PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 32, Fasc. 2 (2012), pp. 215–225

ON ELEMENTARY CHARACTERIZATIONS OF THE α-MODIFIED POISSON DISTRIBUTION

BY

KATARZYNA STELIGA (LUBLIN) AND DOMINIK SZYNAL (STALOWA WOLA)

Abstract. In this article we give a characterization of α-modified Poisson distributions extending Chatterji's result. Moreover, we consider the α-modified Poisson distributions of type j which are known as Delaporte distributions.

2000 AMS Mathematics Subject Classification: Primary: 60E05; Secondary: 62E10.

Key words and phrases: Poisson distribution, α-modified Poisson distribution, α-modified binomial distribution, Delaporte distribution, Riordan α symbol.

1. INTRODUCTION

There is an extensive literature on characterizations of the Poisson distribution. Feller [5] pointed out that if X and Y are independent Poisson random variables, then the conditional distribution of X given $X+Y$ is binomial. More precisely, if X and Y are independent random variables and

$$X \sim P(\lambda), \quad P[X=k] = e^{-\lambda}\frac{\lambda^k}{k!}, \quad Y \sim P(\mu), \quad P[Y=k] = e^{-\mu}\frac{\mu^k}{k!},$$
$$k = 0, 1, 2, \ldots;\ \lambda > 0,\ \mu > 0,$$

then for each $t \geqslant 0$

$$P[X = k | X+Y = t] = \binom{t}{k} p^k (1-p)^{t-k}, \quad 0 \leqslant k \leqslant t,$$

with $p = \lambda/(\lambda+\mu)$. We note that these conditional distributions depend only on the ratio $\lambda/(\lambda+\mu)$.

Chatterji [3] dealt with the inverse problem. He showed that if X and Y are independent random variables such that

$$P[X = i] = f(i), \quad P[Y = i] = g(i),$$

where $f(i)>0,\ g(i)>0,\ \sum_{i=0}^{\infty} f(i)=\sum_{i=0}^{\infty} g(i)=1,$ and if for each $t \geqslant 0$

$$P[X=k|X+Y=t]=\begin{cases}\binom{t}{k}p_t^k(1-p_t)^{t-k}, & 0\leqslant k\leqslant t,\\ 0, & k>t,\end{cases}$$

then $p_t \equiv p,\ t=0,1,2,\ldots$, and

$$f(i)=e^{-\theta\delta}\frac{(\theta\delta)^i}{i!},\quad g(i)=e^{-\theta}\frac{\theta^i}{i!},$$

where $\delta = p/(1-p)$ and $\theta>0$ is arbitrary.

We extend Chatterji's characterization of Poisson distributions to a characterization of α-modified Poisson distributions. Moreover, we extend this characterization to α-modified Poisson distributions of type j. These are known as Delaporte distributions, and are widely applied in actuarial investigations (cf. Delaporte [4], Willmot and Sundt [13], Johnson et al. [6], Murat and Szynal [8], Sundt and Vernic [12]). Another characterization of one-parametric α-modified Poisson distribution was discussed by Steliga and Szynal in [11].

First we recall the definitions of α-modified binomial distributions and α-modified Poisson distributions introduced by Berg and Jaworski [1]. A random variable X is said to have an *α-modified binomial distribution* $(X\sim MB(N,p,\phi))$ if the following holds:

$$P[X=x]=\binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{(1+\alpha\phi)^N},\quad x=0,1,\ldots,N, \tag{1.1}$$

where $q=1-p,\ \phi\geqslant 0,\ p+\phi\geqslant 0$, are parameters, and $\alpha_k\equiv\alpha^k=k!,\ k\in\mathbb{N}$, is Riordan's symbol. For $p>0$ and $\phi=0$, (1.1) reduces to the common binomial distribution.

We mention here that Chakraborty in [2] introduced a new class of α-modified binomial distributions. He considered, among other things, α-modified binomial distributions $(X\sim MB(N,p,q,\phi))$ defined by

$$P[X=x]=\binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{(q+p+\alpha\phi)^N},\quad x=0,1,\ldots,N, \tag{1.2}$$

where in this case $q+p\neq 1$ is allowed, $\phi\geqslant 0,\ p+\phi\geqslant 0,\ q>0$. We shall use the formulae (1.1) and (1.2) in the form

$$P[X=x]=\binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{1+\sum_{i=1}^{N}N!\big((N-i)!\big)^{-1}\phi^i},\quad x=0,1,\ldots,N, \tag{1.3}$$

and

$$P[X=x]= \binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum_{i=1}^{N}N!\big((N-i)!\big)^{-1}\phi^i(q+p)^{N-i}},\quad x=0,1,\ldots,N. \tag{1.4}$$

A random variable X is said to have an *α-modified Poisson distribution* $(X \sim MP(\lambda, \psi))$ if

$$P[X = x] = \frac{(\lambda + \alpha\psi)^x}{x!}(1 - \psi)e^{-\lambda}, \quad x = 0, 1, 2, \dots,$$

where $\lambda > 0$ and ψ are parameters such that $\lambda + \psi \geqslant 0,\ |\psi| < 1$.

