ that we have only to sum up those $(\alpha, \beta) \in B(A_\gamma) \times B(A_\gamma)$ for which $0 \leqslant l(\alpha^*) = l(\beta^*) \leqslant \gamma+1$. Further, we know that, for all $\alpha \in B(A_\gamma)$, $l(\alpha)+n(\alpha) = l(\alpha^*)+$

(³) $[a]$ denotes the greatest integer not greater than a.

$+2n(\alpha) = \gamma+1$ or $\gamma+2$. Therefore, we have $n(\alpha) = (\gamma+1-l(\alpha^*))/2$ or $(\gamma+2-l(\alpha^*))/2$, respectively. Since $n(\alpha)$ is an integer, we get

$$n(\alpha) = [(\gamma+2-l(\alpha^*))/2].$$

Thus we have to sum up all (α, β) for which

$$0 \leqslant l(\alpha^*) = l(\beta^*) \leqslant \gamma+1 \quad \text{and} \quad n(\alpha) = n(\beta) = [(\gamma+2-l(\alpha^*))/2].$$

This proves the first part of the theorem.

2. For $k_i, r_i \in \{0, 1, \ldots\}$, $i \in \{0, 1, \ldots, l\}$, $l \in \{0, 1, \ldots\}$, we can show by induction with respect to $k_0+r_0+k_1+r_1$ that

$$C_{k_0+r_0}^{k_0} C_{k_1+r_1}^{k_1} \leqslant C_{k_0+r_0+k_1+r_1}^{k_0+k_1}.$$

We have

$$\prod_{i=0}^{l} C_{k_i+r_i}^{k_i} \leqslant C_{(k_0+r_0)+\ldots+(k_l+r_l)}^{k_0+\ldots+k_l}. \tag{8}$$

By Theorem 1 and (3) we obtain

$$\|x_t - x_t^{(\gamma)}\| \leqslant \sum_{\alpha\in B(A_\gamma)} \|I_\alpha(f_\alpha(\cdot, x_\cdot), t_0, t)\|,$$

and by the Minkowski inequality and Proposition 2 we get

$$(\mathrm{E}\|x_t - x_t^{(\gamma)}\|^2)^{1/2} \leqslant \sum_{\alpha\in B(A_\gamma)} \left(\mathrm{E}\|I_\alpha(f_\alpha(\cdot, x_\cdot), t_0, t)\|^2\right)^{1/2}$$

$$\leqslant \sum_{\alpha\in B(A_\gamma)} \left(K_{f_\alpha}\left(\prod_{i=0}^{l(\alpha^*)} C_{2k_i(\alpha)}^{k_i(\alpha)}\right) \frac{(t-t_0)^{l(\alpha)+n(\alpha)}}{(l(\alpha)+n(\alpha))!}\right)^{1/2}.$$

We noted above that $l(\alpha)+n(\alpha) = \gamma+1$ or $\gamma+2$. Now by (8) we obtain

$$\mathrm{E}\|x_t - x_t^{(\gamma)}\|^2 \leqslant \Big(\sum_{\alpha\in B(A_\gamma)} K_{f_\alpha}^{1/2} (C_{2n(\alpha)}^{n(\alpha)})^{1/2}\Big)^2 \frac{(t-t_0)^{\gamma+1}}{(\gamma+1)!} \max\left(1, \frac{t-t_0}{\gamma+2}\right),$$

which completes the proof.

3. Almost sure convergence. In the following we give a condition for uniform almost sure convergence of the approximations.

THEOREM 3. *If for all $\alpha \in M$ there exists f_α such that $f_\alpha(t_0, x_{t_0}), f_\alpha(\cdot, x_\cdot) \in \mathfrak{M}_m$ and $K_{f_\alpha} \leqslant C_1 C_2^{l(\alpha)+n(\alpha)}$, then the approximations $x_t^{(\gamma)}$ converge for $\gamma \to \infty$ P-a.s. to x_t, uniformly with respect to $t \in [t_0, T]$, and*

$$x_t = \lim_{\gamma\to\infty} x_t^{(\gamma)} = x_{t_0} + \sum_{\alpha\in M} f_\alpha(t_0, x_{t_0}) I_\alpha(t_0, t) \quad P\text{-a.s.}$$

Remark 3. For example, the assumptions of Theorem 3 are fulfilled for the linear Itô process

$$x_t = x_{t_0} + \sum_{j=0}^{n} \int_{t_0}^{t} \left((a^{(j)}, x_u) + b^{(j)}\right) dw_u^{(j)},$$

$t \in [t_0, T]$ with $\mathrm{E}\|x_{t_0}\|^2 < \infty$.

