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EXPLICIT REPRESENTATION OF THE SKOROKHOD MAP WITH TIME DEPENDENT BOUNDARIES

BY

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Abstract. An explicit formula for the Skorokhod type reflection map for real-valued càdlàg functions is developed in the general case of constraining set $[\alpha, \beta]$, where α and β are not constant but change with time. In addition, a number of properties of the reflection map, including continuity and Lipschitz conditions under uniform, J_1 and M_1 metrics, are studied.

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1. INTRODUCTION

The original Skorokhod map was introduced in [7] as a tool for solving stochastic differential equations.

DEFINITION 1.1 (one-sided Skorokhod map). Let α and ψ be real-valued càdlàg functions on $[0, \infty)$. A pair of real-valued càdlàg functions (ϕ, η), where η is nondecreasing, is said to be a *solution of the Skorokhod problem* (SP) on $[\alpha, \infty)$ for ψ if the following two properties are satisfied:

(i) For every $t \in [0, \infty)$, $\phi(t) = \psi(t) + \eta(t) \geqslant \alpha(t)$.

(ii) $\eta(0-) = 0, \eta(0) \geqslant 0$ and $\int_0^\infty I_{\{\phi(s) > \alpha(s)\}} d\eta(s) = 0$.

The map $\Gamma_\alpha : D[0, \infty) \to D_+[0, \infty)$ defined by $\Gamma_\alpha(\psi) = \phi$ is called the *one-sided reflection map* or *Skorokhod map* (SM) on $[\alpha, \infty)$, and the pair (ϕ, η) is called a *solution of the Skorokhod problem* on $[\alpha, \infty)$ for ψ. The condition $\eta(0-) = 0$ is a traditional convention indicating merely that η has a jump at 0 whenever $\eta(0) > 0$. In the above, $D[0, \infty)$ denotes the set of real-valued right-continuous functions having left limits (usually called *càdlàg functions*). Similarly, we shall use $I[0, \infty)$, $C[0, \infty)$, $BV[0, \infty)$ and $AC[0, \infty)$ to denote subspaces of $D[0, \infty)$ consisting of nondecreasing functions, continuous functions, functions of bounded variation and absolutely continuous functions, respectively. The plus sign as a subscript will indicate nonnegative functions.

The work presented in this paper was inspired mainly by [4], where Kruk, Lehoczky, Ramanan and Shreve provided an explicit formula and studied the properties of the two-sided Skorokhod map (reflection map) $\Gamma_{0,a}$ from $D\,[0,\infty)$ to $D_+\,[0,\infty)$ constraining the process in $D[0,\infty)$ to remain in the interval $[0,a]$, where a is a positive constant. From the applications point of view, it is desirable to allow the reflection boundary to be dependent on time. Therefore, we are going to generalize their results by replacing the constant 0 and a with a function $\alpha, \beta \in D\,[0,\infty)$ such that $\alpha(t) < \beta(t)$ for every $t \geqslant 0$.

The concept of Skorokhod map with time dependent boundaries has been studied recently by Burdzy et al. in [2]. In fact, their analysis includes a more relaxed version of the Skorokhod map called the *extended Skorokhod map*. By using different methods based on the approach in [4] we provide in Section 2 a somewhat different and independently derived explicit representation formula for the two-sided Skorokhod map with time dependent boundaries. In Section 3 we establish the so-called non-antiparticipatory property of the SM. In Section 4 we study continuity and Lipschitz conditions for the SM under three metrics on $D[0,\infty)$.

The original Skorokhod map developed in [7] was a one-sided map with $\alpha = 0$, i.e. Γ_0. Its existence and uniqueness are well known. The following result provides an explicit formula for a one-sided SM with a general boundary α.

LEMMA 1.1. *Let* $\alpha \in D[0,\infty)$. *Then*

$$\Gamma_\alpha(\psi) = \Gamma_0(\psi - \alpha) + \alpha \quad \textit{for every } \psi \in D[0,\infty). \tag{1.1}$$

Proof. Let $\psi' = \psi - \alpha$ and consider the SP for ψ' on $[0,\infty)$. Let (ϕ', η') be its solution. Then

$$\phi' = \psi' + \eta' \geqslant 0 \quad \text{and} \quad \int_0^\infty I_{\{\phi'(s)>0\}}\, d\eta'(s) = 0.$$

It suffices to show that (ϕ, η) defined by $\phi = \phi' + \alpha$ and $\eta = \eta'$ is the solution of the SP for ψ on $[\alpha, \infty)$. Indeed,

$$\phi = \phi' + \alpha = \psi' + \eta' + \alpha = \psi + \eta \geqslant \alpha$$

and

$$\int_0^\infty I_{\{\phi(s)>\alpha(s)\}}\, d\eta(s) = \int_0^\infty I_{\{\phi'(s)>0\}}\, d\eta'(s) = 0. \quad \blacksquare$$

REMARK 1.1. It follows immediately from Lemma 1.1 and equation (1.4) in [4] that

$$\Gamma_\alpha(\psi)(t) = \psi(t) + \sup_{0\leqslant s\leqslant t} [\alpha(s) - \psi(s)]^+ . \tag{1.2}$$

DEFINITION 1.2 (two-sided Skorokhod map). Let $\alpha, \beta \in D\,[0,\infty)$ be such that $\alpha(t) < \beta(t)$ for every $t \geqslant 0$. Given $\psi \in D[0,\infty)$, a pair of functions $(\bar{\phi}, \bar{\eta}) \in D[0,\infty) \times BV[0,\infty)$ is said to be a *solution of the Skorokhod problem* on $[\alpha, \beta]$ for ψ if the following two properties are satisfied:

(i) For every $t \in [0,\infty]$, $\bar{\phi}(t) = \psi(t) + \bar{\eta}(t) \in [\alpha(t), \beta(t)]$.

(ii) $\bar{\eta}\,(0-) = 0$ and $\bar{\eta}$ has the decomposition

$$\bar{\eta} = \bar{\eta}_l - \bar{\eta}_u,$$

where $\bar{\eta}_l, \bar{\eta}_u \in I[0,\infty)$, and

$$\int_0^\infty I_{\{\bar{\phi}(s) > \alpha(s)\}} d\bar{\eta}_l\,(s) = 0 \quad \text{and} \quad \int_0^\infty I_{\{\bar{\phi}(s) < \beta(s)\}} d\bar{\eta}_u\,(s) = 0. \tag{1.3}$$

To show the uniqueness of the solution of the Skorokhod problem when α and β are functions of time we will need the following result. It generalizes part (i) of Lemma 2.2 in [8].

LEMMA 1.2. *Assume that $\beta, \psi_1, \psi_2 \in D[0,\infty)$, $\beta(t) > 0$ for every $t \geqslant 0$ and let $(\bar{\phi}_1, \bar{\eta}_1)$ and $(\bar{\phi}_2, \bar{\eta}_2)$ be the corresponding solutions of the Skorokhod problems for ψ_1 and ψ_2 on $[0,\beta]$. Let $\bar{\eta}_1 = \bar{\eta}_l^1 - \bar{\eta}_u^1$ and $\bar{\eta}_2 = \bar{\eta}_l^2 - \bar{\eta}_u^2$ be the corresponding decompositions satisfying* (1.3). *Then for $t \in [0,\infty)$ the following inequality holds*:

$$\begin{aligned} &|\bar{\phi}_1(t) - \bar{\phi}_2(t)|^2 \\ &\leqslant |\psi_1(t) - \psi_2(t)|^2 + 2\int_0^t \big(\psi_1(t) - \psi_2(t) - \psi_1(s) + \psi_2(s)\big)\big(\bar{\eta}_1(ds) - \bar{\eta}_2(ds)\big). \end{aligned} \tag{1.4}$$

P r o o f. It follows from (1.3) that

$$\begin{aligned} &\int_0^t \big(\bar{\phi}_1(s) - \bar{\phi}_2(s)\big)\bar{\eta}_l^1(ds) \leqslant 0, \quad \int_0^t \big(\bar{\phi}_1(s) - \bar{\phi}_2(s)\big)\bar{\eta}_u^1(ds) \geqslant 0, \\ &\int_0^t \big(\bar{\phi}_1(s) - \bar{\phi}_2(s)\big)\bar{\eta}_l^2(ds) \geqslant 0, \quad \int_0^t \big(\bar{\phi}_1(s) - \bar{\phi}_2(s)\big)\bar{\eta}_u^2(ds) \leqslant 0. \end{aligned} \tag{1.5}$$

Therefore

$$\int_0^\infty \big(\bar{\phi}_1(s) - \bar{\phi}_2(s)\big)\big(\bar{\eta}_1(ds) - \bar{\eta}_2(ds)\big) \leqslant 0. \tag{1.6}$$

For every $t \geqslant 0$ we have

$$|\bar{\eta}_1(t) - \bar{\eta}_2(t)|^2 \leqslant 2\int_0^t \big(\bar{\eta}_1(s) - \bar{\eta}_2(s)\big)\big(\bar{\eta}_1(ds) - \bar{\eta}_2(ds)\big) \tag{1.7}$$

and

$$(1.8)\quad \big(\psi_1(t)-\psi_2(t)\big)\big(\bar\eta_1(t)-\bar\eta_2(t)\big)=\int_0^t \big(\psi_1(t)-\psi_2(t)\big)\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big)$$
$$=\int_0^t \big(\psi_1(t)-\psi_2(t)-\psi_1(s)+\psi_2(s)\big)\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big)$$
$$+\int_0^t \big(\psi_1(s)-\psi_2(s)\big)\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big).$$

Using (1.7) and (1.8) we get

$$\begin{aligned}
&|\bar\phi_1(t)-\bar\phi_2(t)|^2\\
&=|\psi_1(t)-\psi_2(t)|^2+2\big(\psi_1(t)-\psi_2(t)\big)\big(\bar\eta_1(t)-\bar\eta_2(t)\big)+|\bar\eta_1(t)-\bar\eta_2(t)|^2\\
&\leqslant |\psi_1(t)-\psi_2(t)|^2+2\int_0^t \big(\psi_1(t)-\psi_2(t)-\psi_1(s)+\psi_2(s)\big)\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big)\\
&\quad +2\int_0^t \big(\bar\phi_1(s)-\bar\phi_2(s)\big)\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big)\\
&\leqslant |\psi_1(t)-\psi_2(t)|^2+2\int_0^t \big(\psi_1(t)-\psi_2(t)-\psi_1(s)+\psi_2(s)\big)\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big),
\end{aligned}$$

where the last inequality follows from (1.6). ■

We provide now sufficient conditions for the existence of the solution to the Skorokhod problem. Later, in Example 2.1, we will establish that the conditions are not redundant.

THEOREM 1.1. *Let $\alpha,\beta\in D[0,\infty)$ be such that $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)]>0$. Then for every $\psi\in D[0,\infty)$ there exists a unique solution of the Skorokhod problem on $[\alpha,\beta]$ for ψ.*

Proof. The existence will follow from Theorem 2.1 and we postpone this part of the proof till then. To establish the uniqueness of the solution, suppose that $(\bar\phi_1,\bar\eta_1)$ and $(\bar\phi_2,\bar\eta_2)$ are two solutions of the Skorokhod problem on $[\alpha,\beta]$ for ψ. Then it is easy to see and it will be verified in Lemma 2.2 that $(\bar\phi_1-\alpha,\bar\eta_1)$ and $(\bar\phi_2-\alpha,\bar\eta_2)$ are two solutions of the Skorokhod problem on $[0,\beta-\alpha]$ for $\psi-\alpha$. Therefore we can assume without loss of generality that $\alpha=0$.

Let $\bar\eta_1=\bar\eta_l^1-\bar\eta_u^1$ and $\bar\eta_2=\bar\eta_l^2-\bar\eta_u^2$ be the respective decompositions satisfying (1.3). Then, by (1.4), for any $t\in[0,\infty)$,

$$|\bar\phi_1(t)-\bar\phi_2(t)|^2\leqslant 0+2\int_0^t 0\cdot\big(\bar\eta_1(ds)-\bar\eta_2(ds)\big)=0,$$

and so $\bar\phi_1(t)=\bar\phi_2(t)$. ■

Suppose that the pair $(\bar{\phi}, \bar{\eta})$ is a solution of the Skorokhod problem on $[\alpha, \beta]$ for ψ. Then the map $\Gamma_{\alpha,\beta} : D[0,\infty) \to D[0,\infty)$ defined by $\Gamma_{\alpha,\beta}(\psi) = \bar{\phi}$ will be called the *two-sided reflection map* or *Skorokhod map* on $[\alpha, \beta]$. Throughout this paper α and β will always denote càdlàg functions possibly satisfying some additional conditions as indicated. $S[0,\infty)$ will denote a subspace of $D[0,\infty)$ consisting of piecewise constant functions with a finite number of jumps and $S_+[0,\infty) = S[0,\infty) \cap D_+[0,\infty)$. We use the projection maps $\pi_{a,b} : \mathbb{R} \to [a,b]$ defined by

$$\pi_{a,b} = \begin{cases} a & \text{if } x \leqslant a, \\ x & \text{if } a \leqslant x \leqslant b, \\ b & \text{if } x \geqslant b. \end{cases}$$

Note that each $\pi_{a,b}$ is monotone and Lipschitz continuous with constant 1. The following is a simple example of a Skorokhod map with a time dependent boundary. It is based on (28) in [3].

EXAMPLE 1.1. Let $\psi \in S[0,\infty)$, let $\alpha, \beta \in S[0,\infty)$ be such that $\alpha(t) < \beta(t)$ for every $t \geqslant 0$ and let $0 = t_0 < t_1 < \ldots < t_k$ be all the jump points of either function. Define $\bar{\phi}(0) = \pi_{\alpha(0),\beta(0)}\big(\psi(0)\big)$ and then, recursively, $\bar{\phi}(t_{i+1}) = \pi_{\alpha(t_{i+1}),\beta(t_{i+1})}\big(\bar{\phi}(t_i) + \psi(t_{i+1}) - \psi(t_i)\big)$. Finally, we set $\bar{\phi}(t) = \bar{\phi}(t_i)$ for any $t \in (t_i, t_{i+1})$ and $\bar{\phi}(t) = \bar{\phi}(t_k)$ for $t \in (t_k, \infty)$. Then $\bar{\phi}$ is a well-defined function in $S[0,\infty)$ and it is easy to verify that conditions (i) and (ii) of Definition 1.2 are satisfied so that $\Gamma_{\alpha,\beta}(\psi) = \bar{\phi}$, i.e. $\bar{\phi}$ is the Skorokhod map for ψ.

