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ON THE ALMOST SURE CONVERGENCE OF THE SQUARE VARIATION OF THE BROWNIAN MOTION

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Abstract. The paper deals with the problem of almost sure (a.s.) convergence of the square variation of the Brownian motion when the diameters d_n of partitions of the time interval tend to zero. It is known that if the diameters converge fast enough, namely if d_n is of order less than $\lg^{-1} n$, then a.s. convergence takes place. On the other hand, we show that there exists a sequence of partitions with diameters d_n of order less than $\lg^{-\alpha} n$ for any $0 < \alpha < 1$ such that the Brownian square variation diverges a.s.

1. Notation. In this paper (w_t), $t \in [0, 1]$, is a standard Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$ (i.e. the increment $w_{t+h}-w_t$ has mean 0 and variance h). If $\pi = (t_r)_{r=1}^m$ is an interval partition of $[0, 1]$ (i.e. $t_r \in [0, 1]$ for $r = 1, \ldots, m$) and f is a real-valued function on $[0, 1]$, then the expression

$$V^2(f, \pi) = \sum_{r=1}^{m-1} (f(t_{r+1})-f(t_r))^2$$

denotes the square variation of f corresponding to π. The random variable $V^2(w, \pi)$ is called the *Brownian square variation* and denoted, shortly, by $V^2(\pi)$. Next, $d(\pi)$ denotes the diameter of the partition π. A sequence of interval partitions of $[0, 1]$ with diameters tending to zero is simply called a *partition sequence*. The integer part of a number x is denoted by $[x]$.

2. Introduction. It is well known that for a.e. Brownian path there exists a descending partition sequence for which the square variation of the given path diverges to infinity (see [2], p. 48-49). Note that this sequence must really depend on the path, since (see [3], p. 243-244) for every descending partition sequence the Brownian square variation converges a.s. (to 1). It also converges in p-th mean for any $p \geqslant 1$ and any (not necessarily descending) partition sequence.

For non-descending partition sequences, the a.s. convergence depends on the rate of convergence to zero of the diameters of partitions. If they are of order $o(\lg^{-1} n)$, then (as shown in [1]) a.s. convergence takes place. We show that there exists a partition sequence with diameters $d(\pi_n)$ of order less than $\lg^{-\alpha} n$ for all $0 < \alpha < 1$ such that the Brownian square variation is a.s. divergent. Note that in a similar theorem in [1] the partitions consist of measurable sets — not of intervals.

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3. Divergence of square Brownian variation. We shall use an idea of Freedman (see [2], p. 48-49) to establish a divergence result in the case where the diameters of partitions do not converge as fast as required in Theorem 4.5 of [1].

THEOREM. *For every* $c \in \boldsymbol{R}_+$ *there exists a partition sequence* (π_n) *with* $d(\pi_n) = o(\lg^{-\alpha} n)$ *for all* $0 < \alpha < 1$ *such that* $V^2(\pi_n) \geqslant c$ *a.s. for infinitely many indices n.*

Proof. We shall sketch the idea first. We shall find (for a fixed c, any natural number m, and positive ε) a finite set of partitions S with the following properties:

$$d(\pi) \leqslant 1/m \quad \text{for all } \pi \in S,$$

$$P\big(\{\omega \in \Omega \mid \exists \pi \in S \ V^2(w(\omega), \pi) \geqslant c\}\big) \geqslant 1-\varepsilon.$$

Such a set will be called an *(m, ε)-set*. We shall derive an estimate of the minimal number of partitions in this set. Then we shall make ε dependent on m so that

$$\sum_{m=1}^{\infty} \varepsilon(m) < \infty$$

and construct a partition sequence by putting elements of successive $(m, \varepsilon(m))$-sets ($m = 1, 2, \ldots$) one after another. The Borel-Cantelli lemma will give the required divergence result. At the same time we shall be able to estimate how fast the diameters of partitions tend to zero.

Now we shall show how to find an (m, ε)-set. Let $\omega \in \Omega$. Like in [2], we say that an interval $[a, b]$ has *ω-weight c* if

$$(w_a(\omega) - w_b(\omega))^2 \geqslant 2c(b-a).$$

Note that

$$P(\{\omega \mid [a, b] \text{ has } \omega\text{-weight } c\}) = p > 0$$

and that this number does not depend on the interval $[a, b]$.

Denote by $J(n, [a, b])$ the set of all intervals obtained from the partition of $[a, b]$ into n equal parts. For natural n, M, j ($j \leqslant M-1$) define sets

$$A(n, j, M, \omega) = \{I \mid I \in J(n, [j/M, (j+1)/M]),\ I \text{ has } \omega\text{-weight } c\}.$$

Take a natural number N such that $(1-p/2)^N \leqslant 1/2$. For sufficiently large m_1 we have

$$(1) \quad P(\{\omega \mid \operatorname{card} A(m_1, j, m, \omega) \geqslant pm_1/2 \text{ for all } j = 0, 1, \ldots, m-1\}) = (1-Q(m_1))^m \geqslant 1-\varepsilon/N,$$

where, for a natural number n, $Q(n)$ is the probability that the total number of successes in a Bernoulli scheme of n trials with probability of success p is smaller than half of its expected value, i.e. smaller than $pn/2$.