Remark 4. It follows from the proof of Theorem 3 that under the assumptions of Theorem 3

$$\mathrm{E} \sup_{t_0 \leqslant s \leqslant T} \|x_s - x_s^{(\gamma)}\|^2 \leqslant C_4 \frac{C_5^{[(\gamma+1)/2]}}{[(\gamma+1)/2]!}.$$

For the proof of Theorem 3 we need

LEMMA 4. *For all* $\alpha \in M \setminus \{v\}$, $t \in [t_0, T]$, *and* $g(\cdot) \in \mathfrak{M}_p$, $p \in \{1, m\}$,

$$J := \mathrm{E} \sup_{t_0 \leqslant s \leqslant t} \|I_\alpha(g(\cdot), t_0, s)\|^2 \leqslant K_g \cdot 4^{l(\alpha)-n(\alpha)} \frac{(t-t_0)^{l(\alpha)+n(\alpha)}}{l(\alpha)!}.$$

Proof. We show this assertion by induction with respect to $l(\alpha)$.

1. $l(\alpha) = 1$, $\alpha = j$.

1.1. $j = 0$.

From (2) we get

$$J = \mathrm{E} \sup_{t_0 \leqslant s \leqslant t} \Big\| \int_{t_0}^{s} g(u)\,du \Big\|^2 \leqslant \mathrm{E} \sup_{t_0 \leqslant s \leqslant t} (s-t_0) \int_{t_0}^{s} \|g(u)\|^2\,du \leqslant K_g (t-t_0)^2.$$

1.2. $j \in \{1, \ldots, n\}$.

By (2) we have

$$I_{(j)}(g(\cdot), t_0, s) = \int_{t_0}^{s} g(u)\,dw_u^{(j)}, \qquad t_0 \leqslant s \leqslant T,$$

which is a square integrable martingale, and by the Doob inequality we obtain

$$J = \mathrm{E} \sup_{t_0 \leqslant s \leqslant t} \Big\| \int_{t_0}^{s} g(u)\,dw_u^{(j)} \Big\|^2 \leqslant 4 \sup_{t_0 \leqslant s \leqslant t} \mathrm{E} \Big\| \int_{t_0}^{s} g(u)\,dw_u^{(j)} \Big\|^2$$

$$\leqslant 4 \sup_{t_0 \leqslant s \leqslant t} \int_{t_0}^{s} \mathrm{E}\|g(u)\|^2\,du \leqslant K_g \cdot 4(t-t_0).$$

2. $l(\alpha) = l \geqslant 2$, $\alpha = (j_1, \ldots, j_l)$.

2.1. $j_l = 0$.

From (2) and the induction assumption it follows that

$$J = \mathrm{E} \sup_{t_0 \leqslant s \leqslant t} \left\|\int_{t_0}^{s} I_{\alpha-}\big(g(\cdot), t_0, u\big) du\right\|^2 \leqslant (t-t_0) \int_{t_0}^{t} \mathrm{E} \sup_{t_0 \leqslant s \leqslant u} \left\|I_{\alpha-}\big(g(\cdot), t_0, s\big)\right\|^2 du$$

$$\leqslant (t-t_0) \int_{t_0}^{t} K_g \cdot 4^{n(\alpha-)} \frac{(u-t_0)^{l(\alpha-)+n(\alpha-)}}{l(\alpha-)!} du$$

$$\leqslant K_g \cdot 4^{l(\alpha)-n(\alpha)} \frac{(t-t_0)^{l(\alpha-)+n(\alpha-)+1}}{l(\alpha-)!\big(l(\alpha-)+n(\alpha-)+1\big)} \leqslant K_g \cdot 4^{l(\alpha)-n(\alpha)} \frac{(t-t_0)^{l(\alpha)+n(\alpha)}}{l(\alpha)!}.$$

2.2. $j_l \in \{1, \ldots, n\}$.