To remain consistent with the notation used in [4] we define the mappings $R_t^{\alpha,\beta}$, $C_{\alpha,\beta}$, $\Lambda_{\alpha,\beta}\colon D[0,\infty) \to D[0,\infty)$ as follows:

$$R_t^{\alpha,\beta}(\phi)(s) = \big(\phi(s) - \beta(s)\big)^+ \wedge \inf_{s \leqslant r \leqslant t} \big(\phi(r) - \alpha(r)\big) \quad \text{for } 0 \leqslant s \leqslant t, \tag{1.9}$$

$$C_{\alpha,\beta}(\phi)(t) = C^{\phi}_{\alpha,\beta}(t) = \sup_{0 \leqslant s \leqslant t} \Big[\big(\phi(s) - \beta(s)\big)^+ \wedge \inf_{s \leqslant r \leqslant t} \big(\phi(r) - \alpha(r)\big)\Big], \tag{1.10}$$

$$\Lambda_{\alpha,\beta}(\phi)(t) = \phi(t) - \sup_{0 \leqslant s \leqslant t} \Big[\big(\phi(s) - \beta(s)\big)^+ \wedge \inf_{s \leqslant r \leqslant t} \big(\phi(r) - \alpha(r)\big)\Big]. \tag{1.11}$$

Using (1.9) we can obtain the following equivalent expressions to (1.11):

$$\Lambda_{\alpha,\beta}(\phi)(t) = \phi(t) - \sup_{0 \leqslant s \leqslant t} R_t^{\alpha,\beta}(\phi)(s) \qquad \text{or} \qquad \Lambda_{\alpha,\beta}(\phi) = \phi - C^{\phi}_{\alpha,\beta}.$$

Note that

$$\phi_1 \leqslant \phi_2 \ \text{ implies } \ C^{\phi_1}_{\alpha,\beta} \leqslant C^{\phi_2}_{\alpha,\beta}. \tag{1.12}$$

As we are often going to consider the special case when $\alpha = 0$, we shall also use the abbreviated notation

$$R_t^{\beta}(\phi) = R_t^{0,\beta}(\phi), \quad C_{\beta}(\phi) = C_{0,\beta}^{\phi}, \quad \Lambda_{\beta}(\phi) = \Lambda_{0,\beta}(\phi).$$

PROPOSITION 1.1. *Let β be a function in $D_+[0,\infty)$. Then*:
(i) *Λ_β maps $D_+[0,\infty)$ into $D_+[0,\infty)$.*
(ii) *If $\beta \in C_+[0,\infty)$, then Λ_β maps $C_+[0,\infty)$ into $C_+[0,\infty)$.*
(iii) *If $\beta \in AC_+[0,\infty)$, then Λ_β maps $AC_+[0,\infty)$ into $AC_+[0,\infty)$.*

Before we prove Proposition 1.1 we recall the definition of a càdlàg function. $\phi \in D_+[0,\infty)$ if and only if the following two conditions hold for any $\varepsilon > 0$:

for each $\theta_1 \geqslant 0$ there is $\theta_2 > \theta_1$ such that $\displaystyle\sup_{s,r\in[\theta_1,\theta_2)} |\phi(s) - \phi(r)| \leqslant \varepsilon$,

for each $\theta_2 \geqslant 0$ there is $0 \leqslant \theta_1 < \theta_2$ such that $\displaystyle\sup_{s,r\in[\theta_1,\theta_2)} |\phi(s) - \phi(r)| \leqslant \varepsilon$.

Note that for $\beta, \phi \in D[0,\infty)$,

(1.13) given $\theta_1 \geqslant 0$ and $\varepsilon > 0$, we can find $\theta_2 > \theta_1$ such that
$$\sup_{s,r\in[\theta_1,\theta_2)} |\phi(s) - \phi(r)| \leqslant \varepsilon \quad \text{and} \quad \sup_{s,r\in[\theta_1,\theta_2)} |\beta(s) - \beta(r)| \leqslant \varepsilon.$$

Similarly,

(1.14) given $\theta_2 \geqslant 0$ and $\varepsilon > 0$, we can find $0 \leqslant \theta_1 \leqslant \theta_2$ such that
$$\sup_{s,r\in[\theta_1,\theta_2)} |\phi(s) - \phi(r)| \leqslant \varepsilon \quad \text{and} \quad \sup_{s,r\in[\theta_1,\theta_2)} |\beta(s) - \beta(r)| \leqslant \varepsilon.$$

The following lemma is a generalization of Lemma 2.1 in [4].

LEMMA 1.3. *Let $\phi, \beta \in D_+[0,\infty)$. Then*:

(i) $$C_\beta^\phi(t_2) - C_\beta^\phi(t_1) \leqslant \sup_{t_1 < s \leqslant t_2} |\phi(s) - \phi(t_1)| + \sup_{t_1 < s \leqslant t_2} |\beta(s) - \beta(t_1)| \quad \textit{for } 0 \leqslant t_1 < t_2;$$

(ii) $$\sup_{t_1,t_2\in[\theta_1,\theta_2)} |\Lambda_\beta(\phi)(t_1) - \Lambda_\beta(\phi)(t_2)| \leqslant 2 \sup_{t_1,t_2\in[\theta_1,\theta_2)} |\phi(t_1) - \phi(t_2)| + \sup_{t_1,t_2\in[\theta_1,\theta_2)} |\beta(t_1) - \beta(t_2)| \quad \textit{for } 0 \leqslant \theta_1 < \theta_2.$$

Proof. From (1.9) we infer that for every $t \geqslant 0$

$$\big(\phi(t) - \beta(t)\big)^+ = \big(\phi(t) - \beta(t)\big)^+ \wedge \phi(t) \leqslant \sup_{0\leqslant s\leqslant t} R_t^\beta(\phi)(s). \tag{1.15}$$

Let $0 \leqslant t_1 < t_2$; then, by (1.9),

$$\begin{aligned}\sup_{t_1<s\leqslant t_2} R^{\beta}_{t_2}(\phi)(s) &\leqslant \sup_{t_1<s\leqslant t_2} \big(\phi(s)-\beta(s)\big)^+ \\ &\leqslant \sup_{t_1<s\leqslant t_2} \big[|\phi(s)-\phi(t_1)| + \big(\phi(t_1)-\beta(t_1)\big)^+ + |\beta(s)-\beta(t_1)|\big] \\ &\leqslant \big(\phi(t_1)-\beta(t_1)\big)^+ + \sup_{t_1<s\leqslant t_2} |\phi(s)-\phi(t_1)| + \sup_{t_1<s\leqslant t_2} |\beta(s)-\beta(t_1)|.\end{aligned}$$

Therefore, by (1.15),

$$\begin{aligned}C^{\phi}_{\beta}(t_2) = \sup_{0<s\leqslant t_2} R^{\beta}_{t_2}(\phi)(s) &= \sup_{0<s\leqslant t_1} R^{\beta}_{t_2}(\phi)(s) \vee \sup_{t_1<s\leqslant t_2} R^{\beta}_{t_2}(\phi)(s) \\ &\leqslant \sup_{0<s\leqslant t_1} R^{\beta}_{t_2}(\phi)(s) \\ &\quad \vee \big[\big(\phi(t_1)-\beta(t_1)\big)^+ + \sup_{t_1<s\leqslant t_2} |\phi(s)-\phi(t_1)| + \sup_{t_1<s\leqslant t_2} |\beta(s)-\beta(t_1)|\big] \\ &\leqslant C^{\phi}_{\beta}(t_1) + \sup_{t_1<s\leqslant t_2} |\phi(s)-\phi(t_1)| + \sup_{t_1<s\leqslant t_2} |\beta(s)-\beta(t_1)|,\end{aligned}$$

which completes the proof of part (i).

To prove part (ii) let $t_1, t_2 \in [\theta_1, \theta_2)$ and assume without loss of generality that $t_1 < t_2$. By part (i),

$$\begin{aligned}|\Lambda_\beta(\phi)(t_2) - \Lambda_\beta(\phi)(t_1)| &\leqslant |\phi(t_1)-\phi(t_2)| + |C^{\phi}_{\beta}(t_1) - C^{\phi}_{\beta}(t_2)| \\ &\leqslant 2 \sup_{t_1,t_2\in[\theta_1,\theta_2)} |\phi(t_1)-\phi(t_2)| + \sup_{t_1,t_2\in[\theta_1,\theta_2)} |\beta(t_1)-\beta(t_2)|. \quad \blacksquare\end{aligned}$$

The arguments from the proof of Lemma 1.3 can be used to show the following, slightly different version.

REMARK 1.2. *If $\phi, \beta \in D_+[0,\infty)$ and $0 \leqslant \theta_1 \leqslant \theta_2$, then*

$$\begin{aligned}\sup_{t_1,t_2\in[\theta_1,\theta_2]} &|\Lambda_\beta(\phi)(t_1) - \Lambda_\beta(\phi)(t_2)| \\ &\leqslant 2 \sup_{t_1,t_2\in[\theta_1,\theta_2]} |\phi(t_1)-\phi(t_2)| + \sup_{t_1,t_2\in[\theta_1,\theta_2]} |\beta(t_1)-\beta(t_2)|. \qquad (1.16)\end{aligned}$$

P r o o f o f P r o p o s i t i o n 1.1. Part (i) follows directly from (1.13), (1.14) and Lemma 1.3. Part (ii) follows directly from (1.16). Part (iii) can be obtained from (1.16) via arguments used in the proof of Corollary 2.3 in [4] with $\nu_{\Lambda_\beta(\phi)}(\varepsilon) = \nu_\phi(\varepsilon/3) \wedge \nu_\beta(\varepsilon/3)$. $\blacksquare$

2. THE EXPLICIT REPRESENTATION OF THE SKOROKHOD MAP AND ITS CONSTRAINING TERM

Burdzy et al. worked out in [2] a representation formula for a generalization of the Skorokhod map called the *extended Skorokhod map* (ESM), which is the solution of the extended Skorokhod problem (ESP). In the definition of the ESP (see Definition 2.2 in [2]) $\bar{\eta}$ is not required to be of bounded variation and (1.3) is replaced by the following weaker conditions:

For every $0 \leqslant s \leqslant t$,

$$\begin{aligned} \bar{\eta}(t) - \bar{\eta}(s) &\geqslant 0 \quad \text{if } \bar{\phi}(r) < \beta(r) \text{ for all } r \in (s,t], \\ \bar{\eta}(t) - \bar{\eta}(s) &\leqslant 0 \quad \text{if } \bar{\phi}(r) > \alpha(r) \text{ for all } r \in (s,t], \\ \bar{\eta}(t) - \bar{\eta}(t-) &\geqslant 0 \quad \text{if } \bar{\phi}(t) < \beta(t), \\ \bar{\eta}(t) - \bar{\eta}(t-) &\leqslant 0 \quad \text{if } \bar{\phi}(t) > \alpha(t). \end{aligned}$$

The ESM is denoted by $\bar{\Gamma}_{\alpha,\beta}$. Thus $\bar{\phi} = \bar{\Gamma}_{\alpha,\beta}(\psi)$. Burdzy et al. established in Theorem 2.6 of [2] that for any $\alpha, \beta \in D[0,\infty)$ such that $\alpha \leqslant \beta$

$$\Gamma_{\alpha,\beta}(\psi) = \psi - \Xi_{\alpha,\beta}(\psi), \tag{2.1}$$

where $\Xi_{\alpha,\beta}(\psi) : D[0,\infty) \to D[0,\infty)$ is given by

$$\begin{aligned} \Xi_{\alpha,\beta}(\psi)(t) = \max \Big\{ &\big[\big(\psi(0) - \beta(0)\big)^+ \wedge \inf_{0 \leqslant r \leqslant t} \big(\psi(r) - \alpha(r)\big)\big], \\ &\sup_{0 \leqslant s \leqslant t} \big[\big(\psi(s) - \beta(s)\big) \wedge \inf_{s \leqslant r \leqslant t} \big(\psi(r) - \alpha(r)\big)\big] \Big\}. \end{aligned} \tag{2.2}$$

They obtained their result first for simple functions and then extended it by the limiting process. The results of this paper are based on the approach developed by Kruk et al. in [4] and are independent of [2]. The following is our generalization of (1.13) in [4].

THEOREM 2.1. *Let $\alpha, \beta \in D[0,\infty)$, $\alpha \leqslant \beta$, and let Γ_α and $\Gamma_{\alpha,\beta}$ be the Skorokhod maps on $[\alpha,\infty)$ and $[\alpha,\beta]$, respectively. If $\inf_{t \geqslant 0} [\beta(t) - \alpha(t)] > 0$, then*

$$\Gamma_{\alpha,\beta} = \Lambda_{\alpha,\beta} \circ \Gamma_\alpha. \tag{2.3}$$

We shall break the proof of Theorem 2.1 into several intermediate results. By a similar argument to the one used in Lemma 1.1 we will show that to solve the SP for ψ on $[\alpha,\beta]$ it is enough to solve the SP for $\psi - \alpha$ on $[0, \beta - \alpha]$. Therefore we focus first on the case of $\alpha = 0$.

LEMMA 2.1. *For any $\beta \in D_+[0,\infty)$ and $\psi \in D[0,\infty)$*

$$0 \leqslant \Lambda_\beta \circ \Gamma_0(\psi) \leqslant \Gamma_0(\psi) \wedge \beta. \tag{2.4}$$

In particular, for $t \in [0,\infty)$,

$$\Gamma_0(\psi)(t) = 0 \textit{ implies } \Lambda_\beta \circ \Gamma_0(\psi)(t) = 0. \tag{2.5}$$

P r o o f. From (1.9) we get $\Lambda_\beta(\phi)(t) = \phi(t) - \sup_{0\leqslant s\leqslant t} R_t^\beta(\phi)(s) \geqslant 0$, and since $\phi = \phi\wedge\beta + \phi\wedge(\phi-\beta)^+$, we have

$$\Lambda_\beta(\phi)(t) \leqslant \phi(t) - \big(\phi(t)-\beta(t)\big)^+ \wedge \phi(t) = \phi(t)\wedge\beta(t). \quad \blacksquare$$

Let us define two increasing sequences of times $\{\sigma_k^\beta \mid k = 0,1,2,\dots\}$ and $\{\tau_k^\beta \mid k = 0,1,2,\dots\}$ as follows. We set

$$\tau_0 = 0, \quad \sigma_0 = \min\{t > 0 \mid \phi(t) - \beta(t) \geqslant 0\} \tag{2.6}$$

and for $k \geqslant 1$

$$\tau_k = \min\{t > \sigma_{k-1} \mid \sup_{\sigma_{k-1}\leqslant s\leqslant t} [\phi(s)-\beta(s)] \geqslant \phi(t)\}, \tag{2.7}$$

$$\sigma_k = \min\{t > \tau_k \mid \phi(t)-\beta(t) \geqslant \inf_{\tau_k\leqslant r\leqslant t} \phi(r)\}. \tag{2.8}$$

The right continuity of ϕ and β guarantees that τ_k and σ_k are well defined. It is easy to see that

$$\phi(\sigma_0) - \beta(\sigma_0) \geqslant 0. \tag{2.9}$$

Furthermore, for $k \geqslant 1$,

$$\sup_{\sigma_{k-1}\leqslant s\leqslant t} [\phi(s)-\beta(s)] < \phi(t) \quad \text{for every } t \in [\sigma_{k-1}, \tau_k), \tag{2.10}$$

$$\sup_{\sigma_{k-1}\leqslant s\leqslant \tau_k} [\phi(s)-\beta(s)] \geqslant \phi(\tau_k), \tag{2.11}$$

$$\phi(s)-\beta(s) < \inf_{\tau_k\leqslant r\leqslant s} \phi(r) \quad \text{for every } s \in [\tau_k, \sigma_k), \tag{2.12}$$

$$\phi(\sigma_k)-\beta(\sigma_k) \geqslant \inf_{\tau_k\leqslant r\leqslant \sigma_k} \phi(r). \tag{2.13}$$

It follows from (2.10) that $\phi(s)-\beta(s) < \phi(t)$ whenever $\sigma_{k-1} \leqslant s \leqslant t < \tau_k$, $k \geqslant 1$. Therefore

$$\phi(s)-\beta(s) \leqslant \inf_{s\leqslant r<\tau_k} \phi(r) \text{ whenever } \sigma_{k-1} \leqslant s < \tau_k. \tag{2.14}$$

Note that $0 = \tau_0 \leqslant \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \dots$ unless one of the times equals ∞, at which point all the following times are also ∞. Also note that the time sequences depend on both ϕ and β.

PROPOSITION 2.1. *Let* $\phi, \beta \in D_+[0,\infty)$. *If* $\inf_{t\geqslant 0}\beta(t) > 0$, *then*

$$\lim_{k\to\infty}\tau_k = \infty \quad \textit{and} \quad \lim_{k\to\infty}\sigma_k = \infty.$$

P r o o f. The conclusion follows from arguments used in the proof of Proposition 3.1 of [4] with $a = \inf_{t\geqslant 0}\beta(t)$. ■

Note that if $\inf_{t\geqslant 0}\beta(t) = 0$, then it is easy to construct ϕ and ψ such that

$$\lim_{k\to\infty}\tau_k = \lim_{k\to\infty}\sigma_k = \gamma < \infty.$$

The next result generalizes Proposition 3.2 in [4].

