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ON LIMIT THEOREMS IN JW-ALGEBRAS

BY

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Abstract. In the present paper, we study bundle convergence in JW-algebra and prove certain ergodic theorems with respect to such convergence. Moreover, conditional expectations of reversible JW-algebras are considered. Using such expectations, the convergence of supermartingales is established.

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1. INTRODUCTION

The concept of almost everywhere convergence and its different variants for sequences in von Neumann algebra were studied by many authors (see, e.g., [18], [21], [25]). There were proved many limit and ergodic theorems with respect to almost everywhere convergence in such algebras with faithful normal state [9], [14], [29]. In [17], a notion of bundle convergence in a von Neumann algebra was firstly defined, which coincides with usual almost everywhere convergence in the case of commutative algebra L_∞. Certain limit theorems with respect to such a convergence were obtained there. Other several results concerning the bundle convergence in a von Neumann algebra and in its L_2-space were studied in [13], [12], [26].

On the other hand, in most mathematical formulations of the foundations of quantum mechanics, the bounded observables of a physical system are identified with a real linear space L of bounded self-adjoint operators on a Hilbert space H. Those bounded observables which correspond to the projections in L form a complete orthomodular lattice P, otherwise known as the lattice of the quantum logic of the physical system. For the self-adjoint operators x and y on H their Jordan product is defined by $x \circ y = (xy + yx)/2 = (x + y)^2 - x^2 - y^2$. Thus $x \circ y$ is self-adjoint, so it is reasonable to assume that L is a Jordan algebra of self-adjoint operators on H, which is closed in the weak operator topology. Hence L is a JW-algebra. It is known that the JW-algebra is a real non-associative analog of a von Neumann algebra that was firstly studied by Topping [27]. He had extended many results from the theory of von Neumann algebras on JW-algebras. Particularly, in [4], a problem of extension of states and traces from JW-algebra to its enveloping von Neumann algebra was solved. In [3], concepts of convergence in measure, almost uniform convergence and convergence s-almost uniform in JW-algebras were introduced, and relations between them were investigated as well. Such kinds of convergence are used to prove various ergodic theorems for Markov operators (see [1], [2], [5], [19], [20]). Asymptotic behavior of positive contractions of Jordan algebras has been studied in [22] and [23]. We refer the reader to the books [6], [7], [16] for the theory of JW-algebras.

The main purposes of this paper is to extend a concept of the bundle convergence from von Neumann algebras to JW-algebras and prove certain limit theorems. The paper is organized as follows. In Section 2, we recall some well-known facts and basic definitions from the theory of Jordan and von Neumann algebras. In Section 3, the bundle convergence and its properties are studied. It is proved that the bundle limit of a sequence of uncorrelated operators is proportional to the identity operator. In the last Section 4, certain properties of conditional expectations of a reversible JW-algebra A with a faithful normal trace are studied, and a martingale convergence theorem is proved as well. Here, we apply a method passing from Jordan algebra to the corresponding enveloping von Neumann algebra [19]. Note that supermartingales and martingales in a von Neumann algebra setting were investigated in [8], [14], [18].

2. PRELIMINARIES

Throughout the paper H denotes a complex Hilbert space, and $B(H)$ the algebra of all bounded linear operators on H.

Recall that a *JW-algebra* is a real linear space of self-adjoint operators from $B(H)$ which is closed under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and also closed in the weak operator topology. Here ab denotes the usual operator multiplication of operators a and b taken from $B(H)$. A JW-algebra A is said to be *reversible* if $a_1 a_2 \dots a_n + a_n a_{n-1} \dots a_1 \in A$, whenever $a_1, a_2, \dots, a_n \in A$. Examples of non-reversible JW-algebras are spin factors which are described in [27] and [7].

Recall that a real $*$-algebra R in $B(H)$ is called a *real W^*-algebra* if it is closed in the weak operator topology and satisfies the conditions $R \cap iR = \{0\}$, $\mathbf{1} \in R$. It is obvious that if R is a real or complex W^*-algebra, then its self-adjoint part $R_{sa} = \{x \in R : x^* = x\}$ forms a reversible JW-algebra.

Given an arbitrary JW-algebra A let $R(A)$ denote the weakly closed real $*$-algebra in $B(H)$ generated by A, and let $W(A)$ denote the W^*- algebra (complex) generated by A.

THEOREM 2.1 ([16]). *Let A be a reversible JW-algebra. Then the following assertions hold*:

(i) $R(A)_{sa} = A$ *and* $W(A) = R(A) + iR(A)$;

(ii) $\|a + ib\| \geqslant \max\{\|a\|, \|b\|\}$ *for every* $a, b \in R(A)$;

(iii) *if* $a + ib \geqslant 0$ *for* $a, b \in R(A)$, *then* $a \geqslant 0$.