From (2), the Doob inequality, and the induction assumption we get

$$J = \mathrm{E} \sup_{t_0 \leqslant s \leqslant t} \left\| \int_{t_0}^{s} I_{\alpha-}\big(g(\cdot), t_0, u\big) dw_u^{(j_l)}\right\|^2$$

$$\leqslant 4 \sup_{t_0 \leqslant s \leqslant t} \mathrm{E} \left\| \int_{t_0}^{s} I_{\alpha-}\big(g(\cdot), t_0, u\big) dw_u^{(j_l)}\right\|^2 \leqslant 4 \sup_{t_0 \leqslant s \leqslant t} \int_{t_0}^{s} \mathrm{E}\left\|I_{\alpha-}\big(g(\cdot), t_0, u\big)\right\|^2 du$$

$$\leqslant 4 \int_{t_0}^{t} \mathrm{E} \sup_{t_0 \leqslant s \leqslant u} \left\|I_{\alpha-}\big(g(\cdot), t_0, s\big)\right\|^2 du \leqslant 4 \int_{t_0}^{t} K_g \cdot 4^{l(\alpha-)-n(\alpha-)} \frac{(u-t_0)^{l(\alpha-)+n(\alpha-)}}{l(\alpha-)!} du$$

$$\leqslant K_g \cdot 4^{l(\alpha)-n(\alpha)} \frac{(t-t_0)^{l(\alpha-)+n(\alpha-)+1}}{l(\alpha-)!\big(l(\alpha-)+n(\alpha-)+1\big)} \leqslant K_g \cdot 4^{l(\alpha)-n(\alpha)} \frac{(t-t_0)^{l(\alpha)+n(\alpha)}}{l(\alpha)!},$$

which completes the proof.

Proof of Theorem 3. We have already noted that the number of elements of $B(A_\gamma)$, $\gamma \in \{1, 2, \ldots\}$, is not greater than $(n+1)^{\gamma+1}$. Therefore, for $\gamma \in \{1, 2, \ldots\}$ we obtain from Theorem 1 and (3) the following:

$$V^{(\gamma)} := \mathrm{E} \sup_{t_0 \leqslant s \leqslant T} \|x_s - x_s^{(\gamma)}\|^2 = \mathrm{E} \sup_{t_0 \leqslant s \leqslant T} \Big\| \sum_{\alpha \in B(A_\gamma)} I_\alpha\big(f_\alpha(\cdot, x_\cdot), t_0, s\big)\Big\|^2$$

$$\leqslant (n+1)^{\gamma+1} \sum_{\alpha \in B(A_\gamma)} \mathrm{E} \sup_{t_0 \leqslant s \leqslant T} \left\|I_\alpha\big(f_\alpha(\cdot, x_\cdot), t_0, s\big)\right\|^2.$$

Using Lemma 4 and the assumption of the theorem we have

$$V^{(\gamma)} \leqslant (n+1)^{\gamma+1} \sum_{\alpha \in B(A_\gamma)} K_{f_\alpha} \cdot 4^{l(\alpha)-n(\alpha)} (T-t_0)^{l(\alpha)+n(\alpha)}/l(\alpha)!$$

$$\leqslant (n+1)^{\gamma+1} \sum_{\alpha \in B(A_\gamma)} C_1 \{4C_2(T-t_0)\}^{l(\alpha)+n(\alpha)}/l(\alpha)!.$$

From the proof of Theorem 2 we know that for $\alpha \in B(A_\gamma)$

$$\gamma+1 \leqslant l(\alpha)+n(\alpha) \leqslant \gamma+2$$

and

$$l(\alpha) = l(\alpha^*)+n(\alpha) = l(\alpha^*)+[(\gamma+2-l(\alpha^*))/2]$$
$$\geqslant l(\alpha^*)/2+[(\gamma+1)/2] \geqslant [(\gamma+1)/2].$$

Therefore we get

$$V^{(\gamma)} \leqslant C_3 \{(n+1)\cdot 4 C_2(T-t_0)\}^{\gamma+1}/[(\gamma+1)/2]! \leqslant C_4 C_5^{[(\gamma+1)/2]}/[(\gamma+1)/2]!.$$

Now for $\varepsilon > 0$ we obtain

$$\sum_{\gamma=1}^{\infty} P(\sup_{t_0\leqslant s\leqslant T} \|x_s - x_s^{(\gamma)}\| > \varepsilon) \leqslant \varepsilon^{-2} \sum_{\gamma=1}^{\infty} V^{(\gamma)} \leqslant 2C_4 \varepsilon^{-2} \sum_{k=1}^{\infty} C_5^k/k!$$
$$\leqslant 2C_4 \varepsilon^{-2} \exp\{C_5\} < \infty,$$

and almost sure convergence holds (see [3], p. 20), which completes the proof.