PROPOSITION 2.2. *Let* $\phi, \beta \in D_+[0,\infty)$. *If* $\inf_{t\geqslant 0}\beta(t) > 0$, *and* $k \geqslant 1$, *then*

$$C^\phi_\beta(t) = \sup_{\sigma_{k-1}\leqslant s\leqslant t}[\phi(s)-\beta(s)]^+ \quad \textit{for every } t \in [\sigma_{k-1},\tau_k).$$

P r o o f. Both the lower bound and the upper bound can be established by the same arguments that were used in the proof of Proposition 3.2 in [4] with C^ϕ replaced by C^ϕ_β and a replaced by β. ■

The next result corresponds to Proposition 3.3 in [4]. We repeat the steps of the proof from there indicating the necessary changes.

PROPOSITION 2.3. *Let* $\phi, \beta \in D_+[0,\infty)$ *and* $\inf_{t\geqslant 0}\beta(t) > 0$. *Then*

$$C^\phi_\beta(t) = 0 \quad \textit{for } 0 \leqslant t < \sigma_0,$$

$$C^\phi_\beta(t) = \inf_{\tau_k\leqslant r\leqslant t}\phi(r) \quad \textit{for every } t \in [\tau_k,\sigma_k) \textit{ and every } k \geqslant 1.$$

P r o o f. For $0 \leqslant t < \sigma_0$ we get $C^\phi_\beta(t) = 0$, by (1.10). To get the upper bound for $k \geqslant 1$ we let $t \in [\tau_k,\sigma_k)$ and write $C^\phi_\beta(t) = S_1 \vee S_2$, where

$$S_1 = \sup_{0\leqslant s\leqslant\tau_k}\left[\left(\phi(s)-\beta(s)\right)^+ \wedge \inf_{s\leqslant r\leqslant t}\phi(r)\right]$$

and

$$S_2 = \sup_{\tau_k\leqslant s\leqslant t}\left[\left(\phi(s)-\beta(s)\right)^+ \wedge \inf_{s\leqslant r\leqslant t}\phi(r)\right].$$

Then $S_1 \leqslant \inf_{\tau_k\leqslant r\leqslant t}\phi(r)$ trivially and $S_2 \leqslant \inf_{\tau_k\leqslant r\leqslant t}\phi(r)$ by (2.12). To prove the lower bound it suffices to show that for every positive ε

$$C^\phi_\beta(t) \geqslant \inf_{\tau_k\leqslant r\leqslant t}\phi(r) - \varepsilon. \tag{2.15}$$

Let $\varepsilon > 0$ be arbitrary and let $\rho \in [\sigma_{k-1}, \tau_k]$ be such that

$$\sup_{\sigma_{k-1} \leqslant s \leqslant \tau_k} [\phi(s) - \beta(s)] < \phi(\rho) - \beta(\rho) + \varepsilon. \tag{2.16}$$

For $t \in [\tau_k, \sigma_k)$ by (1.10) we have

$$C_\beta^\phi(t) \geqslant \big(\phi(\rho) - \beta(\rho)\big)^+ \wedge \inf_{\rho \leqslant r < \tau_k} \phi(r) \wedge \inf_{\tau_k \leqslant r \leqslant t} \phi(r). \tag{2.17}$$

By (2.11) we obtain

$$\big(\phi(\rho) - \beta(\rho)\big)^+ \geqslant \phi(\rho) - \beta(\rho) \geqslant \phi(\tau_k) - \varepsilon \tag{2.18}$$

and by (2.14) and (2.11) we get

$$\inf_{\rho \leqslant r < \tau_k} \phi(r) \geqslant \phi(\rho) - \beta(\rho) \geqslant \phi(\tau_k) - \varepsilon. \tag{2.19}$$

Thus

$$C_\beta^\phi(t) \geqslant \big(\phi(\tau_k) - \varepsilon\big) \wedge \inf_{\tau_k \leqslant r \leqslant t} \phi(r) \geqslant \inf_{\tau_k \leqslant r \leqslant t} \phi(r) - \varepsilon, \tag{2.20}$$

and so (2.15) holds. ∎

The following result generalizes (3.24) of [4]. It combines the statements of Propositions 2.2 and 2.3 into one representation formula.

THEOREM 2.2. *Let $\phi, \beta \in D_+[0, \infty)$ and $\inf_{t \geqslant 0} \beta(t) > 0$. Then $C_\beta^\phi(t)$ has the following representation*:

$$C_\beta^\phi(t) = \begin{cases} 0 & \textit{if } 0 \leqslant t < \sigma_0, \\ \sup_{\sigma_{k-1} \leqslant s \leqslant t} [\phi(s) - \beta(s)]^+ & \textit{if } \sigma_{k-1} \leqslant t < \tau_k, k \geqslant 1, \\ \inf_{\tau_k \leqslant r \leqslant t} \phi(r) & \textit{if } \tau_k \leqslant t < \sigma_k, k \geqslant 1. \end{cases} \tag{2.21}$$

P r o o f. The formula (2.21) follows directly from the prior propositions. ∎

REMARK 2.1. From (2.21), (2.13) and (2.9) we obtain immediately the following:

(i) $C_\beta^\phi(\tau_k) = \phi(\tau_k)$ for every $k \geqslant 1$,

(ii) $C_\beta^\phi(\sigma_k) = \phi(\sigma_k) - \beta(\sigma_k)$ for every k.

REMARK 2.2. It is clear from Theorem 2.2 that, for every $k \geqslant 1$, C_β^ϕ is nondecreasing on $[\sigma_{k-1}, \tau_k)$ and nonincreasing on $[\tau_k, \sigma_k)$. It can be shown as in [4] that C_β^ϕ has a possible upward jump at σ_{k-1} and a possible downward jump at τ_k.

The next remark corresponds to (3.33) and some following observations in [4].

REMARK 2.3. *If* $\sigma_{k-1} \leqslant t < \tau_k$ *for some* k, *then*

$$\phi(t) > 0, \tag{2.22}$$

$$C_\beta^\phi(t) < \phi(t). \tag{2.23}$$

Proof. Suppose that $\sigma_{k-1} \leqslant t < \tau_k$ for some k. Then, by (2.10) and (2.13),

$$\phi(t) > \sup_{\sigma_{k-1} \leqslant s \leqslant t} [\phi(s) - \beta(s)] \geqslant \phi(\sigma_{k-1}) - \beta(\sigma_{k-1}) \geqslant 0,$$

which proves (2.22). From (2.22) and (2.21) we get

$$C_\beta^\phi(t) = \sup_{\sigma_{k-1} \leqslant s \leqslant t} [\phi(s) - \beta(s)]^+ = \sup_{\sigma_{k-1} \leqslant s \leqslant t} [\phi(s) - \beta(s)] < \phi(t),$$

which proves part (2.23). ■

REMARK 2.4. *If* $\tau_k \leqslant t < \sigma_k$ *for some* $k \geqslant 1$, *then*

$$C_\beta^\phi(t) > \phi(t) - \beta(t). \tag{2.24}$$

Proof. Let $\tau_k \leqslant t < \sigma_k$ for some $k \geqslant 1$; then, by (2.21) and (2.12), $C_\beta^\phi(t) = \inf_{\tau_k \leqslant r \leqslant t} \phi(r) > \phi(t) - \beta(t)$. ■

The following is a generalization of Theorem 3.4 in [4].

THEOREM 2.3. *Let* $\phi, \beta \in D_+[0, \infty)$, $\inf_{t \geqslant 0} \beta(t) > 0$ *and let* $\bar{\phi} = \phi - C_\beta^\phi$. *Then*

(i) $C_\beta^\phi \in BV[0, \infty)$,

(ii) $\bar{\phi} \in D[0, \infty)$ *and* $0 \leqslant \bar{\phi}(t) \leqslant \beta(t)$ *for every* $t \in [0, \infty)$,

(iii) $|C_\beta^\phi(t)| = \int_0^t I_{\{\bar{\phi}(s)=0 \text{ or } \bar{\phi}(s)=\beta(s)\}} d|C_\beta^\phi|(s)$,

(iv) $C_\beta^\phi(t) = -\int_0^t I_{\{\bar{\phi}(s)=0\}} d|C_\beta^\phi|(s) + \int_0^t I_{\{\bar{\phi}(s)=\beta(s)\}} d|C_\beta^\phi|(s)$.

Proof. Part (i) follows directly from (2.21) and part (ii) follows from the definition of C_β^ϕ. To prove (iii) we can easily adopt the arguments from the proof of (3.28) in [4]. It is enough to show that

$$\int_A d|C_\beta^\phi| = 0, \quad \text{where } A = \{t \geqslant \sigma_0 \mid \bar{\phi}(t) \in (0, \beta(t))\}. \tag{2.25}$$

As in [4], for every $t \in A$, we define $a(t) = \inf\{s \in [\sigma_0, t] | (s, t] \subset A\}$ and $b(t) = \sup\{s \geqslant t | [t, s) \subset A\}$. Then $a(t) \leqslant t < b(t)$, where $b(t) \notin A$ while $a(t)$

might or might not be in A. Also the open interval $\big(a(t), b(t)\big)$ is nonempty and is contained in A. It can be shown that A is a countable union of such open non-overlapping intervals and some of their left endpoints. That is, $A = \bigcup_{i\in I}(a_i, b_i) \cup \{a_j | j \in J\}$, where I is a countable set and $J \subset I$. Consider $a_j \in A$, where $j \in J$. Then, by Remark 2.1, a_j must be an interior point of either (σ_{k-1}, τ_k) or (τ_k, σ_k) for some k. By extending the argument from [4] we can show that C^ϕ_β is continuous at a_j. To show (2.25) it is enough now to prove that $\int_{a_i}^{b_i} d|C^\phi_\beta| = 0$ for every $i \in I$. For this it suffices to show that C^ϕ_β is constant on (a_i, b_i) for every i. Since $\bar\phi(t) \in \big(0, \beta(t)\big)$ for every $t \in (a_i, b_i)$, it follows from Remark 2.1 that either $(a_i, b_i) \subset (\sigma_{k-1}, \tau_k)$ or $(a_i, b_i) \subset (\tau_k, \sigma_k)$ for some k. We shall consider only the latter. It is enough to show that C^ϕ_β is constant on every closed interval $[c_i, d_i] \subset (a_i, b_i)$, while $C^\phi_\beta(t) = \inf_{\tau_k \leqslant r \leqslant t} \phi(r)$ for every $t \in (a_i, b_i)$. Let $\rho = \inf\{t \in [c_i, d_i] \mid C^\phi_\beta(t) < C^\phi_\beta(c_i)\}$. Suppose that $\rho < \infty$. Since C^ϕ_β is right-continuous and, by Remark 2.2, nonincreasing on $[c_i, d_i]$, we must have $C^\phi_\beta(t) = C^\phi_\beta(c_i)$ for every $t \in [c_i, \rho)$. At ρ we would have $C^\phi_\beta(\rho) = \phi(\rho)$, which is impossible for $\rho \in A$. Therefore $\rho = \infty$, and so C^ϕ_β is constant on the interval $[c_i, d_i]$.

To prove (iv), first consider t such that $\phi(t) = 0$. By Remark 2.3, $\tau_k \leqslant t < \sigma_k$ for some k. Thus, by Remark 2.2, C^ϕ_β is nonincreasing on $\{t \geqslant 0 \mid \bar\phi(t) = 0\}$. Similarly, C^ϕ_β is nondecreasing on $\{t \geqslant 0 \mid \bar\phi(t) = \beta(t)\}$. Therefore (iv) follows from (iii). ■

We are now ready to complete the proofs of Theorems 2.1 and 1.1.

P r o o f o f T h e o r e m 2.1 f o r $\alpha = 0$. Suppose that $\alpha = 0$. Let $\psi \in D[0, \infty)$, $\phi = \Gamma_0(\psi)$, $\eta = \phi - \psi$ and let $\bar\phi = \Lambda_\beta \circ \Gamma_0(\psi)$, $\bar\eta = \bar\phi - \psi$. Then $\bar\phi = \phi - C^\phi_\beta = \psi + \eta - C^\phi_\beta$ and $\bar\eta = \eta - C^\phi_\beta$. Since $\eta \in I[0, \infty)$, by Theorem 2.3 (i), $\eta - C^\phi_\beta \in BV[0, \infty)$. Also, by Theorem 2.3 (ii), $\bar\phi \in D[0, \infty)$ and $0 \leqslant \bar\phi \leqslant \beta$. It follows from Lemma 2.1 that $\bar\phi(t) = 0$ whenever $\phi(t) = 0$. Consequently, by Definition 1.1, $\int_0^\infty I_{\{\bar\phi(s)>0\}} d\eta(s) = 0$. It is enough, therefore, to define

$$\bar\eta_l(t) = \eta(t) + \int_0^t I_{\{\bar\phi(s)=0\}} d|C^\phi_\beta|(s) \quad \text{and} \quad \bar\eta_u(t) = \int_0^t I_{\{\bar\phi(s)=\beta(s)\}} d|C^\phi_\beta|(s)$$

and (1.3) follows immediately from parts (iii) and (iv) of Theorem 2.3. ■

At this point the existence of a solution to the Skorokhod problem on $[0, \beta]$ is established, and so the proof of Theorem 1.1 is also complete in the case of $\alpha = 0$. The next two lemmas will allow us to reduce the general case to the case of $\alpha = 0$.

LEMMA 2.2. *Let $\alpha, \beta \in D[0,\infty)$ be such that* $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)] > 0$. *Then*

$$\Gamma_{\alpha,\beta}(\psi) = \Gamma_{0,\beta-\alpha}(\psi-\alpha)+\alpha \quad \text{for every } \psi \in D[0,\infty). \tag{2.26}$$

P r o o f. Let $\beta' = \beta - \alpha$ and consider the SP for $\psi' = \psi - \alpha$ on $[0, \beta']$. We already know that it has a unique solution satisfying (2.3). Let $(\bar{\phi}', \bar{\eta}')$ be that solution. Then $\bar{\phi}' = \psi' + \bar{\eta}' \in [0, \beta']$ and $\bar{\eta}' = \bar{\eta}_l' - \bar{\eta}_u'$, where $\bar{\eta}_u', \bar{\eta}_l' \in I[0,\infty)$ and

$$\int_0^\infty I_{\{\bar{\phi}'(s)>0\}} d\bar{\eta}_l'(s) = 0 \quad \text{and} \quad \int_0^\infty I_{\{\bar{\phi}'(s)<\beta'(s)\}} d\bar{\eta}_u'(s) = 0. \tag{2.27}$$

Then $(\bar{\phi}, \bar{\eta})$ defined by $\bar{\phi} = \bar{\phi}' + \alpha$ and $\bar{\eta} = \bar{\eta}'$ is the solution of SP for ψ on $[\alpha, \beta]$. Indeed, $\bar{\phi} = \bar{\phi}' + \alpha = \psi' + \bar{\eta}' + \alpha = \psi + \bar{\eta}' = \psi + \bar{\eta}$. Also $\bar{\eta} = \bar{\eta}' = \bar{\eta}_l' - \bar{\eta}_u'$, where

$$\int_0^\infty I_{\{\bar{\phi}(s)>\alpha(s)\}} d\bar{\eta}_l(s) = \int_0^\infty I_{\{\bar{\phi}'(s)>0\}} d\bar{\eta}_l'(s) = 0,$$

$$\int_0^\infty I_{\{\bar{\phi}(s)<\beta(s)\}} d\bar{\eta}_u(s) = \int_0^\infty I_{\{\bar{\phi}'(s)<\beta'(s)\}} d\bar{\eta}_u'(s) = 0. \quad \blacksquare$$

LEMMA 2.3. *Let $\alpha, \beta \in D[0,\infty)$. Then*

$$\Lambda_{\alpha,\beta}(\phi) = \Lambda_{0,\beta-\alpha}(\phi-\alpha)+\alpha \quad \text{for every } \phi \in D[0,\infty). \tag{2.28}$$

P r o o f. We have

$$\begin{aligned}\Lambda_{\alpha,\beta}(\phi)(t) &= \phi(t)-\alpha(t) - \sup_{0\leqslant s\leqslant t}\Big[\big(\phi(s)-\beta(s)\big)^+ \wedge \inf_{s\leqslant r\leqslant t}\big(\phi(r)-\alpha(r)\big)\Big] + \alpha(t)\\ &= \Lambda_{0,\beta-\alpha}(\phi-\alpha)(t)+\alpha(t). \quad \blacksquare\end{aligned}$$

P r o o f o f T h e o r e m 2.1 f o r g e n e r a l $\alpha \in D[0,\infty)$. By using Lemmas 1.1, 2.2 and 2.3 we get

$$\begin{aligned}\Gamma_{\alpha,\beta}(\psi) &= \Gamma_{0,\beta-\alpha}(\psi-\alpha)+\alpha = \Lambda_{0,\beta-\alpha}\circ\Gamma_0(\psi-\alpha)+\alpha\\ &= \Lambda_{0,\beta-\alpha}\big(\Gamma_\alpha(\psi)-\alpha\big)+\alpha = \Lambda_{\alpha,\beta}\circ\Gamma_\alpha(\psi). \quad \blacksquare\end{aligned}$$

With the completion of the proof of Theorem 2.1 the existence of a solution to the Skorokhod problem on $[\alpha, \beta]$ is established, and so the proof of Theorem 1.1 is complete. Indeed, according to Theorem 2.1, $\bar{\phi} = \Lambda_{\alpha,\beta}\circ\Gamma_\alpha(\psi)$ and $\bar{\eta} = \bar{\phi} - \psi$ define a solution $(\bar{\phi}, \bar{\eta})$ for ψ.