Throughout the paper we always assume that a JW-algebra A is reversible, therefore we do not stress on it, if it is not necessary.

Let A be a reversible JW-algebra and ρ be a faithful normal (f.n.) state (resp. a faithful normal semifinite (f.n.s.) trace) on A. Then ρ can be extended to an f.n. state ρ_1 (resp. an f.n.s. trace) on the W^*-algebra $W(A)$. Namely, for every $x \in W(A)^+$ (W^+ means the positive part of W), we have $x = a + ib$, where $a, b \in R(A)$, $a \in A^+$, $b^* = -b$ (skew-symmetric element). Then, we put $\rho_1(x) = \rho(a)$ (see [7], Theorem 1.2.9, for more details).

Let A be a JW-algebra with an f.n.s. trace τ and τ_1 be its extension to $W(A)$. Set $\mathfrak{N}_\tau = \{x \in A : \tau(|x|) < \infty\}$, $\mathfrak{N}_{\tau_1} = \{x \in W(A) : \tau_1(|x|) < \infty\}$. Completion of $\mathfrak{N}_\tau$ (resp. $\mathfrak{N}_{\tau_1}$) with respect to the norm $\|x\|_1 = \tau(|x|)$, $x \in \mathfrak{N}_\tau$ (resp. $\|x\|_1 = \tau_1(|x|)$, $x \in \mathfrak{N}_{\tau_1}$) is denoted by $L_1(A, \tau)$ (resp. $L_1(W(A), \tau_1)$). It is obvious that $L_1(A, \tau) \subset L_1\big(W(A), \tau_1\big)$.

Let W be a von Neumann algebra with an f.n. state ρ. By $\mathrm{Proj}W$ we denote the set of all projections in W.

We recall some facts about almost uniform convergence.

A sequence $\{x_n\} \subset W$ is said to be *almost uniformly convergent* to $x \in W$ ($x_n \xrightarrow{a.u.} x$) if, for each $\varepsilon > 0$, there exists $p \in \mathrm{Proj}W$ with $\rho(p^\perp) < \varepsilon$ such that $\|(x_n - x)p\| \to 0$ as $n \to \infty$.

Further, we need the following

LEMMA 2.1 ([18]). *For a uniformly bounded sequence*, *almost uniform convergence implies strong convergence.*

Now we give some necessary definitions and results from [17].

Suppose that $\{D_m\} \subset W^+$ with $\sum_{m=1}^\infty \rho(D_m) < \infty$. The *bundle* (determined by the sequence $\{D_m\}$) is the set

$$P_{(D_m)} = \big\{p \in \mathrm{Proj}W : \\ p \neq 0,\ \sup_m \big\|p\big(\sum_{k=1}^m D_k\big)p\big\| < \infty \text{ and } \|pD_mp\| \to 0 \text{ as } m \to \infty\big\}.$$

Let $x_n, x \in W$ $(n = 1, 2, \dots)$. We say that $\{x_n\}$ is *bundle convergent* to x ($x_n \xrightarrow{b,W} x$) if there exists a bundle $P_{(D_m)}$ with $D_m \in W^+, \sum_{m=1}^\infty \rho(D_m) < \infty$ such that

$$p \in P_{(D_m)} \text{ implies } \|(x_n - x)p\| \to 0.$$

LEMMA 2.2 ([17]). *Let* $0 < \varepsilon < 1/16$, $D_m \in W^+$ $(m = 1, 2, \dots)$ *and*

$$\sum_{m=1}^{\infty} \rho(D_m) < \varepsilon.$$

Then there exists $p \in \mathrm{Proj}\, W$ *such that*

$$\rho(p^{\perp}) < \varepsilon^{1/4}, \quad \big\|p\big(\sum_{k=1}^{m} D_k\big)p\big\| < 4\varepsilon^{1/2},\ m = 1, 2, \dots$$

3. THE BUNDLE CONVERGENCE IN JW-ALGEBRAS

In this section, we study bundle convergence and its properties in JW-algebras. We shall prove that the bundle limit of a sequence of uncorrelated operators is proportional to the identity operator. Throughout the paper A denotes a reversible JW-algebra.

Let A be a JW-algebra with an f.n. state ρ and ρ_1 be its extension to the enveloping von Neumann algebra $W(A)$.

Suppose that $\{D_m\} \subset A^+$ with $\sum_{m=1}^{\infty} \rho(D_m) < \infty$. The *bundle* is the set

$$\mathcal{P}_{(D_m)} = \big\{p \in \mathrm{Proj} A\colon \\ p \neq 0,\ \sup_m \big\|p\big(\sum_{k=1}^{m} D_k\big)p\big\| < \infty \text{ and } \|p(D_m)p\| \to 0 \text{ as } m \to \infty\big\}.$$

REMARK 3.1. *Let* $\{D_m\}_{m=1}^{\infty} \subset A^+$, $\sum_{m=1}^{\infty} \rho(D_m) < \infty$. *Then* $\mathcal{P}_{(\{D_m\}_{m=1}^{\infty})} = \mathcal{P}_{(\{D_m\}_{m=k}^{\infty})}$ *and in the definition of bundle we may suppose that* $\sum_{m=1}^{\infty} \rho(D_m) < \varepsilon$ *for some positive number* ε.