4. Weak convergence. For the investigation of bounded continuous functionals of x_t, $t\in[t_0, T]$, it is useful to know whether the sequence of approximations is weakly convergent on $C[t_0, T]$ (see [1]). The following theorem presents a sufficient condition for weak convergence.

THEOREM 4. *If for $\gamma \to \infty$ the family of finite-dimensional distributions of $x_t^{(\gamma)}$, $t\in[t_0, T]$, is convergent to that of x_t, $t\in[t_0, T]$, and*

$$\mathrm{E}\,\|f_\alpha(t_0, x_{t_0})\|^4 \leqslant C_1 C_2^{l(\alpha)+n(\alpha)} \quad \textit{for all } \alpha\in M,$$

then for $\gamma \to \infty$ the sequence of approximations is weakly convergent on $C[t_0, T]$ to x_t, $t\in[t_0, T]$.

Remark 5. For example, the family of finite-dimensional distributions is convergent in the above-discussed cases of mean square convergence.

In the proof of Theorem 4 we will use

LEMMA 5. *For all $\alpha\in M$ and $t\in[t_0, T]$ we have*

$$\mathrm{E}(|I_\alpha(t_0, t)|^4 \mid \mathcal{F}_{t_0}) \leqslant 6^{2(l(\alpha)-n(\alpha))}(t-t_0)^{2(l(\alpha)+n(\alpha))}/l(\alpha)!.$$

Proof. We prove this assertion by induction with respect to $l(\alpha)$.

1. $l(\alpha) = 0$.

We have $\alpha = v$ and by (2) we obtain $I_\alpha(t_0, t) \equiv 1$ and $l(\alpha) = n(\alpha) = 0$, so that the inequality in the lemma is fulfilled.

2. $l(\alpha) = l \geqslant 1$, $\alpha = (j_1, \ldots, j_l)$.

2.1. $j_l = 0$.

From (2) and the induction assumption we get

$$\begin{aligned} E\left(|I_\alpha(t_0, t)|^4 \mid \mathscr{F}_{t_0}\right) &\leqslant E\left(\left(\left|\int_{t_0}^{t} I_{\alpha-}(t_0, u)\,du\right|^2\right)^2 \mid \mathscr{F}_{t_0}\right) \\ &\leqslant E\left(\left((t-t_0)\int_{t_0}^{t} |I_{\alpha-}(t_0, u)|^2\,du\right)^2 \mid \mathscr{F}_{t_0}\right) \\ &\leqslant E\left((t-t_0)^3 \int_{t_0}^{t} |I_{\alpha-}(t_0, u)|^4\,du \mid \mathscr{F}_{t_0}\right) \\ &\leqslant (t-t_0)^3 \int_{t_0}^{t} E\left(|I_{\alpha-}(t_0, u)| \mid \mathscr{F}_{t_0}\right) du \\ &\leqslant (t-t_0)^3\, 6^{2(l(\alpha-)-n(\alpha-))} \frac{(t-t_0)^{2(l(\alpha-)+n(\alpha-))+1}}{l(\alpha-)!\left(2\left(l(\alpha-)+n(\alpha-)\right)+1\right)} \\ &\leqslant 6^{2(l(\alpha)-n(\alpha))} \frac{(t-t_0)^{2(l(\alpha)+n(\alpha))}}{l(\alpha)!}. \end{aligned}$$

2.2. $j_l \in \{1, \ldots, n\}$.

