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ON COMPLETENESS OF RANDOM TRANSITION COUNTS FOR MARKOV CHAINS. II

BY

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Abstract. It is shown that the random transition count is complete for Markov chains with a fixed length and a fixed initial state, for some subsets of the set of all transition probabilities. The main idea is to apply graph theory to prove completeness in a more general case than in Palma [5].

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1. INTRODUCTION

Let us consider the set $\mathbf{X}$ of all trajectories of homogeneous Markov chains with a fixed length $N \geqslant 2$, a finite state space $S = \{1, \ldots, n\}$, and a fixed initial state $x_1 = i'$. Let $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbf{X}$ denote a trajectory from the chain. The parameter space for $\mathbf{X}$ is

$$\mathcal{P} = \Big\{p = (p_{i,j}) : \forall_{i,j \in S}\, p_{i,j} \geqslant 0,\ \sum_{j=1}^{n} p_{i,j} = 1 \text{ for each } i \in S\Big\}.$$

Let $\mathcal{Z} \subset S \times S$ denote a fixed subset satisfying

$$\forall_{i \in S}\, \exists_{j \in S}\, (i,j) \notin \mathcal{Z}. \tag{1.1}$$

We denote by $\mathcal{P}_{\mathcal{Z}}$ the set of stochastic matrices $p \in \mathcal{P}$ such that

$$\forall_{(i,j) \in \mathcal{Z}}\, p_{i,j} = 0. \tag{1.2}$$

This condition feels pretty well to the characterization of some classical types of Markov chains.

EXAMPLE 1.1. Assume that $S = \mathbb{Z}$. Then $(x_1, \ldots, x_N)$ is a random walk with stationary transition probability, possibly depending on position and direction, if and only if the matrix p is taken from $\mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(i, i+\epsilon); i \in S,\ \epsilon = \pm 1\}.$$

The generalizations for random walks with reflexing and absorbing barriers can be obtained by adding some elements to $\mathcal{Z}$.

Throughout the paper we are dealing with a statistical space of the form

$$\bigl(\mathbf{X}, \{P_{(p_{i,j})} :\ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\}\bigr),$$

where for $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbf{X}$

$$P_{(p_{i,j})}(\{\mathbf{x}\}) = p_{x_1,x_2} \cdot\ \ldots\ \cdot p_{x_{N-1},x_N}.$$

The *random transition count* F is defined in the usual way as a matrix

$$F(\mathbf{x}) = (f_{i,j})_{i,j=1,\ldots,n},$$

$$f_{i,j} = \#\{t = 1, \ldots, N-1 :\ x_t = i,\ x_{t+1} = j\} \quad \text{for } i, j \in S.$$

Obviously, F is a basic tool in any statistical investigation. In particular, F is a sufficient statistic. The concept of completeness of a sufficient statistic is due to Lehmann and Scheffé [4], [3]. F is *complete* if all matrices from $\mathcal{P}$ are allowed (see [2], [1]). Moreover, the statistic F is always complete for a Markov bridge when the initial and final states are fixed (see [6]).

It is worth noting that consequences of the completeness of a statistic for testing theory belong to classical topics. In particular, these play an essential role in the theory of uniformly most powerful unbiased tests and in minimum variance unbiased estimation. An exposition of this theory can be found in [4], [3].

Unfortunately, the random transition count F could be incomplete for some space $\mathcal{P}_{\mathcal{Z}}$.

EXAMPLE 1.2. Let $S = \{1, 2, 3, 4\}$, $N = 5$, and let $x_1 = 1$ be fixed. For the space $\mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(1,2), (1,3), (2,3), (3,4), (4,2), (3,1)\}$$

the statistic F is not complete.

Moreover, note that if the initial state x_1 is not fixed, then the statistic F is not sufficient. The natural sufficient statistic which should be investigated in such a situation is

$$G(\mathbf{x}) = \bigl(x_1, F(\mathbf{x})\bigr),$$

but in general G is not complete.