Once $m_1, \ldots, m_i$ have been chosen, choose m_{i+1} so large that

$$(2) \quad P(\{\omega \mid \operatorname{card} A(m_{i+1}, j, M_i, \omega) \geqslant pm_{i+1}/2 \text{ for all } j = 0, 1, \ldots, M_i-1\}) = (1-Q(m_{i+1}))^{M_i} \geqslant 1-\varepsilon/N,$$

where

$$(3) \qquad M_i = mm_1 \ldots m_i, \quad i = 1, \ldots, N.$$

After N steps we have defined $m_1, \ldots, m_N$.

Now put

$$B_i(\omega) = \bigcup_{j=0}^{M_{i-1}-1} A(m_i, j, M_{i-1}, \omega), \quad i = 1, \ldots, N,$$

and $C_1(\omega) = B_1(\omega)$, $C_{i+1}^{(\omega)}$ are these intervals in $B_{i+1}^{(\omega)}$ which are not subintervals of the intervals in $\bigcup_{j=1}^{i} C_j(\omega)$.

Finally, define $\pi(\omega)$ to consist of the endpoints of the intervals in $\bigcup_{i=1}^{N} C_i(\omega)$ and of the points j/m for $j = 0, 1, \ldots, m$. From the choice of N and m_i's it is clear that with probability at least $1-\varepsilon$ we have

$$V^2(w(\omega), \pi(\omega)) \geqslant c.$$

The diameters of all $\pi(\omega)$ are clearly not greater than $1/m$. Thus we have constructed a finite (m, ε)-set (consisting of all $\pi(\omega)$ defined above).

Put

$$K(m, \varepsilon) = \min\{\operatorname{card} S \mid S \text{ is an } (m, \varepsilon)\text{-set}\}.$$

To estimate this number note that $Q(n)$ decreases exponentially, i.e. there exists a positive number r such that

$$(4) \qquad Q(n) \leqslant \exp(-n/r).$$

Write $\eta = (1-\varepsilon/N)^{1/m}$. By (4), to make (1) hold it suffices to put

$$(5) \qquad m_1 = -[r \lg(1-\eta)].$$

Once $m_1, \ldots, m_i$ have been chosen, to prove (2) it suffices to put

$$(6) \qquad m_{i+1} = -[r \lg(1-\eta^{m/M_i})], \quad i = 1, \ldots, N-1,$$

where M_i is defined by (3). It is easy to calculate from (5) and (6) that

(7) $$m_{i+1}-m_i \leqslant 1+r\lg m_i, \quad i = 1, \ldots, N-1.$$

From (3), (5), and (7) we get

(8) $$M_N \leqslant \beta m m_1^N$$

for some $\beta > 1$, all m large enough and ε small enough.

Now put

(9) $$\varepsilon(m) = N/m^2.$$

Fix any $\gamma \in (0, 1)$. For large m we have

(10) $$1-(1-1/m^2)^{1/m} \geqslant \gamma m^{-3}.$$

By (5) and (8)-(10) we obtain

(11) $$K(m, \varepsilon(m)) \leqslant 2^{M_N} \leqslant a^{m(\lg m)^N}$$

for all m large enough and some constant a. By (11), for large m and some constant $b > a$ we have

(12) $$\sum_{j=1}^{m} K(j, \varepsilon(j)) \leqslant b^{m(\lg m)^N}.$$

Now take any $\delta > 0$ and put

(13) $$m(n) = \max\{m \mid b^{m(\lg m)^N} < n\},$$

(14) $$\bar{m}(n) = \max\{m \mid b^{m^{1+\delta}} < n\}.$$

For large n we have

(15) $$m(n) > \bar{m}(n).$$

Now construct a partition sequence by putting elements of successive $(m, \varepsilon(m))$-sets ($m = 1, 2, \ldots$) one after another. For large n it follows from (12) and (13) that π_n does not belong to any of the $(j, \varepsilon(j))$-sets, $j = 1, \ldots, m(n)$. By (14) and (15), this implies

$$d(\pi_n) \leqslant \frac{1}{m(n)+1} \leqslant \frac{1}{\bar{m}(n)+1} \leqslant (\lg_b n)^{-1/(1+\delta)} \quad \text{for all } n \text{ large enough,}$$

which is equivalent to

$$d(\pi_n) = o(\lg^{-\alpha} n) \quad \text{for all } \alpha \ (0 < \alpha < 1).$$

From the Borel-Cantelli lemma, the definition of an $(m, \varepsilon(m))$-set, and the choice of $\varepsilon(m)$ it readily follows that with probability 1 the inequality $V^2(\pi_n) \geqslant c$ holds for infinitely many indices n. Thus the proof of the theorem is complete.

Remark. We have used (15) to simplify the calculation. It is easy to see that, in fact,

$$d(\pi_n) = O\left(\frac{(\lg\lg n)^N}{\lg n}\right).$$

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