COROLLARY 2.1. *If $\alpha, \beta \in D[0,\infty)$ and* $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)] > 0$, *then $\Gamma_{\alpha,\beta}$ maps $BV[0,\infty)$ into $BV[0,\infty)$.*

P r o o f. Let $\psi, \alpha, \beta \in D[0,\infty)$ and let $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)]>0$. By Theorem 2.1, $\bar\phi = \Gamma_{\alpha,\beta}(\psi)$ is the Skorokhod map on $[\alpha,\beta]$ for ψ. In particular $\bar\eta = \bar\phi - \psi \in BV[0,\infty)$. Thus, $\bar\phi = \psi + \bar\eta \in BV[0,\infty)$ whenever $\psi \in BV[0,\infty)$. ∎

The following example shows that without the assumption of $\inf_{t\geqslant 0}\beta(t) > 0$ formula (2.3) of Theorem 2.1 does not produce the Skorokhod map as defined by Definition 1.2.

EXAMPLE 2.1. Consider a function ψ defined for $0 \leqslant t < 1$ as follows:

$$\psi(t) = \begin{cases} (2n+1)^{-1} & \text{if } t \in [1-2^{-2n}, 1-2^{-(2n+1)}),\ n = 0,1,2,\ldots, \\ -(2n)^{-1} & \text{if } t \in [1-2^{-(2n-1)}, 1-2^{-2n}),\ n = 1,2,\ldots \end{cases}$$

and set $\psi(t) = 0$ for $t \geqslant 1$. Let $\alpha = 0$ and define β for $0 \leqslant t < 1$ as follows: $\beta(t) = |\psi(t)| = n^{-1}$ for $t \in [1-2^{-(n-1)}, 1-2^{-n}),\ n = 1,2,\ldots$, and $\beta(t) = 1$ for $t \geqslant 1$. Note that $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)] = 0$. We will show that if $\bar\phi$ is defined by (2.3), then $\bar\eta = \bar\phi - \psi \notin BV[0,\infty)$. Using (2.7) and (2.8), it is easy to verify that $\sigma_n = 1-2^{-2(n+1)}$ for $n = 0,1,2,\ldots$, and $\tau_n = 1-2^{-(2n+1)}$ for $n = 1,2,\ldots$ Applying the explicit formula (1.3) from [4] to calculate $\phi(t)$ and (2.21) to calculate C^{ϕ}_{β} we can obtain $\bar\phi(t)$ and $\bar\eta(t)$ as follows:

If $t \in [1-2^{-2n}, 1-2^{-(2n+1)})$ for some $n = 1,2,\ldots$, then $\phi(t) = 2^{-1} + (2n+1)^{-1}$, $C^{\phi}_{\beta}(t) = 2^{-1}$, and so $\bar\phi(t) = (2n+1)^{-1}$ and $\bar\eta(t) = 0$.

If $t \in [1-2^{-(2n+1)}, 1-2^{-2(n+1)})$ for some $n = 1,2,\ldots$, then $\phi(t) = 2^{-1} - (2n+2)^{-1}$, $C^{\phi}_{\beta}(t) = 2^{-1} - (2n+2)^{-1}$, and so $\bar\phi(t) = 0$ and $\bar\eta(t) = (2n+2)^{-1}$. Since $\sum_{n=2}^{\infty} |\bar\eta(1-2^{n}) - \bar\eta(1-2^{-(n+1)})| = 2\sum_{n=2}^{\infty}(2n)^{-1} = \infty$, $\bar\eta$ is of unbounded variation.

We end this section with the following reflection property.

REMARK 2.5. *If* $\alpha, \beta \in D[0,\infty)$ *and* $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)] > 0$, *then for every* $\psi \in D[0,\infty)$

$$\Gamma_{\alpha,\beta}(-\psi) = -\Gamma_{-\beta,-\alpha}(\psi). \tag{2.29}$$

P r o o f. Let $(\bar\phi, \bar\eta)$ be the solution of the Skorokhod problem on $[\alpha,\beta]$ for $-\psi$ with $\bar\eta = \bar\eta_l - \bar\eta_u$ satisfying (1.3). Then $-\bar\phi = \psi - \bar\eta$, $-\beta(t) \leqslant -\bar\phi(t) \leqslant -\bar\alpha(t)$ for every $t \geqslant 0$, and

$$\int_0^\infty I_{\{-\bar\phi(s) > -\beta(s)\}} d\bar\eta_u(s) = 0 \quad \text{and} \quad \int_0^\infty I_{\{-\bar\phi(s) < -\alpha(s)\}} d\bar\eta_l(s) = 0.$$

Thus $(-\bar\phi, -\bar\eta)$ is the solution of the Skorokhod problem on $[-\beta, -\alpha]$ for ψ. ∎

3. NON-ANTICIPATORY PROPERTIES

We shall use the shift operators $T_d, H_d : D[0,\infty) \to D[0,\infty)$ indexed by $d \in [0,\infty)$ and defined by $T_d(f)(t) = f(d+t) - f(d)$ and $H_d(f)(t) = f(d+t)$ for $t \in [0,\infty)$. Let $\psi^d = T_d(\psi)$, $\bar\eta^d = T_d(\bar\eta)$, $\bar\eta_l^d = T_d(\bar\eta_l)$, $\bar\eta_u^d = T_d(\bar\eta_u)$, $\bar\phi^d = H_d(\bar\phi)$, $\alpha^d = H_d(\alpha)$ and $\beta^d = H_d(\beta)$.

THEOREM 3.1 (non-anticipatory property). *If $(\bar\phi,\bar\eta)$ solves the Skorokhod problem for ψ on $[\alpha,\beta]$, then $(\bar\phi^d,\bar\eta^d)$ solves the Skorokhod problem for $\bar\phi(d)+\psi^d$ on $[\alpha^d,\beta^d]$. In particular,*

$$\Gamma_{\alpha,\beta}(\psi)(d+s) = \Gamma_{\alpha^d,\beta^d}\big(\bar\phi(d) + T_d(\psi)\big)(s). \tag{3.1}$$

P r o o f. Let $(\bar\phi,\bar\eta)$ be a solution to the Skorokhod problem for ψ on $[\alpha,\beta]$. Then

$$\begin{aligned}\bar\phi(d) + \psi^d(t) &= \psi(d+t) + \bar\phi(d) - \psi(d) = \bar\phi(d+t) - \bar\eta(d+t) + \bar\eta(d)\\ &= \bar\phi^d(t) - \bar\eta^d(t).\end{aligned}$$

Also, $\bar\phi^d(t) = \bar\phi(d+t) \in [\alpha(d+t),\beta(d+t)] = [\alpha^d(t),\beta^d(t)]$, which establishes part (i) of Definition 1.2. To establish part (ii) note that

$$\int_0^\infty I_{\{\bar\phi^d(s)>\alpha^d\}}\, d\bar\eta_l^d(s) = \int_d^\infty I_{\{\bar\phi(t)>\alpha\}}\, d\bar\eta_l(t) = 0$$

and

$$\int_0^\infty I_{\{\bar\phi^d(s)<\beta^d(s)\}}\, d\bar\eta_u^d(s) = \int_d^\infty I_{\{\bar\phi(t)<\beta(t)\}}\, d\bar\eta_u(t) = 0. \quad \blacksquare$$

As a consequence of the anti-participatory property of the Skorokhod map we obtain the following property of the constraining term.

COROLLARY 3.1. *Let $\phi,\alpha,\beta \in D[0,\infty)$ be such that $\inf_{t\geqslant 0}\big(\beta(t)-\alpha(t)\big)>0$ and $\phi \geqslant \alpha$. Then for any $d,h \geqslant 0$*

$$\begin{aligned}C^\phi_{\alpha,\beta}(d+h) = C^\phi_{\alpha,\beta}(d) + \sup_{0\leqslant s\leqslant h} \Big[\big(\phi(d+s) - C^\phi_{\alpha,\beta}(d) - \beta(d+s)\big)^+ \\ \wedge \inf_{s\leqslant r\leqslant h}\big(\phi(d+r) - C^\phi_{\alpha,\beta}(d) - \alpha(d+r)\big)\Big].\end{aligned}$$

P r o o f. Let $(\bar\phi,\bar\eta)$ be the solution of the Skorokhod problem for ϕ on $[\alpha,\beta]$. Then, by Theorem 3.1, $(\bar\phi^d,\bar\eta^d)$ is the solution of the Skorokhod problem for

$\bar{\phi}(d) + \phi^d$ on $[\alpha^d, \beta^d]$. Therefore

$$\begin{aligned}
&C^{\phi}_{\alpha,\beta}(d+h) - C^{\phi}_{\alpha,\beta}(d) = \bar{\eta}(d+h) - \bar{\eta}(d) = C^{\bar{\phi}(d)+\phi^d}_{\alpha^d,\beta^d}(h) \\
&= \sup_{0\leqslant s\leqslant h} \Big[\Big(\big(\bar{\phi}(d) + \phi(d+s) - \phi(d)\big) - \beta(d+s)\Big)^+ \\
&\qquad\qquad \wedge \inf_{s\leqslant r\leqslant h} \big(\bar{\phi}(d) + \phi(d+r) - \phi(d) - \alpha(d+r)\big)\Big] \\
&= \sup_{0\leqslant s\leqslant h} \Big[\Big(\big(\phi(d+s) - \bar{\eta}(d)\big) - \beta(d+s)\Big)^+ \\
&\qquad\qquad \wedge \inf_{s\leqslant r\leqslant h} \big(\phi(d+r) - \bar{\eta}(d) - \alpha(d+r)\big)\Big] \\
&= \sup_{0\leqslant s\leqslant h} \big[\big(\phi(d+s) - C^{\phi}_{\alpha,\beta}(d) - \beta(d+s)\big)^+ \\
&\qquad\qquad \wedge \inf_{s\leqslant r\leqslant h} \big(\phi(d+r) - C^{\phi}_{\alpha,\beta}(d) - \alpha(d+r)\big)\big]. \quad \blacksquare
\end{aligned}$$

4. THE CONTINUITY PROPERTIES OF THE SKOROKHOD MAP AND THE METRICS ON $D[0,T]$

In this section we need to consider functions on $D[0,T]$. The notation of $\Lambda_{\alpha,\beta}$, Γ_α, $\Gamma_{\alpha,\beta}$, $R^{\alpha,\beta}_t$, $C^{\phi}_{\alpha,\beta}$, that we used before in relation to $D[0,\infty]$, will be applied now for $D[0,T]$. We discuss continuity and Lipschitz conditions for $\Lambda_{\alpha,\beta}$ and $\Gamma_{\alpha,\beta}$ under three metrics: the uniform metric, J_1 metric d_0 and M_1 metric d_1. We begin with the uniform metric.

PROPOSITION 4.1. *For any $\psi_1, \psi_2, \phi_1, \phi_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in D[0,T]$ such that $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ we have*

$$\|\Gamma_{\alpha_1}(\psi_1) - \Gamma_{\alpha_2}(\psi_2)\|_T \leqslant 2\|\psi_1 - \psi_2\|_T + \|\alpha_1 - \alpha_2\|_T, \tag{4.1}$$

$$\|C^{\phi_1}_{\alpha_1,\beta_1} - C^{\phi_2}_{\alpha_2,\beta_2}\|_T \leqslant \|\phi_1 - \phi_2\|_T + [\|\alpha_1 - \alpha_2\|_T \vee \|\beta_1 - \beta_2\|_T], \tag{4.2}$$

$$\begin{aligned}
&\|\Lambda_{\alpha_1,\beta_1}(\phi_1) - \Lambda_{\alpha_2,\beta_2}(\phi_2)\|_T \\
&\qquad \leqslant 2\|\phi_1 - \phi_2\|_T + [\|\alpha_1 - \alpha_2\|_T \vee \|\beta_1 - \beta_2\|_T].
\end{aligned} \tag{4.3}$$

If also $\inf_{t\geqslant 0}[\beta_1(t) - \alpha_1(t)] > 0$, *and* $\inf_{t\geqslant 0}[\beta_2(t) - \alpha_2(t)] > 0$, *then*

$$\begin{aligned}
&\|\Gamma_{\alpha_1,\beta_1}(\psi_1) - \Gamma_{\alpha_1,\beta_2}(\psi_2)\|_T \\
&\qquad \leqslant 4\|\psi_1 - \psi_2\|_T + 2\|\alpha_1 - \alpha_2\|_T + [\|\alpha_1 - \alpha_2\|_T \vee \|\beta_1 - \beta_2\|_T].
\end{aligned} \tag{4.4}$$

P r o o f. We shall use the following easily verifiable inequalities:

$$|x_2 \wedge y_2 - x_1 \wedge y_1| \leqslant |x_2 - x_1| \vee |y_2 - y_1|, \tag{4.5}$$

$$|(y_1 - x_1)^+ - (y_2 - x_2)^+| \leqslant |x_1 - x_2| + |y_1 - y_2|. \tag{4.6}$$

Let $\psi_1, \psi_2, \phi_1, \phi_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in D_+[0,T]$, $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ and let $t \in [0,T]$. Using (1.2) and (4.6) we get

$$\begin{aligned}
&|\Gamma_{\alpha_1}(\psi_1)(t) - \Gamma_{\alpha_2}(\psi_2)(t)| \\
&\leqslant \sup_{0\leqslant t\leqslant T} \left[\psi_1(t) - \psi_2(t) - \sup_{0\leqslant s\leqslant t} \big(\alpha_1(s) - \psi_1(s)\big)^+ + \sup_{0\leqslant s\leqslant t} \big(\alpha_2(s) - \psi_2(s)\big)^+\right] \\
&\leqslant \sup_{0\leqslant t\leqslant T} [\psi_1(t) - \psi_2(t)] + \sup_{0\leqslant t\leqslant T} \left|\big(\alpha_1(s) - \psi_1(s)\big)^+ - \big(\alpha_2(s) - \psi_2(s)\big)^+\right| \\
&\leqslant \sup_{0\leqslant t\leqslant T} [\psi_1(t) - \psi_2(t)] + \sup_{0\leqslant t\leqslant T} [|\alpha_1(s) - \alpha_2(s)| + |\psi_1(s) - \psi_2(s)|] \\
&\leqslant 2\|\psi_1 - \psi_2\|_T + \|\alpha_1 - \alpha_2\|_T.
\end{aligned}$$

Taking $\sup_{0\leqslant t\leqslant T}$ we obtain (4.1).