EXAMPLE 1.3. Let $S = \{1, 2\}$, $N = 2$. The statistic $G(x_1, x_2) = \bigl(x_1, (f_{i,j})\bigr)$ is sufficient, but not complete (see [5]).

Thus it is necessary to put some extra assumptions on the set $\mathcal{Z}$ and S (cf. conditions (I) and (II) in Section 4).

In theoretical investigation we will assume that there exists a completeness of the random transition count for some special classes of Markov chains for which the state space S splits into one class S_0 of inessential states and into classes of equivalence $S_1, \dots, S_{\beta_0},\ \beta_0 \in \mathbb{N}$, of essential states for the class $\mathcal{P}_{\mathcal{Z}} \subset \mathcal{P}$.

The following proposition shows that condition (I) in Section 4 is necessary to obtain completeness of F.

PROPOSITION 1.1. *Let $S = \{1,2,3,4,5\}, N = 6$, and let $x_1 = 1$ be fixed. For some space $\mathcal{P}_{\mathcal{Z}}$ the statistic F is not complete.*

P r o o f. Let the set $\mathcal{Z}$ be defined by

$$(S \times S) \setminus \mathcal{Z} = \{(1,2),(2,1),(2,3),(3,4),(4,3),(4,5),(5,4)\}.$$

Then the state space S splits into one class $S_0 = \{1,2\}$ of inessential states and into one class $S_1 = \{3,4,5\}$ of essential states for the class $\mathcal{P}_{\mathcal{Z}} \subset \mathcal{P}$. Any matrix $p \in \mathcal{P}_{\mathcal{Z}}$ can be written as

$$p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1-r & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-q & 0 & q \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad r, q \in [0,1].$$

The statistic $F(\mathbf{x}) = (f_{i,j})$ takes on the following values with corresponding probabilities:

$$M_1 = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P\big(F(\mathbf{x}) = M_1\big) = (1-r)^2, \qquad P\big(F(\mathbf{x}) = M_2\big) = (1-r) \cdot r,$$

$$M_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad M_4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P\big(F(\mathbf{x}) = M_3\big) = r \cdot q, \qquad P\big(F(\mathbf{x}) = M_4\big) = r \cdot (1-q).$$

Then the expectation $\mathbb{E}(g \circ F)$ vanishes for any non-zero function g satisfying

$$g(M_1) = g(M_2) = 1, \quad g(M_3) = g(M_4) = -(1-r)/r. \quad \blacksquare$$

Our purpose is to find possibly the largest class of Markov chains for which the random transition count will be a complete statistic. The main result of the paper is Theorem 4.2 in Section 4, where we describe some classes of sets $\mathcal{Z}$ such that F is a complete statistic for distributions parametrized by matrices from $\mathcal{P}_{\mathcal{Z}} \subset \mathcal{P}$. The main idea is to apply graph theory to prove completeness in a more general case than in [5]. We suppose that our result is the strongest possible in the given setup. Its proof involves polynomial theory and graph theory as well. Section 3 presents some definitions and lemmas of graph theory. In Section 2 we give some auxiliary results, the classification of states and we introduce a specific notation concerning "tables" of numbers.

2. AUXILIARY RESULTS AND THE CLASSIFICATION OF STATES

We start with proving the following auxiliary results, which will be needed in Section 4. Fix $i', i'' \in S = \{1, \ldots, n\}$.

For any matrix f of dimension $n \times n$, let $\bar{f} = (\bar{f}_{i,j})$ denote a "table" being the matrix f with deleted elements $f_{i,i+1}$, $i = 1, \ldots, n$, $i \neq i''$ ($f_{n,n+1}$ is another notation for $f_{n,1}$). More precisely, we use a set of indices

$$S_2^{i',i''} = \big\{(i,j) \in S \times S : \ j \in S \setminus \{i+1\} \text{ for } i \in S \setminus \{i'', n\},$$
$$j \in S \text{ for } i = i'', \ j \in S \setminus \{1\} \text{ for } i = n \text{ in the case } n \neq i''\big\}$$