By (2), a well-known inequality (see [2], p. 458), and the induction assumption we have

$$\begin{aligned} E\left(|I_\alpha(t_0, t)|^4 \mid \mathscr{F}_{t_0}\right) &= E\left(\left|\int_{t_0}^{t} I_{\alpha-}(t_0, u)\,dw_u^{(j_l)}\right|^4 \mid \mathscr{F}_{t_0}\right) \\ &\leqslant 36(t-t_0) \int_{t_0}^{t} E\left(|I_{\alpha-}(t_0, u)|^4 \mid \mathscr{F}_{t_0}\right) du \\ &\leqslant 6^2 (t-t_0)\, 6^{2(l(\alpha-)-n(\alpha-))} \frac{(t-t_0)^{2(l(\alpha-)+n(\alpha-))+1}}{l(\alpha-)!\left(2\left(l(\alpha-)+n(\alpha-)\right)+1\right)} \\ &\leqslant 6^{2(l(\alpha)-n(\alpha))} \frac{(t-t_0)^{2(l(\alpha)+n(\alpha))}}{l(\alpha)!}, \end{aligned}$$

which completes the proof of the lemma.

Proof of Theorem 4. For all $t_1, t_2 \in [t_0, T]$, $t_1 \geqslant t_2$, and $\gamma \in \{1, 2, \ldots\}$ it follows from (3) that

$$\begin{aligned} V := E\|x_{t_1}^{(\gamma)} - x_{t_2}^{(\gamma)}\|^4 &\leqslant E\left\|\sum_{r=1}^{\gamma} \sum_{\alpha \in A_r \setminus A_{r-1}} f_\alpha(t_0, x_{t_0})\left(I_\alpha(t_0, t_1) - I_\alpha(t_0, t_2)\right)\right\|^4 \\ &\leqslant \sum_{r=1}^{\gamma} 2^{3r} E\left\|\sum_{\alpha \in A_r \setminus A_{r-1}} f_\alpha(t_0, x_{t_0})\left(I_\alpha(t_0, t_1) - I_\alpha(t_0, t_2)\right)\right\|^4. \end{aligned}$$

Since the number of elements of $A_r \backslash A_{r-1}$, $r \in \{1, \ldots, \gamma\}$, is not greater than $(n+1)^r$, by (2) we have

$$V \leqslant \sum_{r=1}^{\gamma} 2^{3r}(n+1)^{3r} \sum_{\alpha \in A_r \backslash A_{r-1}} \mathrm{E}\left\|f_\alpha(t_0, x_{t_0})\big(I_\alpha(t_0, t_1) - I_\alpha(t_0, t_2)\big)\right\|^4$$

$$\leqslant \sum_{r=1}^{\gamma} (2(n+1))^{3r} \sum_{\alpha \in A_r \backslash A_{r-1}} \mathrm{E}\|f_\alpha(t_0, x_{t_0})\|^4 \mathrm{E}\Big(\Big|\int_{t_2}^{t_1} I_{\alpha-}(t_0, u)\, dw_u^{(j_{l(\alpha)})}\Big|^4 \,\Big|\, \mathscr{F}_{t_0}\Big)$$

$$\leqslant \sum_{r=1}^{\gamma} (2(n+1))^{3r} \sum_{\alpha \in A_r \backslash A_{r-1}} \mathrm{E}\|f_\alpha(t_0, x_{t_0})\|^4 (t_1 - t_2) \times$$

$$\times\big((T-t_0)^2 + 36\big) \int_{t_2}^{t_1} \sup_{t_0 \leqslant s \leqslant T} \mathrm{E}\big(|I_{\alpha-}(t_0, s)|^4 \,\big|\, \mathscr{F}_{t_0}\big)\, du.$$

For $\alpha \in A_r \backslash A_{r-1}$, $r \in \{1, \ldots, \gamma\}$, we obtain

$$l(\alpha) + n(\alpha) = l(\alpha^*) + 2n(\alpha) = r$$

and

$$l(\alpha) = r - \big(r - l(\alpha^*)\big)/2 = r/2 + l(\alpha^*)/2 \geqslant [r/2].$$

It follows from Lemma 5 together with the assumptions of the theorem that

$$V \leqslant (t_1 - t_2)^2 C_4 \sum_{r=1}^{\gamma} C_5^r (n+1)^r/[r/2]!$$

$$\leqslant (t_1 - t_2)^2 C_6 \sum_{k=1}^{\infty} C_7^k/k! \leqslant (t_1 - t_2)^2 C_6 \exp\{C_7\}.$$

From this estimation we infer that the weak convergence of the sequence of approximations holds (see [1] or [3], p. 485).

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Akademie der Wissenschaften der DDR
Institut für Mathematik
DDR-1086 Berlin, Mohrenstrasse 39

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