By (4.5) and (4.6),

$$\begin{aligned}
C^{\phi_1}_{\alpha,\beta_1}(t) - C^{\phi_2}_{\alpha,\beta_2}(t) &\leqslant \sup_{0\leqslant s\leqslant t} [R^{\alpha_1,\beta_1}_t(\phi_1)(s) - R^{\alpha_2,\beta_2}_t(\phi_2)(s)] \\
\leqslant \sup_{0\leqslant s\leqslant t} &\Big[\left|\big(\phi_1(s) - \beta_1(s)\big)^+ - \big(\phi_2(s) - \beta_2(s)\big)^+\right| \\
&\qquad \vee \Big|\inf_{s\leqslant r\leqslant t} \big(\phi_1(r) - \alpha_1(r)\big) - \inf_{s\leqslant r\leqslant t} \big(\phi_2(r) - \alpha_2(r)\big)\Big|\Big] \\
\leqslant \sup_{0\leqslant s\leqslant t} &\{[|\phi_1(s) - \phi_2(s)| + |\beta_1(s) - \beta_2(s)|] \\
&\qquad \vee \sup_{s\leqslant r\leqslant t} [|\phi_1(r) - \phi_2(r)| + |\alpha_1(r) - \alpha_2(r)|]\} \\
\leqslant (\|\phi_1 &- \phi_2\|_T + \|\beta_1 - \beta_2\|_T) \vee (\|\phi_1 - \phi_2\|_T + \|\alpha_1 - \alpha_2\|_T) \\
\leqslant \|\phi_1 &- \phi_2\|_T + (\|\alpha_1 - \alpha_2\|_T \vee \|\beta_1 - \beta_2\|_T).
\end{aligned}$$

Now, taking $\sup_{0\leqslant t\leqslant T}$ we conclude (4.2).

Since

$$\|\Lambda_{\alpha_1,\beta_1}(\phi_1) - \Lambda_{\alpha_2,\beta_2}(\phi_2)\|_T \leqslant \|\phi_1 - \phi_2\|_T + \|C^{\phi_1}_{\alpha_1,\beta_1} - C^{\phi_2}_{\alpha_1,\beta_2}\|_T,$$

(4.3) follows immediately from (4.2). Finally, using (4.1) combined with (4.3) and (2.3) we obtain (4.4). ∎

COROLLARY 4.1. *Both mappings $\Lambda_{\alpha,\beta}$ and $\Gamma_{\alpha,\beta}$ are Lipschitz continuous in the uniform metric with corresponding constants* 2 *and* 4, *respectively.*

The following scaling properties will be useful in dealings with Skorokhod metric d_0. Let $\mathcal{M} = \{\lambda : [0,T] \to [0,T] \mid \lambda \text{ is increasing, continuous and onto}\}$. Then for every $\lambda \in \mathcal{M}$ the following properties hold:

$$\Gamma_\alpha(\psi) \circ \lambda = \Gamma_{\alpha\circ\lambda}(\psi \circ \lambda), \tag{4.7}$$

$$\Lambda_{\alpha\circ\lambda,\beta\circ\lambda}(\phi \circ \lambda) = \Lambda_{\alpha,\beta}(\phi) \circ \lambda. \tag{4.8}$$

Indeed, (4.7) follows directly from the definition of Γ_α. To prove (4.8) note first that

$$\begin{aligned} &C^{\phi}_{\alpha,\beta}\big(\lambda(t)\big) \\ &\quad = \sup_{0\leqslant s\leqslant \lambda(t)} R^{\alpha,\beta}_{\lambda(t)}(\phi)(s) = \sup_{0\leqslant s\leqslant t} R^{\alpha\circ\lambda,\beta\circ\lambda}_{t}(\phi\circ\lambda)(s) = C^{\phi\circ\lambda}_{\alpha\circ\lambda,\beta\circ\lambda}(t). \end{aligned} \tag{4.9}$$

Therefore, $\Lambda_{\alpha,\beta}(\phi)\big(\lambda(t)\big) = \phi\big(\lambda(t)\big) - C^{\phi}_{\alpha,\beta}\big(\lambda(t)\big) = \phi\circ\lambda(t) - C^{\phi\circ\lambda}_{\alpha\circ\lambda,\beta\circ\lambda}(t) = \Lambda_{\alpha\circ\lambda,\beta\circ\lambda}(\phi\circ\lambda)(t)$.

Note that (4.8) holds also on $[0,\infty)$ with λ being an increasing continuous bijection on $[0,\infty)$. The Skorokhod metric d_0 on $D[0,\infty)$ ($D[0,T]$, respectively) is defined by

$$d_0(f,g) = \inf_{\lambda}\,(\|\lambda - I\| \vee \|f - g\circ\lambda\|),$$

where the infimum is over all strictly increasing continuous bijections of $[0,\infty)$ ($D[0,T]$, respectively).

PROPOSITION 4.2. *Let $T>0$ and let $\alpha,\beta \in D[0,T]$ be such that $\alpha(t)\leqslant\beta(t)$ for every $t\in[0,T]$. Then*:

(i) *For any $\psi_1,\psi_2 \in D[0,T]$*

$$d_0\big(\Gamma_\alpha(\psi_1),\Gamma_\alpha(\psi_2)\big) \leqslant 2d_0(\psi_1,\psi_2) + \sup_{r,s\in[0,T]}|\alpha(r)-\alpha(s)|. \tag{4.10}$$

(ii) *For any $\phi_1,\phi_2 \in D[0,T]$*

$$\begin{aligned} &d_0\big(\Lambda_{\alpha,\beta}(\phi_1),\Lambda_{\alpha,\beta}(\phi_2)\big) \\ &\quad \leqslant 2d_0(\phi_1,\phi_2) + \sup_{r,s\in[0,T]}|\alpha(r)-\alpha(s)| \vee \sup_{r,s\in[0,T]}|\beta(r)-\beta(s)|. \end{aligned} \tag{4.11}$$

(iii) *If $\inf_{t\geqslant 0}[\beta(t)-\alpha(t)]>0$, then for any $\psi_1,\psi_2\in D[0,T]$*

$$\begin{aligned} d_0\big(\Gamma_{\alpha,\beta}(\psi_1),\Gamma_{\alpha,\beta}(\psi_2)\big) &\leqslant 4d_0(\psi_1,\psi_2) + 2\sup_{r,s\in[0,T]}|\alpha(r)-\alpha(s)| \\ &\quad + \sup_{r,s\in[0,T]}|\alpha(r)-\alpha(s)| \vee \sup_{r,s\in[0,T]}|\beta(r)-\beta(s)|. \end{aligned} \tag{4.12}$$

P r o o f. We begin with (i). Let $\lambda \in \mathcal{M}$. By (4.7) and (4.1) we have

$$\begin{aligned}\|\lambda - I\|_T \vee \|\Gamma_\alpha(\psi_1) - \Gamma_\alpha(\psi_2)\circ\lambda\|_T &= \|\lambda - I\|_T \vee \|\Gamma_\alpha(\psi_1) - \Gamma_{\alpha\circ\lambda}(\psi_2\circ\lambda)\|_T \\ &\leqslant \|\lambda - I\|_T \vee (2\|\psi_1 - \psi_2\circ\lambda\|_T + \|\alpha - \alpha\circ\lambda\|_T) \\ &\leqslant \|\lambda - I\|_T \vee 2\|\psi_1 - \psi_2\circ\lambda\|_T + \sup_{r,s\in[0,T]}|\alpha(r) - \alpha(s)|.\end{aligned}$$

Taking $\inf_\lambda$ we conclude (4.10).

To prove (ii) we can assume without loss of generality that $\phi_1 \neq \phi_2$. Given $\phi_1, \phi_2 \in D[0,T]$, $\phi_1 \neq \phi_2$, for every $\delta > 0$ there is $\lambda \in \mathcal{M}$ such that

$$\sup_{0\leqslant t\leqslant T}|\lambda(t) - t| \leqslant d_0(\phi_1,\phi_2) + \delta[1 \wedge d_0(\phi_1,\phi_2)]$$

and

$$\sup_{0\leqslant t\leqslant T}\big|\phi_1(t) - \phi_2\big(\lambda(t)\big)\big| \leqslant d_0(\phi_1,\phi_2) + \delta[1 \wedge d_0(\phi_1,\phi_2)].$$

By (4.3),

$$\begin{aligned}&\|\Lambda_{\alpha,\beta}(\phi_1) - \Lambda_{\alpha,\beta}(\phi_2)\circ\lambda\|_T = \|\Lambda_{\alpha,\beta}(\phi_1) - \Lambda_{\alpha\circ\lambda,\beta\circ\lambda}(\phi_2\circ\lambda)\|_T \\ &\leqslant 2\|\phi_1 - \phi_2\circ\lambda\|_T + \|\alpha - \alpha\circ\lambda\|_T \vee \|\beta - \beta\circ\lambda\|_T \\ &\leqslant 2\|\phi_1 - \phi_2\circ\lambda\|_T + \sup_{r,s\in[0,T]}|\alpha(r) - \alpha(s)| \vee \sup_{r,s\in[0,T]}|\beta(r) - \beta(s)| \\ &\leqslant 2\big[d_0(\phi_1,\phi_2) + \delta\big(1 \wedge d_0(\phi_1,\phi_2)\big)\big] \\ &\quad + \sup_{r,s\in[0,T]}|\alpha(r) - \alpha(s)| \vee \sup_{r,s\in[0,T]}|\beta(r) - \beta(s)|.\end{aligned} \tag{4.13}$$

Since (4.13) holds for every $\delta > 0$, we can conclude (4.11).

Part (iii) follows from (2.3), (4.11) and (4.10). ∎

The following example shows that the terms representing the oscillations of α and β, $\sup_{r,s\in[0,T]}|\alpha(r) - \alpha(s)|$ and $\sup_{r,s\in[0,T]}|\beta(r) - \beta(s)|$, appearing in (4.11) and (4.12), cannot be omitted. In particular, the example shows that, in general, neither Λ_β nor $\Gamma_{0,\beta}$ are continuous in d_0 metric.

EXAMPLE 4.1. Define

$$\beta = aI_{[0,b)} + 2aI_{[b,N]}, \qquad \phi_1 = 2aI_{[b-\varepsilon,N]}, \qquad \phi_2 = 2aI_{[b,N]}.$$

Then it is easy to see that $\bar\phi_1 = aI_{[b-\varepsilon,N]}$ and $\bar\phi_2 = \phi_2 = 2aI_{[b,N]}$. Hence $d_0(\bar\phi_1,\bar\phi_2) = a$, while $d_0(\phi_1,\phi_2) = \varepsilon$. Since ε can be chosen arbitrarily small, Λ_β is not continuous as a map $\Lambda_\beta\colon \big(D_+[0,\infty), d_0\big) \to \big(D_+[0,\infty), d_0\big)$ and neither is $\Gamma_{0,\beta}$. Note that in this example the oscillation of β is the dominating term of the distance. By applying (2.29) we could easily create an example with the oscillation of α as the dominating term.

We consider next the continuity of the Skorokhod map under M_1 metric d_1. The time dependent nature of the constraints α and β makes this issue somewhat technically involved. We shall need a few technical results before addressing it. The next result examines the monotonicity of the Skorokhod map.

PROPOSITION 4.3. *Let $\phi, \beta \in D_+[0,T]$ and $\phi(0) < \beta(0)$. Then*:

(i) *If ϕ is nondecreasing and there is $S \in (0,T]$ and $a > 0$ such that $\phi \leqslant \beta$ on $[0,S]$ and $\beta(s) = a \leqslant \phi(s)$ for $s \in [S,T]$, then $\Lambda_\beta(\phi)$ is nondecreasing.*

(ii) *If ϕ is nondecreasing, β nonincreasing and there is $S \in (0,T]$ and $a > 0$ such that $\phi(s) = a \leqslant \beta(s)$ for $s \in [0,S]$ and $\beta(s) \leqslant a \leqslant \phi(s)$ for $s \in [S,T]$, then $\Lambda_\beta(\phi)$ is nonincreasing.*

(iii) *If both ϕ and β are nonincreasing and there is $S \in (0,T]$ and $a > 0$ such that $a \leqslant \phi(s) \leqslant \beta(s)$ for $s \in [0,S]$ and $\beta(s) \leqslant \phi(s) = a$ for $s \in [S,T]$, then $\Lambda_\beta(\phi)$ is nonincreasing.*

P r o o f. Suppose that the assumptions in (i) hold. Then $\inf_{s \leqslant r \leqslant t} \phi(r) = \phi(s) \geqslant \big(\phi(s) - \beta(s)\big)^+$, and so $C^\phi_\beta(t) = \sup_{0 \leqslant s \leqslant t}[\phi(s) - \beta(s)]^+ = \big(\phi(t) - a\big)^+$ for every $t \in [0,T]$. Therefore $\Lambda_\beta(\phi) = \phi - C^\phi_\beta = \phi - (\phi - a)^+ = \phi \wedge a$, and so it is nondecreasing.

When ϕ is nondecreasing, and β is nonincreasing, then $\inf_{s \leqslant r \leqslant t} \phi(r) = \phi(s) \geqslant \big(\phi(s) - \beta(s)\big)^+$ and $\phi - \beta$ is nondecreasing. Hence, under the assumptions of (ii), $C^\phi_\beta(t) = \sup_{0 \leqslant s \leqslant t}[\phi(s) - \beta(s)]^+ = \big(\phi(t) - \beta(t)\big)^+$. Therefore $\Lambda_\beta(\phi) = \phi - C^\phi_\beta = \phi - (\phi - \beta)^+ = \phi \wedge \beta = a \wedge \beta$, and so it is nonincreasing.

Finally, if the assumptions of (iii) hold, then

$$C^\phi_\beta(t) = \sup_{0 \leqslant s \leqslant t} \big[\big(\phi(s) - \beta(s)\big)^+ \wedge \phi(t)\big],$$

and so $C^\phi_\beta(t) = 0$ for $t \in [0,S]$ and $C^\phi_\beta(t) = \phi(t) - \beta(t)$ for $t \in [S,T]$. Therefore $\Lambda_\beta(\phi) = \phi - C^\phi_\beta = \phi - (\phi - \beta)^+ = \phi \wedge \beta$ and it is nonincreasing. ■

We shall use the following conventions: for $\phi \in D[0,T]$, $\phi(0-)$ will stand for $\phi(0)$ and $G_\phi = \{(t,y) \in [0,T] \times \mathbb{R} \mid y \in [\phi(t-) \wedge \phi(t), \phi(t-) \vee \phi(t)]\}$ will represent the graph of ϕ ordered by the standard relation $\leqslant$ defined as follows:

$$(4.14) \quad (t_1, y_1) \leqslant (t_2, y_2) \quad \text{if either } t_1 < t_2 \text{ or } t_1 = t_2 \text{ and } |\phi(t_1-) - y_1| \leqslant |\phi(t_2-) - y_2|.$$

For each $t \in [0,T]$, $G^t_\phi = \{(t,y) \mid y \in [\phi(t-) \wedge \phi(t), \phi(t-) \vee \phi(t)]\}$ represents a vertical segment of the graph if ϕ is discontinuous at t or a point $\big(t, \phi(t)\big)$ otherwise. Then, $G_\phi = \bigcup_{0 \leqslant t \leqslant T} G^t_\phi$. A continuous nondecreasing mapping (r,g) from $[0,1]$ onto G_ϕ will be called a *parametric representation* of G_ϕ, and $\Pi(\phi)$

will denote the set of all *parametric representations* of G_ϕ. The M_1 metric d_1 is defined for $\phi_1, \phi_2 \in D[0,\infty)$ by

$$d_1(\phi_1,\phi_2) = \inf\{\|r_1 - r_2\|_T \vee \|g_1 - g_2\|_T \mid (r_i, g_i) \in \Pi(\phi_i),\ i = 1,2\}.$$

Before we examine Lipschitz conditions under this metric we need to understand how graph parameterizations are being transformed by Λ_β. For $\phi \in D[0,T]$, D_ϕ will denote the set of all points of discontinuity of ϕ in $[0,T]$.