and define

$$\bar{f}_{i,j} = f_{i,j} \quad \text{for } (i,j) \in S_2^{i',i''}, \tag{2.1}$$

$$\bar{f} = (\bar{f}_{i,j})_{(i,j) \in S_2^{i',i''}}. \tag{2.2}$$

LEMMA 2.1. *Fix $i', i'' \in S$. Let $\mathcal{M}^{i',i''}$ denote a set of matrices $f = (f_{i,j})$ of dimension $n \times n$ satisfying*

$$\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i) \quad \textit{for } i \in S, \tag{2.3}$$

with $\delta_{i'}(i)$ being the Kronecker delta. The function $f \to \bar{f}$ defined by (2.1) *and* (2.2) *is one-to-one on the class $\mathcal{M}^{i',i''}$. There exist functions $\psi_i^{i',i''}$ for $i \neq i''$, defined on the tables $(f_{i,j})_{(i,j) \in S_2^{i',i''}}$, with non-negative integer values satisfying*

$$\psi_i^{i',i''}(\bar{f}) = f_{i,i+1} \quad \textit{for } i \in S \setminus \{i''\} \textit{ and any } f \in \mathcal{M}^{i',i''}.$$

P r o o f. C a s e 1. Assume that $i'' = n$. We define the functions $\psi_i^{i',i''}$ by induction for $i = 1, 2, \ldots, n-1$. By formula (2.3) with $i = 1$, one can obviously put

$$\psi_1^{i',i''}(\bar{f}) = \sum_j \bar{f}_{j,1} + \delta_{i'}(1) - \sum_{j \neq 2} \bar{f}_{1,j}.$$

Then formula (2.3) gives successively

$$\psi_i^{i',i''}(\bar{f}) = \psi_{i-1}^{i',i''}(\bar{f}) + \sum_{j \neq i-1} \bar{f}_{j,i} + \delta_{i'}(i) - \sum_{j \neq i+1} \bar{f}_{i,j} \quad \text{for } i = 2, \ldots, n-1.$$

C a s e 2. Assume that $i'' \neq n$. We define $\psi_i^{i',i''}$ by induction for $i = i''+1, \ldots, n, 1, \ldots, i''-1$. Formula (2.3) with $i = i''+1$ gives

$$\psi_{i''+1}^{i',i''}(\bar{f}) = \sum_j \bar{f}_{j,i''+1} + \delta_{i'}(i''+1) - \sum_{j \neq i''+2 (\mathrm{mod}\ n)} \bar{f}_{i''+1,j}.$$

Then by formula (2.3) we get successively

$$\begin{aligned} (2.4) \quad \psi_i^{i',i''}(\bar{f}) &= \psi_{i-1}^{i',i''}(\bar{f}) \\ &+ \sum_{j \neq i-1} \bar{f}_{j,i} + \delta_{i'}(i) - \sum_{j \neq i+1 (\mathrm{mod}\ n)} \bar{f}_{i,j} \quad \text{for } i = i''+2, \ldots, n. \end{aligned}$$

Taking $i = 1$ in (2.3), we obtain

$$\psi_1^{i',i''}(\bar{f}) = \psi_n^{i',i''}(\bar{f}) + \sum_{j \neq n} \bar{f}_{j,1} + \delta_{i'}(1) - \sum_{j \neq 2} \bar{f}_{1,j}.$$

Finally, we use (2.4) for $i = 2, \ldots, i''-1$. ∎

COROLLARY 2.1. *The function* $f \to \bar{f}$ *defined by* (2.1) *and* (2.2) *is one-to-one on the class* $\{F(\bar{\mathbf{x}}) : \bar{\mathbf{x}} = (x_1, \ldots, x_t),\ x_1 = i',\ x_t = i'',\ 1 \leqslant t < N\}$ *of all values of random transition counts for the trajectories* $\bar{\mathbf{x}}$ *with a fixed initial state* i' *and a fixed final state* i'' (*and any length*).