Let $x_n,\ x \in A$ $(n = 1, 2, \dots)$. We say that $\{x_n\}$ is *bundle convergent* to x ($x_n \xrightarrow{b,A} x$) if there exists a bundle $\mathcal{P}_{(D_m)}$ such that

$$p \in \mathcal{P}_{(D_m)} \text{ implies } \|p(x_n - x)^2 p\| \to 0.$$

Clearly, an intersection of two bundles is a bundle as well. Consequently, the bundle convergence in A is additive, and the bundle limit in A is unique. In the case when A is a self-adjoint part of some von Neumann algebra, bundle convergence in Jordan algebra coincides with convergence in a von Neumann algebra setting.

Similarly, a sequence $\{x_n\} \subset A$ is said to be *almost uniformly convergent* to $x \in A$ ($x_n \xrightarrow{a.u.} x$) if, for each $\varepsilon > 0$, there exists $p \in \mathrm{Proj} A$ with $\rho(p^{\perp}) < \varepsilon$ such that $\|p(x_n - x)^2 p\| \to 0$ as $n \to \infty$ (see [3]).

In the sequel we need the following auxiliary result.

LEMMA 3.1. *Let* $\sigma, \delta > 0,\ D \in A^+$ *and suppose that* $p \in \mathrm{Proj} W(A)$ *with* $\rho_1(p^{\perp}) < \sigma$ *such that* $\|pDp\| < \delta$. *Then there exists a projection* $E \in \mathrm{Proj} A$ *with* $\rho(E^{\perp}) < 2\sigma$ *such that* $\|EDE\| < 4\delta$.

Proof. Suppose that $p = x + iy,\ x, y \in R(A)$; one finds $0 \leqslant x \leqslant \mathbf{1}$ and let $x = \int_0^1 \lambda de_\lambda$ be the spectral resolution of x. Put

$$b = \int\limits_{1/2}^{1} \lambda^{-1} de_\lambda.$$

Then

$$E = xb = \int\limits_0^1 \lambda de_\lambda \int\limits_{1/2}^{1} \lambda^{-1} de_\lambda = s(x) - e_{1/2},$$

where $s(x)$ means the support projector of x, and

$$\begin{aligned} 2\big(s(x) - x\big) &= 2\int\limits_0^1 (1-\lambda) de_\lambda \geqslant 2\int\limits_0^{1/2} (1-\lambda) de_\lambda \\ &> \int\limits_0^{1/2} de_\lambda = e_{1/2}. \end{aligned}$$

Since $s(x)^\perp \leqslant 2s(x)^\perp$, we have

$$\mathbf{1} - E = s(x)^\perp + e_{1/2} \leqslant 2\Big(s(x)^\perp + \big(s(x) - x\big)\Big) = 2(\mathbf{1} - x)$$

and

$$\rho(\mathbf{1} - E) \leqslant 2\rho(\mathbf{1} - x) = 2\rho_1(\mathbf{1} - x - iy) = 2\rho_1(\mathbf{1} - p) < 2\sigma.$$

This means that $E \neq 0$.

Then the inequality (ii) of Theorem 2.1 with commutativity of b and x implies

$$\begin{aligned} \|EDE\| &= \|xbDxb\| \leqslant \|b\|^2 \|xDx\| \\ &= 4\|\sqrt{D}x\|^2 \leqslant 4\|\sqrt{D}(x + iy)\|^2 \\ &= 4\|pDp\|, \end{aligned}$$

which means that $\|EDE\| < 4\delta$. This completes the proof. ■

Using Lemmas 2.2 and 3.1 we get the following

LEMMA 3.2. *Let* $0 < \varepsilon < 1/16$, $D_m \in A^+\ (m = 1, 2, \dots)$ *and*

$$\sum_{m=1}^{\infty} \rho(D_m) < \varepsilon.$$

Then there exists $E \in \mathrm{Proj} A$ *such that*

$$\rho(E^\perp) < 2\varepsilon^{1/4}, \quad \big\|E\big(\sum_{k=1}^{m} D_k\big)E\big\| < 4\varepsilon^{1/2},\ m = 1, 2, \dots$$

COROLLARY 3.1. *For each bundle $\mathcal{P}_{(D_m)}$ with*

$$D_m \in A^+ \quad \textit{and} \quad \sum_{m=1}^{\infty} \rho(D_m) < \infty$$

and for each $\varepsilon > 0$ there exists $p \in \mathcal{P}_{(D_m)}$ such that $\rho(p^{\perp}) < 2\varepsilon$.