LEMMA 4.1. *Let* $\phi_1, \phi_2, \beta \in D[0,T]$, *let* $(r_1, g_1) \in \Pi_{\phi_1}$ *and let* $(r_2, g_2) \in \Pi_{\phi_2}$. *Then there is a continuous, nondecreasing mapping* ξ *from* $[0,1]$ *onto* $[0,1]$ *such that for every* $i = 1,2$ *and* $t \in D_\beta$ *there are* $0 \leqslant u_t^i < v_t^i \leqslant 1$ *such that the following conditions hold*:

$$v_s^i < u_t^i \text{ whenever } s < t, \tag{4.15}$$

$$(r_i \circ \xi, g_i \circ \xi)^{-1}(G_{\phi_i}^t) = \big[u_t^i, v_t^i\big]. \tag{4.16}$$

Proof. Since D_β is a countable subset of $[0,T]$, for each $t \in D_\beta$ we can choose $[u_t, v_t] \subset [0,1]$, so that $u_t < v_t$ for every $t \in [0,T]$ and $v_s < u_t$ for $s < t$. (Remark 12.3.3 in [9] describes such a construction.) For each $t \in D_\beta$ let $[a_t^i, b_t^i] = (r_i, g_i)^{-1}(G_{\phi_i}^t)$. We define $\xi_1 : [0,1] \to [0,1]$ by setting $\xi_1(u_t) = a_t^1$, $\xi_1(v_t) = b_t^1$ and extending it by linear interpolation onto $[u_t, v_t]$ for every $t \in D_\beta$. Then we extend it to a continuous function on the closure of $\bigcup_{t \in D_\beta}[u_t, v_t]$. Finally, we set $\xi_1(0) = 0$, $\xi_1(1) = 1$ and extend it onto the rest of $[0,1]$ by linear interpolation. Similarly, we let $[c_t^2, d_t^2] = \xi_1^{-1}([a_t^2, b_t^2])$ and define $\xi_2 : [0,1] \to [0,1]$ by setting $\xi_2(0) = 0$, $\xi_2(1) = 1$, $\xi_2(u_t) = c_t^2$, $\xi_2(v_t) = d_t^2$ for every $t \in D_\beta$ and extending it onto $[0,1]$ by continuity and linear interpolation. Finally, we define $\xi = \xi_1 \circ \xi_2$. Then

$$(r_1 \circ \xi, g_1 \circ \xi)^{-1}(G_{\phi_1}^t) = \xi^{-1}([a_t^1, b_t^1]) = \xi_2^{-1}\Big(\xi_1^{-1}\big(([a_t^1, b_t^1])\big)\Big) = \xi_2^{-1}([u_t, v_t])$$

and

$$(r_2 \circ \xi, g_2 \circ \xi)^{-1}(G_{\phi_2}^t) = \xi_2^{-1}\big(\xi_1^{-1}([a_t^2, b_t^2])\big) = \xi_2^{-1}([c_t^2, d_t^2]) = [u_t, v_t].$$

Therefore it is enough to take $[u_t^1, v_t^1] = \xi_2^{-1}([u_t, v_t])$ and $[u_t^2, v_t^2] = [u_t, v_t]$ and (4.16) is satisfied. Because ξ is nondecreasing, (4.15) follows. ∎

LEMMA 4.2. *Let* $\phi_1, \phi_2, \beta \in D[0,T]$, *let* $(r_1, g_1) \in \Pi_{\phi_1}$ *and let* $(r_2, g_2) \in \Pi_{\phi_2}$. *Then there is a continuous, nondecreasing mapping* ξ *from* $[0,1]$ *onto* $[0,1]$ *such that for every* $i = 1,2$ *and* $t \in D_\beta$ *there are* $0 \leqslant a_t^i < b_t^i < c_t^i < d_t^i \leqslant 1$ *such that the following conditions hold*:

$$d_s^i < a_t^i \text{ whenever } s < t, \tag{4.17}$$

$$(4.18)\qquad (r_i\circ\xi, g_i\circ\xi)^{-1}(G^t_{\phi_i}) = \big[a^i_t, d^i_t\big],$$

$$(4.19)\qquad (g_i\circ\xi)^{-1}\big(\phi_i(t-)\big) = \big[a^i_t, b^i_t\big],$$

$$(4.20)\qquad (g_i\circ\xi)^{-1}\big(\phi_i(t)\big) = \big[c^i_t, d^i_t\big].$$

P r o o f. Let $\big[u^i_t, v^i_t\big] = (r_i, g_i)^{-1}(G^t_{\phi_i})$. We assume first that

$$(4.21)\qquad u^i_t < v^i_t \quad \text{for every } t\in D_\beta \text{ and each } i=1,2.$$

We define $\xi_1 : [0,1]\to[0,1]$ by setting $\xi_1(0)=0$, $\xi_1(1)=1$, and for each $t\in D_\beta$

$$\xi_1(u^1_t) = \xi_1\big((2u^1_t+v^1_t)/3\big) = u^1_t \quad\text{and}\quad \xi_1\big((u^1_t+2v^1_t)/3\big) = \xi_1(v^1_t) = v^1_t.$$

Finally, we extend it onto $[0,1]$ by linear interpolation.

For each $t\in D_\beta$, let $[a^2_t, d^2_t] = \xi_1^{-1}([u^2_t, v^2_t])$. We define $\xi_2 : [0,1]\to[0,1]$ similarly to ξ_1. We set $\xi_2(0)=0$, $\xi_2(1)=1$, and for each $t\in D_\beta$ we set

$$\xi_2(a^2_t) = \xi_2\big((2a^2_t+d^2_t)/3\big) = a^2_t \quad\text{and}\quad \xi_2\big((a^2_t+2d^2_t)/3\big) = \xi_2(d^2_t) = d^2_t,$$

and we extend it onto $[0,1]$ by linear interpolation. Then we define $\xi = \xi_1\circ\xi_2$. Since both ξ_1 and ξ_2 are continuous nondecreasing mappings from $[0,1]$ onto $[0,1]$, so is ξ. We define a^1_t, d^1_t so that $[a^1_t, d^1_t] = \xi_2^{-1}([u^1_t, v^1_t])$.

Consider $s<t$. Since for each $i=1,2$, r_i is nondecreasing, we have $v^i_s < u^i_t$. Because ξ_1 and ξ_2 are nondecreasing, we also have $d^i_s \leqslant a^i_t$. Since $[u^1_s, v^1_s]$ and $[u^1_t, v^1_t]$ are disjoint, so must be $\xi_2^{-1}([u^1_s, v^1_s])$ and $\xi_2^{-1}([u^1_t, v^1_t])$. Thus we have established (4.17). The remaining properties will be shown separately for $i=1,2$. If $i=1$, then

$$(r_1\circ\xi, g_1\circ\xi)^{-1}(G^t_{\phi_1}) = (\xi_1\circ\xi_2)^{-1}([u^1_t, v^1_t]) = \xi_2^{-1}([u^1_t, v^1_t]) = [a^1_t, d^1_t].$$

Similarly, if $i=2$, then

$$(r_2\circ\xi, g_2\circ\xi)^{-1}(G^t_{\phi_2}) = (\xi_1\circ\xi_2)^{-1}([u^2_t, v^2_t]) = \xi_2^{-1}([a^2_t, d^2_t]) = [a^2_t, d^2_t].$$

We define b^i_t, c^i_t by $(g_i\circ\xi)^{-1}\big(\phi_i(t-)\big) = \big[a^i_t, b^i_t\big]$, and $(g_i\circ\xi)^{-1}\big(\phi_i(t)\big) = \big[c^i_t, d^i_t\big]$ so that (4.19) and (4.20) hold. Then

$$\begin{aligned}[a^1_t, b^1_t] &= (r_1\circ\xi, g_1\circ\xi)^{-1}\Big(\big\{\big(t,\phi_1(t-)\big)\big\}\Big)\\ &\qquad\supset (\xi_1\circ\xi_2)^{-1}(\{u^1_t\}) = \xi_2^{-1}\left([u^1_t, (2u^1_t+v^1_t)/3]\right),\end{aligned}$$

and so $a_t^1 < b_t^1$. Similarly,

$$[c_t^1, d_t^1] = (r_1 \circ \xi, g_1 \circ \xi)^{-1}\Big(\big\{\big(t, \phi_1(t)\big)\big\}\Big)$$
$$\supset (\xi_1 \circ \xi_2)^{-1}(\{v_t^1\}) = \xi_2^{-1}\big([(u_t^1 + 2v_t^1)/3, v_t^1]\big),$$

so that $c_t^1 < d_t^1$. Finally, because the mappings $(r_1 \circ \xi, g_1 \circ \xi)^{-1}\big(\big\{\big(t, \phi_1(t-)\big)\big\}\big)$ and $(r_1 \circ \xi, g_1 \circ \xi)^{-1}\big(\big\{\big(t, \phi_1(t)\big)\big\}\big)$ are disjoint, we get $b_t^1 < c_t^1$.

Analogously, $a_t^2 < b_t^2$, because

$$[a_t^2, b_t^2] = (r_2 \circ \xi, g_2 \circ \xi)^{-1}\Big(\big\{\big(t, \phi_2(t-)\big)\big\}\Big) \supset (\xi_1 \circ \xi_2)^{-1}(\{u_t^2\}) \supset \xi_2^{-1}(\{a_t^2\})$$
$$= [a_t^2, (2a_t^2 + d_t^2)/3],$$

and $c_t^2 < d_t^2$, because

$$[c_t^2, d_t^2] = (r_2 \circ \xi, g_2 \circ \xi)^{-1}\Big(\big\{\big(t, \phi_2(t)\big)\big\}\Big) \supset (\xi_1 \circ \xi_2)^{-1}(\{v_t^2\}) \supset \xi_2^{-1}(\{d_t^2\})$$
$$= [(a_t^2 + 2d_t^2)/3, d_t^2].$$

Thus the proof is complete in the case when $u_t^i < v_t^i$ for every $t \in D_\beta$ and $i = 1, 2$. If this is not the case, then by Lemma 4.1 there is ξ_0 such that (4.21) holds for $(r_i \circ \xi_0, g_i \circ \xi_0)$. Then we can apply the already proven part to $(r_i \circ \xi_0, g_i \circ \xi_0)$ and so there is ξ' and $0 \leqslant a_t^i < b_t^i < c_t^i < d_t^i \leqslant 1$ such that (4.17)–(4.20) hold for $(r_i \circ \xi_0 \circ \xi', g_i \circ \xi_0 \circ \xi')$. ■

REMARK 4.1. *If* $\phi, \beta \in D_+[0, T]$, $\inf_{0 \leqslant t \leqslant T} \beta(t) > 0$, $t \in (0, T]$ *and* $\phi(t) \leqslant \beta(t)$, *then* $\bar{\phi}(t) - \bar{\phi}(t-) = \phi(t) - \phi(t-)$.

P r o o f. Note that

$$[\phi(t) - \phi(t-)] - [\bar{\phi}(t) - \bar{\phi}(t-)] = [\phi(t) - \bar{\phi}(t)] - [\phi(t-) - \bar{\phi}(t-)]$$
$$= C_\beta^\phi(t) - C_\beta^\phi(t-).$$

Therefore, it is enough to show that $C_\beta^\phi(t) - C_\beta^\phi(t-) = 0$. Suppose that $C_\beta^\phi(t) - C_\beta^\phi(t-) > 0$. Then

$$C_\beta^\phi(t) = R_t^\beta(\phi)(t) = \big(\phi(t) - \beta(t)\big)^+ \wedge \phi(t) = \big(\phi(t) - \beta(t)\big)^+ > 0,$$

which contradicts our assumption. ■

REMARK 4.2. *If* $\phi, \beta \in D_+[0, T]$, $\inf_{0 \leqslant t \leqslant T} \beta(t) > 0$, $t \in (0, T]$ *and* $\phi(t) - \beta(t) \geqslant \phi(t-) - \bar{\phi}(t-)$, *then* $\bar{\phi}(t) = \beta(t)$.

P r o o f. Let $\phi(t) - \beta(t) \geqslant \phi(t-) - \bar{\phi}(t-)$. Then, by (2.4), $\phi(t) - \beta(t) \geqslant 0$. Suppose $\bar{\phi}(t) < \beta(t)$. Then $C^{\phi}_{\beta}(t) - C^{\phi}_{\beta}(t-) = [\phi(t) - \bar{\phi}(t)] - [\phi(t-) - \bar{\phi}(t-)] \geqslant \beta(t) - \bar{\phi}(t)$. Thus, as in the proof of Remark 4.1,

$$R^{\beta}_t(\phi)(t) = \sup_{0 \leqslant s \leqslant t} R^{\beta}_t(\phi)(s) = \big(\phi(t) - \beta(t)\big)^+ \wedge \phi(t) = \phi(t) - \beta(t),$$

and therefore $\bar{\phi}(t) = \phi(t) - \sup_{0 \leqslant s \leqslant t} R^{\beta}_t(\phi)(s) = \beta(t)$, which contradicts our assumption. ■

LEMMA 4.3. *Let $\phi_1, \phi_2, \beta \in D_+[0,T]$ be such that $\inf_{0 \leqslant t \leqslant T} \beta(t) > 0$, and let $(r_1, g_1) \in \Pi_{\phi_1}$ and $(r_2, g_2) \in \Pi_{\phi_2}$. Then there is a continuous, nondecreasing mapping ξ from $[0,1]$ onto $[0,1]$ and parametric representations $(r^{\beta}_1, g^{\beta}_1)$ and $(r^{\beta}_2, g^{\beta}_2)$ in Π_β such that*

$$r_1 \circ \xi = r^{\beta}_1 \quad \text{and} \quad r_2 \circ \xi = r^{\beta}_2, \tag{4.22}$$

$$\big(r_1 \circ \xi, \Lambda_{g^{\beta}_1}(g_1 \circ \xi)\big) \in \Pi_{\Lambda_\beta(\phi_1)} \quad \text{and} \quad \big(r_2 \circ \xi, \Lambda_{g^{\beta}_2}(g_2 \circ \xi)\big) \in \Pi_{\Lambda_\beta(\phi_2)}. \tag{4.23}$$

P r o o f. By Lemma 4.2 there is $\{(a^i_t, b^i_t, c^i_t, d^i_t) \mid t \in D_\beta,\ i = 1,2\}$ and a mapping ξ such that (4.17)–(4.20) hold. For $i = 1,2$, let $\tilde{r}_i = r_i \circ \xi$ and $\tilde{g}_i = g_i \circ \xi$. Then $(\tilde{r}_i, \tilde{g}_i) \in \Pi_{\phi_i}$.

For $i = 1,2$ we let $\Delta^t_i = \phi_i(t-) - \bar{\phi}_i(t-)$. Then we define parameterizations $(r^{\beta}_i, g^{\beta}_i) \in \Pi_\beta$ so that parameterizations $(\tilde{r}_i, \tilde{g}_i)$ and $(r^{\beta}_i, g^{\beta}_i)$ are synchronized in a particular way.