For the sake of completeness we give the following lemma which was used (in almost the same form) by Denny and Wright [1]. Assume that integers $n \geqslant 1$, $j(1) \geqslant 1, \ldots, j(n) \geqslant 1,\ q \geqslant 0$ and real $c > 0$ are fixed. Denote by $\mathcal{U}$ the set of all systems of positive numbers $u = (u_{i,j})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant j(i)$, such that

$$\sum_{j=1}^{j(i)} u_{i,j} \leqslant c \quad \text{for any } 1 \leqslant i \leqslant n.$$

Denote by M the set of systems of non-negative integers $m = (m_{i,j})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant j(i)$, satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{j(i)} m_{i,j} \leqslant q.$$

Let $\varphi_i\colon M \to \{0,1,2,\ldots\}$, $i=1,\ldots,n$, be any given functions. For each $m \in M$ let us define a function $W_m : \mathcal{U} \to \mathbb{R}$ as follows:

$$W_m(u) = \prod_{i=1}^{n} \Big[\prod_{j=1}^{j(i)} u_{i,j}^{m_{i,j}} \cdot (c - u_{i,1} - \ldots - u_{i,j(i)})^{\varphi_i((m_{i,j}))}\Big].$$

LEMMA 2.2. *The system of functions $W_m : \mathcal{U} \to \mathbb{R}$, indexed by $m \in M$, is linearly independent.*

P r o o f. Cf. Lemma 2 in Denny and Wright [1]. ∎

For the reader's convenience we repeat the classification of states for the class $\mathcal{P}_{\mathcal{Z}}$ from Palma [5].

Let us fix a set $\mathcal{Z}$ satisfying (1.1). For $p \in \mathcal{P}_{\mathcal{Z}}$ (cf. (1.2)), we use standard notation $\big(p_{i,j}(n)\big)_{i,j\in S} = p(n) = p^n$. To simplify our considerations, let us observe that there exists a matrix $p^{\mathcal{Z}} \in \mathcal{P}_{\mathcal{Z}}$ with a maximal set of positive elements:

$$p^{\mathcal{Z}}_{i,j} > 0 \quad \text{if only } (i,j) \notin \mathcal{Z}.$$

We say that a state i is *inessential* for the class $\mathcal{P}_{\mathcal{Z}}$ if

$$\exists_j \,\big[\big(\exists_{t_0\geqslant 1}\ p^{\mathcal{Z}}_{i,j}(t_0) > 0\big) \wedge \big(\forall_{t\geqslant 1}\ p^{\mathcal{Z}}_{j,i}(t) = 0\big)\big].$$

A state is *essential* if it is not inessential. We define an equivalence relation in the class of essential states:

$$i \sim j \quad \text{if } \exists_{s,t\geqslant 1}\,\big(p^{\mathcal{Z}}_{i,j}(s) > 0 \wedge p^{\mathcal{Z}}_{j,i}(t) > 0\big).$$

Consequently, the set of essential states for the class $\mathcal{P}_{\mathcal{Z}}$ splits into classes of equivalence. We denote them by $S_1, S_2, \ldots, S_{\beta_0}$, $\beta_0 \in \mathbb{N}$.

Now we recall the main result from paper [5] which will be useful in proving our main theorem in Section 4.

Let the set $\mathcal{Z}$ be such that the state space S is the whole class of essential states. Thus we assume that

(2.5) there exists a permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of the set S satisfying

$$\{(\pi_1,\pi_2),\ldots,(\pi_{n-1},\pi_n),(\pi_n,\pi_1)\} \cap \mathcal{Z} = \emptyset.$$

THEOREM 2.1. *Let $\mathcal{Z}$ satisfy* (1.1) *and* (2.5). *Let $\big(\mathbf{X}, \{P_{(p_{i,j})} : (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\}\big)$, with $\mathcal{P}_{\mathcal{Z}}$ given by* (1.2), *be the statistical space of all trajectories of Markov chains with the state space $S = \{1,\ldots,n\}$, a fixed initial state $x_1 = i'$, and a fixed trajectory size $N \geqslant 2$. Then the random transition count F is complete.*

3. SOME DEFINITIONS AND LEMMAS OF GRAPH THEORY

The following lemmas from graph theory go back to the work by Paszkiewicz [6], and will be needed to prove the completeness of F in a more general case.