Proof. According to Remark 3.1 let us suppose that $\sum_{m=1}^{\infty} \rho(D_m) < \varepsilon$. Let $0 < \alpha_m \nearrow \infty$ be a sequence of numbers such that $\sum_{m=1}^{\infty} \alpha_m \rho(D_m) < \varepsilon$. Put $B_m = \alpha_m D_m$, $m = 1, 2, \dots$, and applying Lemma 3.2 to B_m and ε^4 we get the existence of $p \in \mathcal{P}_{(B_m)}$ such that

$$\rho(p^{\perp}) < 2\varepsilon, \quad \Big\| p\big(\sum_{k=1}^{m} \alpha_k D_k\big) p \Big\| < 4\varepsilon^2 < \infty,$$

but

$$\|pD_m p\| \leqslant \alpha_m^{-1} \Big\| \sum_{k=1}^{m} p(\alpha_k D_k) p \Big\| \leqslant 4\varepsilon^2 \alpha_m^{-1} \quad \text{and} \quad \|pD_m p\| \to 0,$$

so $p \in \mathcal{P}_{(D_m)}$. ∎

Obviously, this corollary implies the following

PROPOSITION 3.1. *If $x_n \xrightarrow{b,A} x$, then $x_n \xrightarrow{a.u.} x$.*

REMARK 3.2. *Let $\{x_n\}$ be uniformly bounded and $x_n \xrightarrow{a.u.} x$ in A, $x \in A$. Then $x_n \to x$ in the strong operator topology.*

Indeed, let x_n, $x \in A$. Then we obviously have $x_n \xrightarrow{a.u.} x$ in $W(A)$. From Lemma 2.1 we derive that $x_n \to x$ in the strong operator topology, i.e. for every $\xi \in H$, $\|x_n\xi - x\xi\|_H \to 0$.

PROPOSITION 3.2. *Let $\{x_n\} \subset A$ and $\sum_{n=1}^{\infty} \rho(|x_n|^2) < \infty$. Then $x_n \xrightarrow{b,A} 0$.*

Proof. The sequence $D_m = |x_m|^2,\ m = 1, 2, \dots$, defines a bundle $\mathcal{P}_{(D_m)}$. Let $p \in \mathcal{P}_{(D_m)}$. Then

$$\|pD_m p\| \to 0 \quad \text{and} \quad \|px_n^2 p\| = \|x_n p\|^2 = \|p|x_n|^2 p\| = \|pD_m p\|. \ ∎$$

THEOREM 3.1. *Let A be a reversible JW-algebra with f.n. state ρ, and ρ_1 be its extension to the enveloping von Neumann algebra $W(A)$. If a sequence $(x_n) \subset A$ is bundle convergent in $\big(W(A); \rho_1\big)$, then it is bundle convergent in $(A; \rho)$.*

P r o o f. Suppose that $x_n \overset{b,W(A)}{\longrightarrow} x$. Then there is a sequence $\{D_m\} \subset W(A)^+$, $m = 1, 2, \dots$, with $\sum_{m=1}^{\infty} \rho_1(D_m) < \infty$ and the corresponding bundle

$$P_{(D_m)} = \big\{p \in \mathrm{Proj}W(A), p \neq 0\colon \\ \sup_m \big\|p\big(\sum_{k=1}^{m} D_k\big)p\big\| < \infty \text{ and } \|pD_m p\| \to 0 \text{ as } m \to \infty\big\}$$

such that for each $p \in P_{(D_m)}$ we have $\|(x_n - x)p\| \to 0$. Due to Theorem 2.1 we get $D_m = K_m + iL_m \geqslant 0$, and $K_m \in A^+$. Hence, according to the definition of ρ_1 we derive $\rho(K_m) = \rho_1(K_m + iL_m)$. Therefore,

$$\sum_{m=1}^{\infty} \rho(K_m) < \infty.$$

By $P_{(K_m)}$ we denote the bundle in $W(A)$, generated by the sequence $\{K_m\}$. Let $p \in P_{(D_m)}$. Then (ii) of Theorem 2.1 yields that $\|pK_m p\| \leqslant \|pK_m p + ipL_m p\| = \|pD_m p\| \to 0$. Similarly, one can establish that

$$\sup_m \big\|p\big(\sum_{i=1}^{m} K_i\big)p\big\| < \infty.$$

Hence, $P_{(D_m)} \subset P_{(K_m)}$, and so $P_{(K_m)}$ is nonempty.