It is easy to verify that if $X \sim MP(\lambda_x, \psi),\ Y \sim P(\lambda_y)$, and X and Y are independent, then for each $t \geqslant 0$

$$\begin{aligned} P[X = k|X + Y = t] &= \binom{t}{k} \frac{(\lambda_x + \alpha\psi)^k \lambda_y^{t-k}}{(\lambda_x + \lambda_y + \alpha\psi)^t} \\ &= \binom{t}{k} \frac{(p + \alpha\phi)^k (1 - p)^{t-k}}{(1 + \alpha\phi)^t} \\ &= \binom{t}{k} \frac{(p + \alpha\phi)^k (1 - p)^{t-k}}{1 + \sum_{i=1}^t t!\big((t - i)!\big)^{-1} \phi^i} \end{aligned}$$

with $p = \lambda_x/(\lambda_x + \lambda_y)$ and $\phi = \psi/(\lambda_x + \lambda_y)$, i.e. the conditional distribution of X given $X + Y$ has the α-modified binomial distribution $(MB(t, p, \phi))$.

We say that a random variable X has an *α-modified discrete distribution with a support in* $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, \dots\}$ if its probability function depends on the α-Riordan symbol.

2. MAIN RESULT

Let C_α denote the class of random variables X such that $0 < P[X = k] = f_\alpha(k),\ k = 0, 1, 2, \dots$, and $\sum_{k=0}^\infty f_\alpha(k) = 1$, and C denote the class of random variables Y such that $P[Y = l] = g(l) > 0,\ l = 0, 1, 2, \dots$, and $\sum_{l=0}^\infty g(l) = 1$.

THEOREM 2.1. *Let X and Y be random variables from C_α and C, respectively. Suppose that X and Y are independent and for each $t \geqslant 0$*

$$P[X = k|X + Y = t] = \begin{cases} \binom{t}{k} \dfrac{(1 - p_t)^{t-k}(p_t + \alpha\phi)^k}{1 + \sum_{i=1}^t t!\big((t - i)!\big)^{-1}\phi^i}, & 0 \leqslant k \leqslant t, \\ 0, & k > t. \end{cases} \tag{2.1}$$

Then $p_t \equiv p,\ 0 < p < 1,\ t = 0, 1, 2 \dots$, and

$$f_\alpha(k) = \frac{1}{k!}(\lambda + \alpha\psi)^k(1 - \psi)e^{-\lambda}, \quad g(k) = \frac{\theta^k}{k!}e^{-\theta}, \tag{2.2}$$

where $\lambda = \theta p/(1 - p),\ \psi = \theta\phi/(1 - p),\ |\psi| < 1$, and $\theta > 0$ is arbitrary.

P r o o f. By the independence of X and Y we have

$$P[X=k|X+Y=t]=\frac{P[X=k]P[Y=t-k]}{P[X+Y=t]}$$
$$=\frac{f_\alpha(k)g(t-k)}{\sum_{i=0}^{t} f_\alpha(i)g(t-i)}.$$

Hence for $0\leqslant k\leqslant t$ we get

$$\text{(2.3)}\qquad \frac{f_\alpha(k)g(t-k)}{\sum_{i=0}^{t} f_\alpha(i)g(t-i)}=\binom{t}{k}\frac{(1-p_t)^{t-k}(p_t+\alpha\phi)^k}{1+\sum_{i=1}^{t} t!\big((t-i)!\big)^{-1}\phi^i}$$
$$=\binom{t}{k}\frac{(1-p_t)^{t-k}\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1}p_t^{k-i}\phi^i}{1+\sum_{i=1}^{t} t!\big((t-i)!\big)^{-1}\phi^i}$$

and

$$\text{(2.4)}\qquad \frac{f_\alpha(k-1)g(t-k+1)}{\sum_{i=0}^{t} f_\alpha(i)g(t-i)}$$
$$=\binom{t}{k-1}\left((1-p_t)^{t-k+1}\sum_{i=0}^{k-1}\frac{(k-1)!}{(k-1-i)!}p_t^{k-1-i}\phi^i\right)\left(1+\sum_{i=1}^{t}\frac{t!}{(t-i)!}\phi^i\right)^{-1},$$