DEFINITION 3.1. Let $Z \subset S \times S$ be a fixed set, fix $i', i'' \in S$. We say that an oriented graph (Y, U), $Y \subset S$, with Y being the set of vertices and $U \subset Y \times Y$ being the set of edges, is defined by Z, i', i'' if

$$Y = \bigcup \{x_1, \ldots, x_N\}, \quad U = \bigcup \{(x_1, x_2), \ldots, (x_{N-1}, x_N)\},$$

where the unions are taken for all sequences (x_t) satisfying $(x_t, x_{t+1}) \notin Z$, $t = 1, \ldots, N-1$, and $x_1 = i'$, $x_N = i''$.

As usual, we say that (Y_0, V_0) is a *cycle* if

$$Y_0 = \{y_1, \ldots, y_s\},$$
$$V_0 = \{(y_1, y_2), \ldots, (y_{s-1}, y_s), (y_s, y_1)\}.$$

In particular, $\big(\{y_1\}, \{(y_1, y_1)\}\big)$ is a cycle. A graph (Y_1, V_1) is a *tree with root* y if for any $z \in Y_1$ there exists exactly one path (Y_z, V_z) of the form

$$Y_z = \{z_1 = z, z_2, \ldots, z_s = y\} \subset Y_1,$$

$$V_z = \{(z_1, z_2), \ldots, (z_{s-1}, z_s)\} \subset V_1,$$

where $z_1, \ldots, z_s$ are mutually different and $s \geqslant 1$, with $V_z = \emptyset$ for $s = 1$. The graph $(Y_1, V_1) = (\{y\}, \emptyset)$ is a tree as well. It is well known that each vertex $z \in Y_1$, $z \neq y$, is the origin of exactly one edge (z, z_1) in V_1.

LEMMA 3.1 (Paszkiewicz [6]). *For any graph (Y, U) defined by Z, i', i'' there exists a tree (Y, W), $W \subset U$, with root i''.*

Let F be the random transition count and let the evolution be given by transition probabilities p from $\mathcal{P}_{\mathcal{Z}}$ with a fixed set $\mathcal{Z}$. In the following lemma we use the notion of a tree to describe some general properties of F.

Let $Y = \{1, \ldots, n\}$ and let a graph (Y, U) be defined by Z, i', i''. The value $F(\mathbf{x}) = (f_{i,j})$ for any trajectory $\mathbf{x}$ and satisfies $f_{i,j} = 0$ for $(i, j) \notin U$. Thus $F(\mathbf{x})$ can be identified with some function $m : U \to \{0, 1, \ldots\}$, $m = f|_U$, and obviously

$$\sum_{j \in Y, (i,j) \in U} m_{i,j} + \delta_{i''}(i) = \sum_{j \in Y, (j,i) \in U} m_{j,i} + \delta_{i'}(i) \quad \text{for } i \in Y.$$

Denote by M the space of all such functions $m : U \to \{0, 1, 2, \ldots\}$.

LEMMA 3.2 (Paszkiewicz [6]). *Let (Y, U) be defined by Z, i', i''. For any tree (Y, V) with root i'' and with $V \subset U$, denote by $j(\cdot)$ a uniquely defined function* *on* $Y \setminus \{i''\}$ *satisfying* $\big(i, j(i)\big) \in V$. *Then there exist functions* Φ_i *on* $\{m|_{U\setminus V};\ m \in M\}$ *with non-negative integer values satisfying*

$$m_{i,j(i)} = \Phi_i(m|_{U\setminus V})$$

for any $i \in Y$, $i \neq i''$, $m \in M$.

4. THE MAIN RESULT

We know that the random transition count F can be incomplete for some space $\mathcal{P}_{\mathcal{Z}}$ (see [5]). Thus in order to obtain completeness it is necessary to make some extra assumptions on the set $\mathcal{Z}$ and S.