Now we show that $P_{(K_m)} \subset P_{(D_m)}$. Let us consider a mapping $\eta : W(A) \to W(A)$ defined by $\eta(x + iy) = x - iy$, which is a $*$-anti-automorphism of $W(A)$ (see [16]). If $x + iy = a^2 \geqslant 0$ for some $a \in W_{sa}(A)$, then $x - iy = \eta(x + iy) = \eta(a^2) = \big(\eta(a)\big)^2 \geqslant 0$, where $x^* = x$, $y = -y^*$. This yields $x - iy \geqslant 0$, and hence $x + iy \leqslant 2x$. Now applying the last inequality to $K_m + iL_m$ we infer that $D_m \leqslant 2K_m$. Assume that $p \in P_{(K_m)}$. Then $\|pK_m p\| \to 0$ as well as $2\|pK_m p\| \to 0$. Consequently, we find $\|pD_m p\| \leqslant 2\|pK_m p\| \to 0$, which implies that $p \in P_{(D_m)}$. So, $P_{(D_m)} = P_{(K_m)}$.

Put $\mathcal{P} = P_{(K_m)} \cap A$, which is the bundle in A generated by $\{K_m\}$. Then $\mathcal{P} \subset P_{D_m}$ and the proof is completed. ■

As a consequence of the last result we have the following

THEOREM 3.2. *Let A be a reversible JW-algebra with f.n. state ρ, and ρ_1 be its extension to the enveloping von Neumann algebra $W(A)$. Suppose that $\{x_n\}$ is a sequence of pairwise orthogonal operators in A, that is $\rho(x_i \circ x_j) = 0$ for $i \neq j$. If*

$$\sum_{n=1}^{\infty} \left(\frac{\log_2(n+1)}{n}\right)^2 \rho(|x_n|^2) < \infty,$$

then

$$\frac{1}{n} \sum_{j=1}^{n} x_j \overset{b,A}{\longrightarrow} 0.$$

P r o o f. Let us note that if $a, b \in A$ are orthogonal in A with respect to ρ, i.e. $\rho(a \circ b) = 0$, then for the extended state ρ_1 we have $\rho_1(ab + ba) = 2\rho(a \circ b) = 0$. Using the fact that $ab - ba$ is skew-symmetric and the definition of the extended state ρ_1, i.e. $\rho_1(x) = 0$ whenever x is skew-symmetric, we obtain $\rho_1(ab - ba) = 0$, which with $\rho_1(ab + ba) = 0$ implies that $\rho_1(ab) = 0$, i.e. a and b are orthogonal in $W(A)$. Hence, the elements x_n, $n \in \mathbb{N}$, are pairwise orthogonal in $W(A)$, so due to Theorem 4.2 in [17] we get

$$\frac{1}{n}\sum_{j=1}^{n} x_j \xrightarrow{b,W(A)} 0.$$

Consequently, by means of Theorem 3.1 we obtain the desired statement. ∎

LEMMA 3.3. *Let* $x_n, x \in A$ $(n = 1, 2, \ldots)$, *and* $x_n \xrightarrow{b,A} x$. *Then*

$$\frac{1}{n}\sum_{k=1}^{n} x_k \xrightarrow{b,A} x.$$

Recall that the *covariance* of two operators $a, b \in A$ is defined by $\operatorname{cov}(a, b) = \rho(a \circ b) - \rho(a)\rho(b)$, and the *variance* of a by $\operatorname{var}(a) = \operatorname{cov}(a, a) = \rho(a \circ a) - |\rho(a)|^2 = \rho(a^2) - |\rho(a)|^2$. If $\operatorname{cov}(a, b) = 0$, then a and b are called *uncorrelated*.

THEOREM 3.3. *Let A be a reversible JW-algebra, acting on the Hilbert space H, with f.n. state ρ. Suppose that $\{x_n\}$ is a sequence of uncorrelated operators in A, and the following conditions are satisfied:*

(i) $x_n \xrightarrow{b,A} x \quad (x \in A)$,

(ii) $\sum_{n=1}^{\infty} n^{-2}\operatorname{var}(x_n)\log^2(n+1) < \infty$.

Then there exists a complex number c such that $x = c\mathbf{1}$, where, as before, $\mathbf{1}$ is the identity in A.