We define r^{β}_i so that $r^{\beta}_i = \tilde{r}_i$. We define g^{β}_i first on each $[a^i_t, d^i_t]$ for $t \in D_\beta$ by setting $g^{\beta}_i(a^i_t) = \beta(t-)$, $g^{\beta}_i(d^i_t) = \beta(t)$. If, furthermore,

$$\begin{aligned}\big[\big(\phi_i(t-) - \Delta^t_i\big) \wedge \big(\phi_i(t) - \Delta^t_i\big), \big(\phi_i(t-) - \Delta^t_i\big) \vee \big(\phi_i(t) - \Delta^t_i\big)\big]&\\ \cap [\beta(t-) \wedge \beta(t), \beta(t-) \vee \beta(t)] \neq \emptyset&,\end{aligned} \tag{4.24}$$

then we require further synchronization of $\tilde{g}_i$ and g^{β}_i that will depend on a particular configuration of $\phi_i(t-) - \Delta^t_i$, $\phi_i(t) - \Delta^t_i$, $\beta(t-)$, $\beta(t)$. Therefore, we consider several cases.

(A) If $\phi_i(t-) - \Delta^t_i \leqslant \beta(t-) \leqslant \phi_i(t) - \Delta^t_i \leqslant \beta(t)$, then we define $g^{\beta}_i(s) = \big(\tilde{g}_i(s) - \Delta^t_i\big) \vee \beta(t-)$ for $s \in [a^i_t, c^i_t]$ and extend it by linear interpolation on $[c^i_t, d^i_t]$. Note that in this case $\tilde{g}_i - \Delta^t_i$ is nondecreasing and $\tilde{g}_i - \Delta^t_i \leqslant g^{\beta}_i$ on $[a^i_t, d^i_t]$.

(B) If $\phi_i(t-) - \Delta^t_i \leqslant \beta(t-) \leqslant \beta(t) \leqslant \phi_i(t) - \Delta^t_i$, then we let

$$s^i_t = \inf\{s \in [a^i_t, d^i_t] \mid \tilde{g}_i(s) - \Delta^t_i \geqslant \beta(t)\}$$

and define $g^{\beta}_i(s) = \big(\tilde{g}_i(s) - \Delta^t_i\big) \vee \beta(t-)$ for $s \in [a^i_t, s^i_t]$ and $g^{\beta}_i(s) = \beta(t)$ on $[s^i_t, d^i_t]$. Note that in this case $\tilde{g}_i - \Delta^t_i$ is nondecreasing, $\tilde{g}_i - \Delta^t_i \leqslant g^{\beta}_i$ on $[a^i_t, s^i_t]$ and $g^{\beta}_i = \beta(t) \leqslant \tilde{g}_i - \Delta^t_i$ on $[s^i_t, d^i_t]$.

(C) If $\phi_i(t-) - \Delta_i^t \leqslant \beta(t) \leqslant \beta(t-) \leqslant \phi_i(t) - \Delta_i^t$ or $\phi_i(t-) - \Delta_i^t \leqslant \beta(t) \leqslant \phi_i(t) - \Delta_i^t \leqslant \beta(t-)$, then we let $s_t^i = \inf\{s \in [a_t^i, d_t^i] | \, \tilde{g}_i(s) - \Delta_i^t > \beta(t)\}$, define $g_i^\beta(s) = \beta(t)$ for $s \in [s_t^i, d_t^i]$ and extend it by linear interpolation on $[a_t^i, s_t^i]$. Then $\tilde{g}_i - \Delta_i^t$ is nondecreasing, $\tilde{g}_i - \Delta_i^t \leqslant g_i^\beta$ on $[a_t^i, s_t^i]$ and $g_i^\beta = \beta(t) \leqslant \tilde{g}_i - \Delta_i^t$ on $[s_t^i, d_t^i]$.

(D) If $\beta(t) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \phi_i(t) - \Delta_i^t \leqslant \beta(t-)$ or $\beta(t) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \beta(t-) \leqslant \phi_i(t) - \Delta_i^t$, then we define $g_i^\beta(b_t^i) = \phi_i(t-) - \Delta_i^t$ and extend it by linear interpolation onto $[a_t^i, b_t^i]$ and $[b_t^i, d_t^i]$. Then $\tilde{g}_i - \Delta_i^t$ is nondecreasing and g_i^β is nonincreasing, $\phi_i(t-) - \Delta_i^t = \tilde{g}_i(s) - \Delta_i^t \leqslant g_i^\beta(s)$ for $s \in [a_t^i, b_t^i]$ and $g_i^\beta(s) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \tilde{g}_i(s) - \Delta_i^t$ for $s \in [b_t^i, d_t^i]$.

(E) If $\beta(t) \leqslant \phi_i(t) - \Delta_i^t \leqslant \phi_i(t-) - \Delta_i^t \leqslant \beta(t-)$, then define $g_i^\beta = \tilde{g}_i - \Delta_i^t$ on $[b_t^i, c_t^i]$ and extend it by linear interpolation onto $[a_t^i, b_t^i]$ and $[c_t^i, d_t^i]$. Then both $\tilde{g}_i - \Delta_i^t$ and g_i^β are nonincreasing, $\phi_i(t-) - \Delta_i^t = \tilde{g}_i(s) - \Delta_i^t \leqslant g_i^\beta(s)$ for $s \in [a_t^i, b_t^i]$ and $g_i^\beta(s) \leqslant \tilde{g}_i(s) - \Delta_i^t = \phi_i(t) - \Delta_i^t$ for $s \in [c_t^i, d_t^i]$.

(F) If $\phi_i(t) - \Delta_i^t \leqslant \beta(t) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \beta(t-)$ holds true, then we define $g_i^\beta = (\tilde{g}_i - \Delta_i^t) \vee \beta(t)$ for $s \in [b_t^i, d_t^i]$ and extend it by linear interpolation on $[a_t^i, b_t^i]$. Note that in this case $\tilde{g}_i - \Delta_i^t$ is nonincreasing and $\tilde{g}_i - \Delta_i^t \leqslant g_i^\beta$ on $[a_t^i, d_t^i]$.

If (4.24) does not hold, then we simply extend g_i^β onto $[a_t^i, d_t^i]$ by linear interpolation.

Outside $\bigcup_{t \in D_\beta} [u_t, v_t]$ we extend r_i^β so that $r_i^\beta = \tilde{r}_i$ and we extend g_i^β so that $g_i^\beta(s) = \beta(t)$ whenever $r_i^\beta(s) = t$ and $t \notin D_\beta$. Note that if $r_i^\beta(s) = t$, then we have $g_i^\beta(s) \in [\beta(t-) \wedge \beta(t), \beta(t-) \vee \beta(t)]$. In fact, it can be easily verified that $(r_i^\beta, g_i^\beta) \in \Pi_\beta$.

In order to prove (4.23) we need to show that

$$\big(r_i \circ \xi, \Lambda_{g_i^\beta}(g_i \circ \xi)\big) \text{ is continuous,} \tag{4.25}$$

$$\big(r_i \circ \xi, \Lambda_{g_i^\beta}(g_i \circ \xi)\big) \text{ maps } [0,1] \text{ onto } G_{\Lambda_\beta(\phi_i)}, \text{ and} \tag{4.26}$$

$$\big(r_i \circ \xi, \Lambda_{g_i^\beta}(g_i \circ \xi)\big) \text{ is increasing in a sense of (4.14).} \tag{4.27}$$

The easiest to establish is (4.25). It follows from Proposition 1.1 (ii) since $r_i \circ \xi$, $g_i \circ \xi$, and g_i^β are all continuous.

We show next that $\big(r_i \circ \xi, \Lambda_{g_i^\beta}(g_i \circ \xi)\big)$ is increasing in a sense of (4.14). Let $0 \leqslant s_1 \leqslant s_2 \leqslant 1$. Then $r_i \circ \xi(s_1) \leqslant r_i \circ \xi(s_2)$ because r_i and ξ are nondecreasing. It remains to show that

$$\big(r_i \circ \xi, \Lambda_{g_i^\beta}(g_i \circ \xi)\big) \text{ is increasing on } [a_t^i, d_t^i]. \tag{4.28}$$

While doing that we will, at the same time, establish (4.26) by showing that

$$\Lambda_{g_i^\beta}(g_i \circ \xi)(a_t^i) = \bar{\phi}_i(t-), \tag{4.29}$$

$$\Lambda_{g_i^\beta}(g_i \circ \xi)(d_t^i) = \bar{\phi}_i(t). \tag{4.30}$$

Consider a fixed i and $t \in D_\beta$. Applying the non-anticipatory property (Theorem 3.1) to $g_i \circ \xi$ and g_i^β, we get $\Lambda_{g_i^\beta}(g_i \circ \xi)(a_t^i + s) = \Lambda_\lambda(\gamma)(s)$, where $\gamma(s) = g_i \circ \xi(a_t^i + s) - \Delta_i^t$ and $\lambda(s) = g_i^\beta(a_t^i + s)$. Note that $\gamma(0) = g_i \circ \xi(a_t^i) - \Delta_i^t = \phi_i(t-) - \phi_i(t-) + \bar{\phi}_i(t-)$ and $\lambda(0) = g_i^\beta(a_t^i) = \beta(t-)$, hence, by (2.4), we have $\gamma(0) \leqslant \lambda(0)$, which means that $\Lambda_\lambda(\gamma)(0) = \gamma(0)$, and so (4.29) holds.

If $\phi_i(t-) \leqslant \phi_i(t)$, then γ is nondecreasing on $[0, d_t^i - a_t^i]$. On the other hand, if $\phi_i(t-) \geqslant \phi_i(t)$, then γ is nonincreasing on $[0, d_t^i - a_t^i]$. Similarly, λ is either nondecreasing or nonincreasing depending whether $\beta(t-) \leqslant \beta(t)$ or $\beta(t-) \geqslant \beta(t)$.

If (4.24) does not hold, then $\gamma \leqslant \lambda$ on $[0, d_t^i - a_t^i]$. Hence $\Lambda_\lambda(\gamma) = \gamma$, and so (4.28) holds. Also, by Remark 4.1, $\Lambda_\lambda(\gamma)(d_t^i - a_t^i) = \gamma(d_t^i - a_t^i) = g_i \circ \xi(d_t^i) - \Delta_i^t = \phi_i(t) - \phi_i(t-) + \bar{\phi}_i(t-) = \bar{\phi}_i(t)$, and so (4.30) holds. Therefore we can assume (4.24).

Further analysis depends on the particular relation among points: $\phi_i(t-) - \Delta_i^t$, $\phi_i(t) - \Delta_i^t$, $\beta(t-)$, $\beta(t)$. We need to consider all configurations possible under (4.24). Since $\phi_i(t-) - \Delta_i^t = \bar{\phi}_i(t-)$, by (2.4), we must always have $\phi_i(t-) - \Delta_i^t \leqslant \beta(t-)$. That leaves eight possible ordered arrangements.

If $\phi_i(t-) - \Delta_i^t \leqslant \beta(t-) \leqslant \phi_i(t) - \Delta_i^t \leqslant \beta(t)$, then, by (A), γ is nondecreasing and $\gamma \leqslant \lambda$ on $[0, d_t^i - a_t^i]$. Similarly, if $\phi_i(t) - \Delta_i^t \leqslant \beta(t) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \beta(t-)$, then, by (F), γ is nonincreasing and $\gamma \leqslant \lambda$ on $[0, d_t^i - a_t^i]$. Therefore, in both cases (4.28) and (4.30) follow, by the same argument we used above, when (4.24) did not hold.

If $\phi_i(t-) - \Delta_i^t \leqslant \beta(t-) \leqslant \beta(t) \leqslant \phi_i(t) - \Delta_i^t$, then, by (B) and Proposition 4.3 (i), $\Lambda_\lambda(\gamma)$ is nondecreasing on $[0, d_t^i - a_t^i]$ and $\Lambda_\lambda(\gamma)(d_t^i - a_t^i) = \lambda(d_t^i - a_t^i) = g_i^\beta(d_t^i) = \beta(t)$. Since, by Remark 4.2, $\beta(t) = \bar{\phi}_i(t)$, (4.30) follows.

If $\phi_i(t-) - \Delta_i^t \leqslant \beta(t) \leqslant \beta(t-) \leqslant \phi_i(t) - \Delta_i^t$ or $\phi_i(t-) - \Delta_i^t \leqslant \beta(t) \leqslant \phi_i(t) - \Delta_i^t \leqslant \beta(t-)$, then, by (C), Proposition 4.3 (i) is applicable. Hence $\Lambda_\lambda(\gamma)$ is nondecreasing on $[0, d_t^i - a_t^i]$ and, as above, $\Lambda_\lambda(\gamma)(d_t^i - a_t^i) = \beta(t)$. Again, by Remark 4.2, $\beta(t) = \bar{\phi}_i(t)$, and so (4.30) follows.

If $\beta(t) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \phi_i(t) - \Delta_i^t \leqslant \beta(t-)$ or $\beta(t) \leqslant \phi_i(t-) - \Delta_i^t \leqslant \beta(t-) \leqslant \phi_i(t) - \Delta_i^t$, then, by (D), Proposition 4.3 (ii), and Remark 4.2, $\Lambda_\lambda(\gamma)$ is nonincreasing on $[0, d_t^i - a_t^i]$ and $\Lambda_\lambda(\gamma)(d_t^i - a_t^i) = g_i^\beta(d_t^i) = \beta(t) = \bar{\phi}_i(t)$.

If $\beta(t) \leqslant \phi_i(t) - \Delta_i^t \leqslant \phi_i(t-) - \Delta_i^t \leqslant \beta(t-)$ holds true, then, by (E), Proposition 4.3 (iii), and Remark 4.2, $\Lambda_\lambda(\gamma)$ is nonincreasing on $[0, d_t^i - a_t^i]$ and

$$\Lambda_\lambda(\gamma)(d_t^i - a_t^i) = g_i^\beta(d_t^i) = \beta(t) = \bar{\phi}_i(t).$$

Thus we have shown that (4.28) and (4.30) hold in each case, and so (4.26) and (4.27) are established, which completes our proof. ∎

The next result generalizes Corollary 1.6 from [4] for d_1 metric. Just like Proposition 4.2 it involves the oscillation of the constraining function β.

PROPOSITION 4.4. *For $T > 0$ let $\beta, \phi_1, \phi_2 \in D_+[0,T]$ be such that*

$$\inf_{0 \leqslant t \leqslant T} \beta(t) > 0.$$

Then

$$d_1\big(\Lambda_\beta(\phi_1), \Lambda_\beta(\phi_2)\big) \leqslant 2\, d_1(\phi_1, \phi_2) + \sup_{s,t\in[0,T]} |\beta(t) - \beta(s)|. \tag{4.31}$$

P r o o f. Let $\phi_1, \phi_2 \in D_+[0,T]$ and let $(r_1, g_1) \in \Pi_{\phi_1}$, $(r_2, g_2) \in \Pi_{\phi_2}$. Let $\xi, (r_1^\beta, g_1^\beta)$, (r_2^β, g_2^β) be as described in Lemma 4.3. Then, by (4.22), (4.23) and (4.3), we have

$$\begin{aligned} d_1\big(\Lambda_\beta(\phi_1), \Lambda_\beta(\phi_2)\big) &\leqslant \|r_1^\beta - r_2^\beta\|_T \vee \|\Lambda_{g_1^\beta}(g_1 \circ \xi) - \Lambda_{g_2^\beta}(g_2 \circ \xi)\|_T \\ &\leqslant \|r_1 \circ \xi - r_2 \circ \xi\|_T \vee [2\, \|g_1 \circ \xi - g_2 \circ \xi\|_T + \|g_1^\beta - g_2^\beta\|_T] \\ &\leqslant \|r_1 - r_2\|_T \vee [2\, \|g_1 - g_2\|_T + \|g_1^\beta - g_2^\beta\|_T] \\ &\leqslant 2\,(\|r_1 - r_2\|_T \vee \|g_1 - g_2\|_T) + \|g_1^\beta - g_2^\beta\|_T. \end{aligned}$$

Because the above inequality holds for any two parameterizations, $(r_1, g_1) \in \Pi_{\phi_1}$ and $(r_2, g_2) \in \Pi_{\phi_2}$, we can conclude (4.31). ∎

Before we examine the distance between two Skorokhod maps in metric d_1 we consider the effect of translation on d_1 distance.