Let $\mathcal{Z} \subset S \times S$ be a fixed set satisfying (1.1), that is

$$\forall_{i\in S}\ \exists_{j\in S}\ \ (i,j) \notin \mathcal{Z}.$$

Let S_0 denote the class of inessential states and let $S_1, \ldots, S_{\beta_0},\ \beta_0 \in \mathbb{N}$, denote classes of equivalence in essential states. Our extra assumptions on S can be formulated as follows:

(I) For each $\beta,\ 1 \leqslant \beta \leqslant \beta_0$, there exists a permutation $(i_1^\beta, \ldots, i_{n(\beta)}^\beta)$ of the set S_β such that

$$\{(i_1^\beta, i_2^\beta), \ldots, (i_{n(\beta)-1}^\beta, i_{n(\beta)}^\beta), (i_{n(\beta)}^\beta, i_1^\beta)\} \cap \mathcal{Z} = \emptyset.$$

(II) For each $\beta,\ 1 \leqslant \beta \leqslant \beta_0$, there exists exactly one pair $(i_\beta, j_\beta) \in S_0 \times S_\beta$ such that

$$(i_\beta, j_\beta) \notin \mathcal{Z}.$$

Fix $\beta,\ 1 \leqslant \beta \leqslant \beta_0$. Our statistical space $\big(\mathbf{X}_{\beta,N},\ \{P^{\beta,N}_{(p_{i,j})} : \ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\}\big)$ is defined as follows:

(i) the space $\mathbf{X}_{\beta,N}$ of trajectories $(x_1, \ldots, x_N)$ is determined by a fixed length N, a fixed initial state $x_1 = i'$, and a fixed essential state class S_β such that the final state x_N belongs to S_β;

(ii) the probability distribution

$$P^{\beta,N}_{(p_{i,j})}(\{\mathbf{x}\}) = p_{x_1,x_2} \cdot\ \ldots\ \cdot p_{x_{N-1},x_N} \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_N) \in \mathbf{X}_{\beta,N}.$$

Without loss of generality we adopt $\beta = 1$.

We first show that the random transition count F is a complete statistic if additionally the following condition holds:

(III) There exists a permutation $(i_1^0, \ldots, i_{n(0)}^0)$ of the class S_0 of inessential states such that

$$\{(i_1^0, i_2^0), \ldots, (i_{n(0)-1}^0, i_{n(0)}^0), (i_{n(0)}^0, i_1^0)\} \cap \mathcal{Z} = \emptyset.$$

THEOREM 4.1. *Let $\mathcal{Z}$ satisfy* (1.1) *and conditions* (I)–(III), *and let the statistical space* $\big(\mathbf{X}_{1,N},\ \{P^{1,N}_{(p_{i,j})}:\ (p_{i,j})\in\mathcal{P}_{\mathcal{Z}}\}\big)$ *be given as above. Then the random transition count F is complete.*

P r o o f. The proof will be divided into two steps.

Assume first that $x_1=i'\in S_1$ is an essential state. Then one can assume that the space S is one class of essential states. Theorem 2.1 completes the proof in this case.

Now, let $x_1=i'\in S_0$ be an inessential state. Let us consider a trajectory $\mathbf{x}\in\mathbf{X}_{1,N}$. Denote by $t(\mathbf{x})<N$ the number of steps on inessential states, so we perform $N-1-t(\mathbf{x})$ steps on S_1.

Let d be any real function defined on all values of the transition count F. To show that the statistic F is complete, it is enough to prove that the condition

$$\forall_{p\in\mathcal{P}_{\mathcal{Z}}}\ \Big(\sum_{f=(f_{i,j})}\big(d(f)\cdot\xi(x_1,f)\cdot\prod_{i,j\in S_0}p_{i,j}^{f_{i,j}}\cdot p_{i'',i'''}\cdot\prod_{i,j\in S_1}p_{i,j}^{f_{i,j}}\big)=0\Big)\tag{4.1}$$

implies

$$d(f)=0\qquad\text{for each value } f \text{ of } F,$$

where $\xi(x_1,f)=\#\{\mathbf{x}: F(\mathbf{x})=f\}$ denotes the number of corresponding trajectories.