P r o o f. Put $y_n = x_n - \rho(x_n)\mathbf{1}$. Then $\rho(y_m \circ y_n) = 0$, $m \neq n$, and $\rho(y_n^2) = \operatorname{var}(x_n)$, $m, n = 1, 2, \ldots$ Consequently, Theorem 3.2 implies

$$\frac{1}{n}\sum_{k=1}^{n} y_k \xrightarrow{b,A} 0. \tag{3.1}$$

Let us put

$$M_n = \frac{1}{n}\sum_{k=1}^{n} x_k \quad \text{and} \quad \lambda_n = \rho(M_n) \quad \text{for every } n \geqslant 1.$$

It then follows from (3.1) that $M_n - \lambda_n\mathbf{1} \xrightarrow{b,A} 0$. From (i) with Lemma 3.3 we obtain $M_n \xrightarrow{b,A} x$. Now the additivity of bundle convergence yields $\lambda_n\mathbf{1} \xrightarrow{b,A} x$, i.e.

there exists a bundle $\mathcal{P}$ such that, for $p \in \mathcal{P}$, $\|p(\lambda_n \mathbf{1} - x)^2 p\| \to 0$. This means that the sequence $\{\lambda_n p\}$ is uniformly bounded in the norm $\|\cdot\|$, which implies that the sequence of complex numbers $\{\lambda_n\}$ is bounded as well. Then from Remark 3.2 we have $\lambda_n \mathbf{1} \to x$ in the strong operator topology, i.e. for every $\xi \in H$ we get

$$\|\lambda_n \xi - x\xi\|_H \to 0.$$

Since $\{\lambda_n\}$ is a bounded sequence, there exists a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \lambda$ for some $\lambda \in \mathbb{C}$. This yields that $\lambda_{n_k} \mathbf{1} \to \lambda \mathbf{1}$ in the strong operator topology. By the strong convergence of $\lambda_{n_k} \mathbf{1}$ to x, we derive $x = \lambda \mathbf{1}$. This completes the proof. ∎

Note that if instead of a state ρ one takes f.n. finte trace, then assumption (i) could be replaced with $x_n \xrightarrow{a.u.} x$. Note that, in general, such a result might not be valid, since almost uniform convergence is not additive (see [24]).

4. CONDITIONAL EXPECTATION IN JW-ALGEBRAS

In this section, certain properties of conditional expectations of reversible JW-algebras are studied. Using such properties we prove a main result of this section, formulated in Theorem 4.4, which is a Jordan analog of the following result:

THEOREM 4.1 ([8]). *Let W be a von Neumann algebra with a faithful normal semifinite trace τ. Let $\{x_\alpha\}$ be a supermartingale in $L_1(W,\tau)$. If $\{x_\alpha\}$ is weakly relatively compact, then there is an $x \in L_1(W,\tau)$ such that $x_\alpha \to x$ in $L_1(W,\tau)$.*

Let A be a reversible JW-algebra with an f.n.s. trace τ. Let A_1 be its JW-subalgebra containing the identity operator $\mathbf{1}$.

Recall that a linear mapping $\phi : A \to A_1$ is said to be a *conditional expectation* with respect to a JW-subalgebra A_1 if the following conditions are satisfied:

(i) $\phi(\mathbf{1}) = \mathbf{1}$;

(ii) if $x \geqslant 0$, then $\phi(x) \geqslant 0$;

(iii) $\phi(xy) = \phi(x)y$ for $x \in A$, $y \in A_1$.

Let $\widetilde{\tau} := \tau\lceil A_1$ be the restriction of the trace τ to A_1 such that $\widetilde{\tau}$ is also semifinite. Then the space $L_1(A_1,\widetilde{\tau})$ of integrable operators with respect to $(A_1,\widetilde{\tau})$ is a subspace of $L_1(A,\tau)$.

THEOREM 4.2 ([10]). *Let A be a reversible JW-algebra with an f.n.s. trace τ and A_1 be its JW-subalgebra with $\mathbf{1}$. Let $\widetilde{\tau} = \tau\lceil A_1$ be the restriction of the trace τ to A_1 such that $\widetilde{\tau}$ is also semifinite. Then there exists a unique positive linear mapping $E(\cdot/A_1) : A \to A_1$ satisfying the condition $\tau\big(E(a/A_1)b\big) = \tau(ab)$ for $a \in A$, $b \in L_1(A_1,\widetilde{\tau})$, and the mapping $E(\cdot/A_1)$ is a conditional expectation with respect to A_1.*

Note that when trace is finite, an analogous result has been proved in [1]. In what follows $E(\cdot/A_1)$ and its extension to $L_1(A,\tau)$ is called a *τ-invariant conditional expectation* with respect to A_1 (see [10]).

We note that if $W(A)$ is the enveloping von Neumann algebra of a JW-algebra A, then the existence of conditional expectations from $W(A)$ onto A has been proved in [15].

Let A be a reversible JW-algebra with f.n.s. trace τ and τ_1 be its extension to $W(A)$. Further, we suppose that the restriction of τ to any considered subalgebras is semifinite. If B is a reversible JW-subalgebra of A, then we denote a τ-invariant conditional expectation with respect to B by $E(\cdot/B)$. Similarly, a τ_1-invariant conditional expectation from $W(A)$ to a subalgebra $W(B)$ is denoted by $\widetilde{E}\big(\cdot/W(B)\big)$, i.e. a conditional expectation with

$$\tau_1\Big(\widetilde{E}\big(a/W(B)\big)x\Big)=\tau_1(ax),\quad a\in W(A),\quad x\in L_1\big(W(B),\tau_1\big), \tag{4.1}$$

which is unique (see [28]). Here $L_1\big(W(B),\tau_1\big)=L_1\big(R(B),\tau_1\big)+iL_1\big(R(B),\tau_1\big)$, and $L_1\big(R(B),\tau_1\big)$ is the completion of $R(B)\cap\mathfrak{N}_{\tau_1}$ with respect to L_1-norm.