PROPOSITION 4.5. *For any ψ_1, ψ_2 and $\alpha \in D[0,T]$*

$$d_1(\psi_1 + \alpha, \psi_2 + \alpha) \leqslant d_1(\psi_1, \psi_2) + \sup_{s,t\in[0,T]} |\alpha(s) - \alpha(t)|. \tag{4.32}$$

P r o o f. Let $\psi_1, \psi_2, \alpha \in D[0,T]$ and choose arbitrary $(r_i, g_i) \in \Pi_{\psi_i}$ for $i = 1, 2$. By Lemma 4.1 there is a continuous, nondecreasing mapping ξ from $[0,1]$ onto $[0,1]$ and intervals $\{[u_t^i, v_t^i] \mid t \in D_\alpha, i = 1, 2\}$ such that (4.15) and (4.16) hold. We can extend the definition of $[u_t^i, v_t^i]$ to all $t \in [0,T]$ by insisting that (4.16) holds for every t. Define $g_i^\alpha : [0,T] \longrightarrow \mathbb{R}$ by setting $g_i^\alpha(u_t^i) = \alpha(t-)$, $g_i^\alpha(v_t^i) = \alpha(t)$ and extending it linearly on each $[u_t^i, v_t^i]$ for each $t \in [0,T]$ and each $i = 1, 2$. Then $(r_i \circ \xi, g_i \circ \xi + g_i^\alpha) \in \Pi_{\phi_i + \alpha}$ for $i = 1, 2$ and

$$\begin{aligned} &\|r_1 \circ \xi - r_2 \circ \xi\|_T \vee \|(g_1 \circ \xi + g_1^\alpha) - (g_2 \circ \xi + g_2^\alpha)\|_T \\ &\qquad \leqslant \|r_1 - r_2\|_T \vee (\|g_1 \circ \xi - g_2 \circ \xi\|_T + \|g_1^\alpha - g_2^\alpha\|_T) \\ &\qquad \leqslant \|r_1 - r_2\|_T \vee \|g_1 - g_2\|_T + \sup_{s,t\in[0,T]} |\alpha(s) - \alpha(t)|. \end{aligned}$$

Thus

$$d_1(\psi_1+\alpha,\psi_2+\alpha)\leqslant \|r_1-r_2\|_T\vee\|g_1-g_2\|_T+\sup_{s,t\in[0,T]}|\alpha(s)-\alpha(t)|,$$

and because $(r_i,g_i)\in\Pi_{\psi_i}$, $i=1,2$, were arbitrary, we conclude (4.32). ∎

THEOREM 4.1. *For $T>0$ let α,β be two functions in $D[0,T]$ such that* $\inf_{0\leqslant t\leqslant T}\big(\beta(t)-\alpha(t)\big)>0$ *and let $\psi_1,\psi_2\in D[0,T]$. Then*

$$\begin{aligned}(4.33)\quad d_1\big(\Gamma_{\alpha,\beta}(\psi_1),\Gamma_{\alpha,\beta}(\psi_2)\big)&\leqslant 4\,d_1(\psi_1,\psi_2)\\&+\sup_{s,t\in[0,T]}|(\beta-\alpha)(s)-(\beta-\alpha)(t)|+4\sup_{s,t\in[0,T]}|\alpha(s)-\alpha(t)|.\end{aligned}$$

Proof. By Lemma 2.2 and Proposition 4.4, we have

$$\begin{aligned}&d_1\big(\Gamma_{\alpha,\beta}(\psi_1),\Gamma_{\alpha,\beta}(\psi_2)\big)=d_1\big(\Gamma_{\beta-\alpha}(\psi_1-\alpha),\Gamma_{\beta-\alpha}(\psi_1-\alpha)\big)\\&=d_1\Big(\Lambda_{\beta-\alpha}\big(\Gamma_0(\psi_1-\alpha)\big),\Lambda_{\beta-\alpha}\big(\Gamma_0(\psi_2-\alpha)\big)\Big)\\&\leqslant 2d_1\big(\Gamma_0(\psi_1-\alpha),\Gamma_0(\psi_2-\alpha)\big)+\sup_{s,t\in[0,T]}|(\beta-\alpha)(s)-(\beta-\alpha)(t)|.\end{aligned}$$

Since, by Theorem 13.5.1 in [9], $d_1\big(\Gamma_0(\psi_1),\Gamma_0(\psi_2)\big)\leqslant 2d_1(\psi_1,\psi_2)$, by applying Proposition 4.5 we conclude (4.33). ∎

REMARK 4.3. Following the standard argument described in Theorem 12.9.4 in [9] we can conclude that the inequalities in (4.1)–(4.4) and (4.31)–(4.33) remain true for $\phi_1,\phi_2,\psi_1,\psi_2,\alpha,\beta\in D[0,\infty)$.

EXAMPLE 4.2. Let β,ϕ_1,ϕ_2 be the functions constructed in Example 4.1. Then $d_1(\bar\phi_1,\bar\phi_2)=a$ while $d_1(\phi_1,\phi_2)=\varepsilon$, and since ε can be arbitrarily small, neither Λ_β nor $\Gamma_{0,\beta}$ are continuous in metric d_1.

We want to make a couple of useful observations about the projection $\pi_{a,b}$ introduced in Section 1 and used in Example 1.1.

REMARK 4.4. *Let $a,b\in\mathbb{R}$ be such that $a<b$. Then for every $x,y\in\mathbb{R}$*

$$(4.34)\qquad \big|\big(\pi_{a,b}(x)-\pi_{a,b}(y)\big)-(x-y)\big|\leqslant|x-y|.$$

Proof. First note that $\pi_{a,b}(x)\leqslant x$ provided $\pi_{a,b}(x)\neq a$. Similarly, we have $\pi_{a,b}(x)\geqslant x$ whenever $\pi_{a,b}(x)\neq b$. Therefore

$$\begin{aligned}(4.35)\qquad \pi_{a,b}(x)-\pi_{a,b}(y)&\leqslant(x-y)I_{\{\pi_{a,b}(x)\neq a\text{ and }\pi_{a,b}(y)\neq b\}}\\&\leqslant(x-y)+|x-y|\end{aligned}$$

and

$$\begin{aligned}(4.36)\qquad \pi_{a,b}(x)-\pi_{a,b}(y)&\geqslant(x-y)I_{\{\pi_{a,b}(x)\neq b\text{ and }\pi_{a,b}(y)\neq a\}}\\&\geqslant(x-y)-|x-y|.\quad ∎\end{aligned}$$

REMARK 4.5. *Let* ψ, α, $\beta \in D[0,\infty)$, $\inf_{t\geqslant 0}\big(\beta(t)-\alpha(t)\big) > 0$ *and* $t > 0$. *Then*

$$\bar{\phi}(0) = \pi_{\alpha(0),\beta(0)}\big(\psi(0)\big), \tag{4.37}$$

$$\bar{\phi}(t) = \pi_{\alpha(t),\beta(t)}\left(\bar{\phi}(t-) + \psi(t) - \psi(t-)\right). \tag{4.38}$$

Proof. Since $\phi(0) = \Gamma_\alpha(\psi)(0) = \psi(0) + \big(\alpha(0)-\psi(0)\big)^+ = \psi(0)\vee\alpha(0)$, $C^{\phi}_{\alpha,\beta}(0) = \big(\phi(0)-\beta(0)\big)^+ \wedge \big(\phi(0)-\alpha(0)\big) = \big(\phi(0)-\beta(0)\big)^+$ and $\bar{\phi}(0) = \phi(0) - C^{\phi}_{\beta}(0) = \phi(0) - \big(\phi(0)-\beta(0)\big)^+ = \phi(0)\wedge\beta(0)$, we get

$$\bar{\phi}(0) = \big(\psi(0)\vee\alpha(0)\big)\wedge\beta(0) = \pi_{\alpha(0),\beta(0)}\big(\psi(0)\big).$$

It is clear from Example 1.1 that (4.38) holds true when ψ, α and β are piecewise constant. For general ψ, α and β with possible discontinuities at t we can find sequences ψ^n α^n and β^n in $S[0,\infty)$ such that

$$\inf_{t\geqslant 0}\big(\beta^n(t)-\alpha^n(t)\big) > 0 \text{ for every } n, \quad \psi^n \overset{n\to\infty}{\longrightarrow} \psi,\ \alpha^n \overset{n\to\infty}{\longrightarrow} \alpha \text{ and } \beta^n \overset{n\to\infty}{\longrightarrow} \beta$$

uniformly on compact sets. Taking the limits on both sides of the equation $\bar{\phi}_n(t) = \pi_{\alpha^n(t),\beta^n(t)}\left(\bar{\phi}_n(t-) + \psi_n(t) - \psi_n(t-)\right)$, by (4.4), we obtain (4.38). ∎

In a special case, when $\alpha = 0$ and $\beta = a > 0$, the statements (4.3), (4.4), (4.11), (4.12), (4.31), and (4.33) coincide with Lipschitz conditions given in Corollary 1.6 in [4]. As it is stated there, it is well known that the smallest Lipschitz constant when β does not depend on time is 2. In the following results we will lower our constant to 2 for time dependent constraining functions α and β by means similar to those used in [9], Theorem 14.8.1.

PROPOSITION 4.6 (Lipschitz continuity). *Let* $\psi_1,\psi_2,\alpha,\beta \in D[0,\infty)$ *and let* $\inf_{t\geqslant 0}\big(\beta(t)-\alpha(t)\big) > 0$. *Then*

$$\|\Gamma_{\alpha,\beta}(\psi_1) - \Gamma_{\alpha,\beta}(\psi_2)\| \leqslant 2\,\|\psi_1-\psi_2\|. \tag{4.39}$$

Proof. We assume first that $\psi_1,\psi_2,\alpha,\beta \in S[0,T]$ and let $0 = t_0 < t_1 < \ldots < t_k$ be all the jump points of ψ_1,ψ_2, α or β. We will show by induction that for $j = 0,1,2,\ldots,k$

$$\left|\big(\bar{\phi}_1(t_j)-\bar{\phi}_2(t_j)\big) - \big(\psi_1(t_j)-\psi_2(t_j)\big)\right| \leqslant \|\psi_1-\psi_2\|. \tag{4.40}$$

Since, by (4.37), $\bar{\phi}_i(0) = \pi_{\alpha(0),\beta(0)}\big(\psi_i(0)\big)$ for $i = 1,2$, it follows immediately from Remark 4.4 that (4.40) holds true for $j = 0$.

Suppose (4.40) holds for j. By Remark 4.5, for $i = 1, 2$ we have

$$\begin{aligned} \bar{\phi}_i(t_{j+1}) &= \Gamma_{\alpha,\beta}(\psi_i)(t_{j+1}) \\ &= \pi_{\alpha(t_{j+1}),\beta(t_{j+1})}\big(\bar{\phi}_i(t_j) + \psi_i(t_{j+1}) - \psi_i(t_j)\big). \end{aligned} \tag{4.41}$$

By (4.35) we obtain

$$\begin{aligned} &\bar{\phi}_1(t_{j+1}) - \bar{\phi}_2(t_{j+1}) = \pi_{\alpha(t_{j+1}),\beta(t_{j+1})}\big(\bar{\phi}_1(t_j) + \psi_1(t_{j+1}) - \psi_1(t_j)\big) \\ &\qquad\qquad\qquad\qquad - \pi_{\alpha(t_{j+1}),\beta(t_{j+1})}\big(\bar{\phi}_2(t_j) + \psi_2(t_{j+1}) - \psi_2(t_j)\big) \\ &\leqslant \begin{cases} 0 & \text{if } \bar{\phi}_1(t_{j+1}) = \alpha(t_{j+1}) \text{ or } \bar{\phi}_2(t_{j+1}) = \beta(t_{j+1}), \\ \big(\bar{\phi}_1(t_j) - \bar{\phi}_2(t_j)\big) + \big(\psi_1(t_{j+1}) - \psi_2(t_{j+1})\big) - \big(\psi_1(t_j) - \psi_2(t_j)\big) & \text{otherwise} \end{cases} \\ &\leqslant \big(\psi_1(t_{j+1}) - \psi_2(t_{j+1})\big) + \|\psi_1 - \psi_2\| \end{aligned}$$

by the inductive assumption.

Similarly, applying (4.36) we obtain the lower bound:

$$\begin{aligned} &\bar{\phi}_1(t_{j+1}) - \bar{\phi}_2(t_{j+1}) = \pi_{\alpha(t_{j+1}),\beta(t_{j+1})}\big(\bar{\phi}_1(t_j) + \psi_1(t_{j+1}) - \psi_1(t_j)\big) \\ &\qquad\qquad\qquad\qquad - \pi_{\alpha(t_{j+1}),\beta(t_{j+1})}\big(\bar{\phi}_2(t_j) + \psi_2(t_{j+1}) - \psi_2(t_j)\big) \\ &\geqslant \begin{cases} 0 & \text{if } \bar{\phi}_1(t_{j+1}) = \beta(t_{j+1}) \text{ or } \bar{\phi}_2(t_{j+1}) = \alpha(t_{j+1}), \\ \big(\bar{\phi}_1(t_j) - \bar{\phi}_2(t_j)\big) + \big(\psi_1(t_{j+1}) - \psi_2(t_{j+1})\big) - \big(\psi_1(t_j) - \psi_2(t_j)\big) & \text{otherwise} \end{cases} \\ &\geqslant \big(\psi_1(t_{j+1}) - \psi_2(t_{j+1})\big) - \|\psi_1 - \psi_2\|. \end{aligned}$$

Thus the proof of (4.40) is complete, and so (4.39) holds true when $\psi_1, \psi_2, \alpha, \beta \in S[0, \infty)$. Consider now general $\psi_1, \psi_2, \alpha, \beta \in D[0, \infty)$ such that

$$\inf_{t \geqslant 0}\big(\beta(t) - \alpha(t)\big) > 0.$$

As in the proof of Remark 4.5, we can find sequences $\psi_1^n, \psi_2^n, \alpha^n, \beta^n \in S[0, \infty)$ converging uniformly on compact sets to ψ_1, ψ_2, α and β, respectively. Since (4.39) holds true for $\psi_1^n, \psi_2^n, \alpha^n, \beta^n$, applying (4.4) and Remark 4.3 we conclude (4.39) for $\psi_1, \psi_2, \alpha, \beta$. ■

REFERENCES

[1] S. V. Anulova and R. Sh. Liptser, *Diffusion approximations for processes with normal reflection*, Theory Probab. Appl. 35 (1990), pp. 411–423.

[2] K. Burdzy, W. Kang and K. Ramanan, *The Skorokhod problem in a time-dependent interval*, Stochastic Process. Appl. 119 (2009), pp. 428–452.

[3] P. Depuis and H. Ishii, *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with application*, Stochastics 35 (1991), pp. 31–62.

[4] L. Kruk, J. Lehoczky, K. Ramanan and S. Shreve, *An explicit formula for the Skorokhod map on* $[0, a]$, Ann. Probab. 35 (2007), pp. 1740–1768.

[5] K. Ramanan, *Reflected diffusions defined via the extended Skorokhod map*, Electron. J. Probab. 11 (2006), pp. 934–992.

[6] K. Ramanan and M. Reiman, *The heavy traffic limit of an unbalanced generalized processor sharing model*, Ann. Appl. Probab. 18 (2008), pp. 22–58.

[7] A. V. Skorokhod, *Stochastic equations for diffusions in a bounded region*, Theory Probab. Appl. 6 (1961), pp. 264–274.

[8] H. Tanaka, *Stochastic differential equations with reflecting boundary condition in convex regions*, Hiroshima Math. J. 9 (1979), pp. 163–177.

[9] W. Whitt, *An Introduction to Stochastic-process Limits and Their Application to Queues*, Springer, New York 2002.

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