which for $t\geqslant 1$ and $1\leqslant k\leqslant t$ leads us to the recursion

$$\text{(2.5)}\qquad \frac{f_\alpha(k)g(t-k)}{f_\alpha(k-1)g(t-k+1)}$$
$$=\frac{t-k+1}{k}\frac{1}{1-p_t}\left(\sum_{i=0}^{k}\frac{k!}{(k-i)!}p_t^{k-i}\phi^i\right)\left(\sum_{i=0}^{k-1}\frac{(k-1)!}{(k-1-i)!}p_t^{k-1-i}\phi^i\right)^{-1}$$

and for $0\leqslant k\leqslant t,\ t\geqslant 0$, we obtain

$$\text{(2.6)}\qquad f_\alpha(k)g(t-k)=\binom{t}{k}\frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1}p_t^{k-i}\phi^i}{(1-p_t)^k}g(t)f_\alpha(0).$$

Now, after setting $k=t$ in (2.5), we have for $t\geqslant 1$

$$\text{(2.7)}\quad f_\alpha(t)=\frac{1}{t}\frac{1}{1-p_t}\frac{\sum_{i=0}^{t} t!\big((t-i)!\big)^{-1}p_t^{t-i}\phi^i}{\sum_{i=0}^{t-1}(t-1)!\big((t-1-i)!\big)^{-1}p_t^{t-1-i}\phi^i}\frac{g(1)}{g(0)}f_\alpha(t-1).$$

It follows that

$$f_\alpha(t)=\prod_{k=1}^{t}\left(\frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1}p_k^{k-i}\phi^i}{\sum_{i=0}^{k-1}(k-1)!\big((k-1-i)!\big)^{-1}p_k^{k-1-i}\phi^i}\right)$$
$$\times\prod_{k=1}^{t}\left(\frac{1}{1-p_k}\right)\left(\frac{g(1)}{g(0)}\right)^t\frac{f_\alpha(0)}{t!}.$$

Letting $g(1)/g(0) = \theta$ we have

$$f_\alpha(t) = \prod_{k=1}^{t}\left(\frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1} p_k^{k-i}\phi^i}{\sum_{i=0}^{k-1}(k-1)!\big((k-1-i)!\big)^{-1} p_k^{k-1-i}\phi^i}\right) \times \prod_{k=1}^{t}\left(\frac{1}{1-p_k}\right)\theta^t \frac{f_\alpha(0)}{t!}. \tag{2.8}$$

Similarly, after setting $k = 1$ in (2.5) we have

$$g(t) = \frac{1}{t}\frac{1-p_t}{p_t+\phi}\frac{f_\alpha(1)}{f_\alpha(0)}g(t-1),$$

which gives (after letting $t = 1$)

$$\frac{f_\alpha(1)}{f_\alpha(0)} = \theta\frac{p_1+\phi}{1-p_1}.$$

Thus

$$g(t) = \frac{1}{t}\frac{1-p_t}{p_t+\phi}\theta\frac{p_1+\phi}{1-p_1}g(t-1) = \frac{g(t-1)}{t}\theta\frac{p_1+\phi}{1-p_1}\left(\frac{p_t+\phi}{1-p_t}\right)^{-1},$$

which leads to

$$g(t) = \frac{g(0)}{t!}\theta^t\frac{(p_1+\phi)^t}{(1-p_1)^t}\left(\prod_{k=1}^{t}\frac{p_k+\phi}{1-p_k}\right)^{-1}, \qquad t \geqslant 1. \tag{2.9}$$

Now from (2.6) we have

$$f_\alpha(k+1)g(t-k-1) = \binom{t}{k+1}\frac{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1}p_t^{k+1-i}\phi^i}{(1-p_t)^{k+1}}g(t)f_\alpha(0),$$

which implies for $k+1 \leqslant t$

$$\frac{g(t-k)}{g(t-k-1)} = \frac{k+1}{t-k}\frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1}p_t^{k-i}\phi^i}{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1}p_t^{k+1-i}\phi^i}(1-p_t)\frac{f_\alpha(k+1)}{f_\alpha(k)}. \tag{2.10}$$

But from (2.7) (letting $t = k+1$) we get

$$\frac{f_\alpha(k+1)}{f_\alpha(k)} = \frac{1}{k+1}\frac{\theta}{1-p_{k+1}}\frac{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1}p_{k+1}^{k+1-i}\phi^i}{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1}p_{k+1}^{k-i}\phi^i},$$

so from (2.10) we have

$$g(t-k) = \frac{\theta}{t-k} g(t-k-1) \frac{1-p_t}{1-p_{k+1}} \times \frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1} p_t^{k-i}\phi^i}{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1} p_t^{k+1-i}\phi^i} \times \frac{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1} p_{k+1}^{k+1-i}\phi^i}{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1} p_{k+1}^{k-i}\phi^i}. \tag{2.11}$$