LEMMA 4.1. *Let $z\in\mathfrak{N}_{\tau_1}\cap R(A)$, and $z^*=-z$. Then $\tau_1(z)=0$.*

P r o o f. Let $z\in\mathfrak{N}_{\tau_1}\cap R(A)$, and $z^*=-z$, so $(iz)^*=iz$, $iz\in\mathfrak{N}_{\tau_1}$. From the definition of $\mathfrak{N}_{\tau_1}$ we have $|z|\in\mathfrak{N}_{\tau_1}$. Hence $|z|\in\mathfrak{N}_{\tau}$. Since $|z|+iz\geqslant 0$, we get $\tau_1(|z|+iz)=\tau_1(|z|)$. According to the linearity of τ_1 on $\mathfrak{N}_{\tau_1}$, $\tau_1(|z|+iz)=\tau_1(|z|)+i\tau_1(z)$, so $\tau_1(|z|)=\tau_1(|z|)+i\tau_1(z)$ and $\tau_1(z)=0$. ∎

It is natural to ask how the restriction of $\widetilde{E}\big(\cdot/W(B)\big)$ to A is related to $E(\cdot/B)$. The next result answers this question.

THEOREM 4.3. *It follows that the restriction of τ_1-invariant conditional expectation $\widetilde{E}\big(\cdot/W(B)\big)$ to A is equal to $E(\cdot/B)$, where, as before, $E(\cdot/B)$ is τ-invariant conditional expectation.*

P r o o f. Due to (4.1) and the uniqueness of $\widetilde{E}\big(\cdot/W(B)\big)$ it is sufficient to prove that for every $a\in A$

$$\tau_1(ax)=\tau_1\big(E(a/B)x\big) \tag{4.2}$$

holds for any $x\in L_1\big(W(B),\tau_1\big)$. First we prove (4.2) for every x taken from $L_1\big(R(B),\tau_1\big)$. A functional $\tau_1(h\cdot)$ is L_1-continuous on $L_1\big(W(B),\tau_1\big)$ for each $h\in W(B)$, therefore it is enough to prove (4.2) for any x taken from $R(B)\cap\mathfrak{N}_{\tau_1}$.

Let $x\in R(B)\cap\mathfrak{N}_{\tau_1}$. Since $x=y+z$, $y,z\in R(B)\cap\mathfrak{N}_{\tau_1}$ such that $y^*=y$, $z^*=-z$, we have

$$\tau_1(ax)=\tau_1(ay)+\tau_1(az).$$

Thus, we need to prove that $\tau_1\big(E(a/B)x\big) = \tau_1\big(E(a/B)y\big) + \tau_1\big(E(a/B)z\big)$. By Lemma 4.1, we obtain $\tau_1(z) = 0$, and due to $(az + za)^* = -(az + za)$, again using Lemma 4.1 we get $\tau_1(az + za) = 0$, which means $\tau_1(az) + \tau_1(za) = 0$, so $2\tau_1(az) = 0$, i.e. $\tau_1(az) = 0$. Consequently, we have $\tau_1\big(E(a/B)z\big) = 0$. Since $y \in B \cap \mathfrak{N}_{\tau_1}$, by Theorem 4.2 we get $\tau_1(ay) = \tau_1\big(E(a/B)y\big)$, that is, $\tau_1(ax) = \tau_1\big(E(a/B)x\big)$ for any $x \in R(B)$. Let x be an arbitrary element taken from $L_1\big(W(B), \tau_1\big)$. Then $x = u + iv$, where $u, v \in L_1\big(R(B), \tau_1\big)$, so

$$\begin{aligned}\tau_1(ax) &= \tau_1\big(a(u + iv)\big) = \tau_1(au) + i\tau_1(av)\\ &= \tau_1\big(E(a/B)u\big) + i\tau_1\big(E(a/B)v\big)\\ &= \tau_1\big(E(a/B)(u + iv)\big) = \tau_1\big(E(a/B)x\big). \quad \blacksquare\end{aligned}$$

Suppose that $\{A_\alpha\}_{\alpha\in\mathbb{R}_+}$ is a family of reversible JW-subalgebras of A containing the identity operator $\mathbf{1}$ such that the set $\bigcup_\alpha A_\alpha$ is weakly dense in A.