Let $(i^0_1,\dots,i^0_{n(0)})$ denote any permutation of S_0 satisfying condition (III). It is obvious that by changing suitably the notation, one can assume that

$$(i^0_1,\dots,i^0_{n(0)})=\big(1,\dots,n(0)\big).$$

Then

$$\big\{(1,2),\dots,\big(n(0)-1,n(0)\big),\big(n(0),1\big)\big\}\cap\mathcal{Z}=\emptyset.$$

We use Lemma 2.1, Corollary 2.1 (and the notation $\bar f_0$) for S_0, $f_0=(f_{i,j})_{i,j\in S_0}$, $n(0)$ instead of $S,\ f,\ n$. Then the factor

$$\prod_{i,j\in S_0}p_{i,j}^{f_{i,j}}\cdot p_{i'',i'''}$$

in (4.1) can be written as

$$\prod_{\substack{i\in S_0\\ i\neq i''}}\ \prod_{\substack{j\in S_0\\ j\neq i+1 (\mathrm{mod}\ n(0))}}p_{i,j}^{\bar f_{i,j}}\cdot\big(1-\sum_{\substack{j\in S_0\\ j\neq i+1 (\mathrm{mod}\ n(0))}}p_{i,j}\big)^{\psi_i(\bar f)}\tag{4.2}$$

$$\times\prod_{j\in S_0}p_{i'',j}^{\bar f_{i'',j}}\cdot\big(1-\sum_{j\in S_0}p_{i'',j}\big)^{\psi_{i''}(\bar f)};$$

obviously, $p_{i'',i'''} = 1 - \sum_{j\in S_0} p_{i'',j}$ and we have put $\psi_{i''}(\bar{f}) = 1$. For simplicity of the notation, we put

$$\begin{aligned} S_0^i &= \big\{ j \in S_0 :\ j \neq i+1 \ \big(\mathrm{mod}\, n(0)\big)\big\} \qquad \text{for } i \in S_0 \text{ and } i \neq i'', \\ S_0^{i''} &= S_0. \end{aligned}$$

Let us observe that by a suitable change of the notation the product (4.2) can be written as some polynomials $W_m(u)$ described in Section 2. Thus it is natural to replace (4.2) by $W_{\bar{f}_0}(p)$, where

$$p \in \Big\{ (p_{i,j})_{i\in S_0,\ j\in S_0^i} : \sum_{j\in S_0^i} p_{i,j} \leqslant 1 \text{ for any } i \in S_0 \Big\}.$$

Using the notation $\bar{d}(f) = d(f)\cdot \xi(x_1, f)$, we can write the equality (4.1) as

$$\sum_f \bar{d}(f)\cdot W_{\bar{f}_0}(p)\cdot \prod_{i,j\in S_1} p_{i,j}^{f_{i,j}} = 0 \qquad \text{for each } (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}},$$

where the sum is taken for all f being values of $F(\cdot)$. Applying Lemma 2.2 we obtain

$$\forall_{p\in\mathcal{P}_{\mathcal{Z}}} \forall_{(\widetilde{f}_{i,j})_{i\in S_0,\ j\in S_0^i}} \sum_{f,\ \bar{f}_0 = \widetilde{f}} \bar{d}(f)\cdot \prod_{i,j\in S_1} p_{i,j}^{f_{i,j}} = 0.$$

Observe that the table $\bar{f}$ for $f = F(\mathbf{x})$ uniquely defines $t(\mathbf{x})$, namely

$$t(\mathbf{x}) = \sum_{i\in S_0,\ j\in S_0^i} \bar{f}_{i,j} + \sum_{i\in S_0\setminus i''} \psi_i(\bar{f}) + 1.$$

Thus, the number of steps in the class S_1 on essential states equals $N - 1 - t(\mathbf{x})$, and is defined by $\bar{f}$. In S_1 the trajectory starts from a fixed essential state $i''' = j_1$ (cf. condition (II)). Thus Theorem 2.1 completes the proof. ∎

We can now formulate our main result.