DEFINITION 4.1. A family $\{x_\alpha\}_{\alpha\in\mathbb{R}_+} \subset L_1(A, \tau)$ is called a *supermartingale* if for every $\alpha \in \mathbb{R}_+$

(1) $x_\alpha \in L_1(A_\alpha, \tau)$;

(2) if $\alpha_1 \leqslant \alpha_2$, then $E(x_{\alpha_2}/A_{\alpha_1}) \leqslant x_{\alpha_1}$.

Note that when we replace $\leqslant$ with the equality sign, then the family $\{x_\alpha\}$ is called *martingale*. If in Definition 4.1 as a JW-algebra A we take a self-adjoint part of a von Neumman algebra W, then we get usual definitions of supermartingale and martingale, respectively, in a von Neumann algebra setting.

It is known [6] that $\big(L_1(W, \tau_1)\big)^* = W$, $\big(L_1(A, \tau)\big)^* = A$, and therefore by $\sigma_W = \sigma_W\big(L_1(W, \tau_1), W\big)$, $\sigma_A = \sigma_A\big(L_1(A, \tau), A\big)$ we denote weak topologies on $L_1(W, \tau_1)$, $L_1(A, \tau)$, respectively. Note that σ_W (resp. σ_A) is generated by a family of seminorms $P_a(x) = |\tau_1(ax)|$, $a \in W$ (resp. $P_a(x) = |\tau(a \circ x)|, a \in A$).

LEMMA 4.2. *Let A be a reversible JW-algebra with an f.n.s. trace τ. Then we have $\sigma_{W(A)}|_{L_1(A,\tau)} = \sigma_A$.*

P r o o f. Let $\{x_\alpha\}$, $x \in L_1(A, \tau)$ and $x_\alpha \overset{\sigma_{W(A)}}{\longrightarrow} x$. Then $\tau_1(ax_\alpha) \to \tau_1(ax)$ for every $a \in W(A)_+$. If, in particular, $a \in A_+ \subset W(A)_+$, then $\tau(a \circ x_\alpha) = \frac{1}{2}\big(\tau(ax_\alpha) + \tau(x_\alpha a)\big) = \tau(ax_\alpha) \to \tau(ax) = \tau(a \circ x)$. Hence $x_\alpha \xrightarrow{\sigma_A} x$.

Conversely, let $x_\alpha \xrightarrow{\sigma_A} x$ and let us take $b \in W(A)_+$ with $b = c + id$. Then $0 \leqslant c \in A$, $d^* = -d$ (see [7]). Thus, $\tau(cx_\alpha) \to \tau(cx)$ since $(dx_\alpha + x_\alpha d)/2$ is a skew-symmetric element in $R(A)$. Then, by Lemma 4.1, we get

$$\tau_1(dx_\alpha) = \tau_1\left(\frac{dx_\alpha + x_\alpha d}{2}\right) = 0.$$

Hence $\tau_1(bx_\alpha) = \tau(cx_\alpha) \to \tau(cx) = \tau_1(bx)$. $\blacksquare$

Now we are ready to prove a main result of this section.

THEOREM 4.4. *Let $\{x_\alpha\}$ be a supermartingale in $L_1(A,\tau)$. If the set $\{x_\alpha\}$ is weakly relatively compact in $L_1(A,\tau)$, then there is $x \in L_1(A,\tau)$ such that $x_\alpha \to x$ in L_1-norm.*

Proof. Let $W(A_\alpha)$ be an enveloping von Neumann algebra of A_α. Then $x_\alpha \in L_1(A_\alpha,\tau) \subset L_1\big(W(A_\alpha),\tau_1\big)$. Moreover, since $\widetilde{E}\big(\cdot\, /W(A_\alpha)\big)$ is a conditional expectation on $W(A)$, by Theorem 4.3, for $\alpha_1 \leqslant \alpha_2$, we have $\widetilde{E}\big(x_{\alpha_2}/W(A_{\alpha_1})\big) = E(x_{\alpha_2}/A_{\alpha_1}) \leqslant x_{\alpha_1}$, i.e. $\{x_\alpha\}$ is a supermartingale in $L_1\big(W(A),\tau_1\big)$. Thus Lemma 4.2 implies that $\{x_\alpha\}$ is weakly relatively compact in $L_1\big(W(A),\tau_1\big)$. Then Theorem 4.1 yields the existence of $x \in L_1\big(W(A),\tau_1\big)$ such that $x_\alpha \to x$ in L_1-norm. The completeness of $L_1(A,\tau)$ with respect to L_1-norm implies that x belongs to $L_1(A,\tau)$. This completes the proof. ∎

Note that in the case when the family $\{x_\alpha\}$ is a martingale a similar result was proved in [10]. Therefore, our result extends the mentioned one. When the trace is finite a similar result was studied in [1] and [5].

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