THEOREM 4.2. *Let $\mathcal{Z} \subset S\times S$ satisfy* (1.1) *and conditions* (I) *and* (II). *Let the stationary space $\big(\mathbf{X}_{1,N},\ \{P^{1,N}_{(p_{i,j})} :\ (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\}\big)$ be given as above. Then the random transition count F is complete.*

P r o o f. For $x_1 \in S_1$ Theorem 2.1 completes proof.

Now, assume that $x_1 = i' \in S_0$. We want to show that the statistic F is complete, that is, that the condition (4.1) implies $d(f) = 0$ for each f.

Let us consider a trajectory $\mathbf{x} \in \mathbf{X}_{1,N}$. Similarly, let $t(\mathbf{x})$ denote the number of steps on inessential states in the class S_0. For the first part $(x_1, \ldots, x_{t(\mathbf{x})})$ of the trajectory **x** we will consider an oriented graph $(\bar{S}_0,\ U),\ U \subset \bar{S}_0 \times \bar{S}_0$, defined by $\mathcal{Z},\ i',\ i''$, that is,

$$\bar{S}_0 = \bigcup\{x_1, \dots, x_{t(\mathbf{x})}\}, \quad U = \bigcup\{(x_1, x_2), \dots, (x_{t(\mathbf{x})-1}, x_{t(\mathbf{x})})\},$$

where the unions are taken for all **x** satisfying

$$x_1 = i', \quad x_{t(\mathbf{x})} = i'' \quad \text{and} \quad (x_t, x_{t+1}) \notin \mathcal{Z}, \quad t = 1, \dots, t(\mathbf{x}) - 1.$$

By Lemma 3.1 there exists a tree $(\bar{S}_0, W),\ W \subset U$, with a root i''. Next, applying Lemma 3.2 and putting $p_{i'',i'''} = 1 - \sum_{j \in \bar{S}_0} p_{i'',j}$ and $f_{i'',i'''} = \Phi_{i''}(m|_{U \setminus W}) = 1$, where $m = f|_U$, we can write the factor

$$\prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'',i'''}$$

in (4.1) in the form

$$\prod_{\substack{i \in \bar{S}_0 \\ i \neq i''}} \prod_{\substack{j \in \bar{S}_0 \\ (i,j) \in U \setminus W}} p_{i,j}^{m_{i,j}} \cdot \big(1 - \sum_{\substack{j \in \bar{S}_0 \\ (i,j) \in U \setminus W}} p_{i,j}\big)^{\Phi_i(m|_{U \setminus W})}$$
$$\times \prod_{j \in \bar{S}_0} p_{i'',j}^{m_{i'',j}} \cdot \big(1 - \sum_{\substack{j \in \bar{S}_0 \\ (i,j) \in U \setminus W}} p_{i'',j}\big)^{\Phi_{i''}(m|_{U \setminus W})}$$

for a system of functions $\Phi_i,\ i \neq i''$, on the space $\{m|_{U \setminus W}\}$. The set $U \setminus W$ can be written as

$$U \setminus W = \{(i,j) :\ i \in \bar{S}_0,\ j \in \bar{S}_0^i\},$$

where

$$\bar{S}_0^i = \big\{j \neq i + 1 \ \big(\mathrm{mod}\ \bar{n}(0)\big)\big\} \quad \text{for } i \in S_0 \text{ and } i \neq i'',$$
$$\bar{S}_0^{i''} = \bar{S}_0.$$

The rest of the proof is analogous to that of Theorem 4.1. ∎

Because the sufficiency of F is obvious, from Bahadur's theorem (see [7]) we have

COROLLARY 4.1. *Under the assumptions of Theorem* 4.2, F *is a minimal sufficient statistic.*

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