From Panjer's recurrence relation (cf. Klugman et al. [7], Panjer and Willmot [9]) we conclude that, for $m = t-k$, $g(m)$ in (2.11) is a probability distribution if

$$\frac{1-p_t}{1-p_{k+1}} \frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1} p_t^{k-i}\phi^i}{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1} p_t^{k+1-i}\phi^i} \times \frac{\sum_{i=0}^{k+1}(k+1)!\big((k+1-i)!\big)^{-1} p_{k+1}^{k+1-i}\phi^i}{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1} p_{k+1}^{k-i}\phi^i} = 1. \tag{2.12}$$

The condition (2.12) will be fulfilled if $p_t = p_{k+1} \equiv p$ for all t and k. Hence, $g(m)$ is a probability function of a Poisson distribution. Hence, letting $p \equiv p_k$ in (2.8), we obtain

$$f_\alpha(t) = \prod_{k=1}^{t} \left(\frac{\sum_{i=0}^{k} k!\big((k-i)!\big)^{-1} p^{k-i}\phi^i}{\sum_{i=0}^{k-1}(k-1)!\big((k-1-i)!\big)^{-1} p^{k-1-i}\phi^i} \right) \prod_{k=1}^{t} \left(\frac{1}{1-p} \right) \theta^t \frac{f_\alpha(0)}{t!}.$$

Hence

$$f_\alpha(t) = \frac{\theta^t}{t!} \frac{f_\alpha(0)}{(1-p)^t} \prod_{k=1}^{t} \frac{(p+\alpha\phi)^k}{(p+\alpha\phi)^{k-1}} = f_\alpha(0) \frac{\theta^t}{t!} \left(\frac{p+\alpha\phi}{1-p} \right)^t. \tag{2.13}$$

Using the condition

$$\sum_{t=0}^{\infty} f_\alpha(t) = 1,$$

we get

$$f_\alpha(0) = \exp\big(-\theta(p+\alpha\phi)/(1-p)\big).$$

Taking into account the property of the α-Riordan symbol, we have

$$\begin{aligned} f_\alpha(0) &= \exp\Big(-\frac{\theta p}{1-p}\Big) \cdot \exp\Big(-\frac{\alpha\theta\phi}{1-p}\Big) \\ &= \exp\Big(-\frac{\theta p}{1-p}\Big) \left[1 + \frac{1}{1!}\left(\frac{\alpha\theta\phi}{1-p}\right) + \frac{1}{2!}\left(\frac{\alpha\theta\phi}{1-p}\right)^2 + \dots \right]^{-1} \\ &= \exp\Big(-\frac{\theta p}{1-p}\Big) \Big(1 - \frac{\theta\phi}{1-p} \Big). \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we obtain

$$f_\alpha(t) = \frac{\theta^t}{t!}\left(\frac{p+\alpha\phi}{1-p}\right)^t \exp\Bigl(-\frac{\theta p}{1-p}\Bigr)\Bigl(1-\frac{\theta\phi}{1-p}\Bigr),$$

which gives

$$f_\alpha(k) = \frac{1}{k!}(\lambda+\alpha\psi)^k(1-\psi)e^{-\lambda}$$

with $\lambda = \theta p/(1-p),\ \psi = \theta\phi/(1-p),\ |\psi| < 1$, completing the proof of (2.2). ∎

REMARK 2.1. Berg and Jaworski [1] pointed out that if X and Y are independent random variables, X has a Poisson distribution $(X \sim P(\lambda))$ and Y has a geometric distribution $(Y \sim G(\psi))$, then the convolution of X and Y has the α-modified Poisson distribution $(MP(\lambda,\psi))$ or a Delaporte distribution. So the statement of Theorem 2.1 gives a characterization of the Delaporte distribution.

The following result generalizes Theorem 2.1.

THEOREM 2.2. *Let X and Y be independent random variables from C_α and C, respectively. If for given $p \in (0,1)$, $\phi \geqslant 0$, and for each $t \geqslant 1$ the following two conditions hold*:

$$(2.15)\qquad \begin{aligned} P[X=t|X+Y=t] &= \frac{(p+\alpha\phi)^t}{1+\sum_{i=1}^t t!\bigl((t-i)!\bigr)^{-1}\phi^i},\\ P[X=t-1|X+Y=t] &= \frac{t(p+\alpha\phi)^{t-1}(1-p)}{1+\sum_{i=1}^t t!\bigl((t-i)!\bigr)^{-1}\phi^i}, \end{aligned}$$

then

$$f_\alpha(k) = \frac{1}{k!}(\lambda+\alpha\psi)^k(1-\psi)e^{-\lambda}, \quad g(k) = \frac{\theta^k}{k!}e^{-\theta},$$

where $\lambda = \theta p/(1-p),\ \psi = \theta\phi/(1-p),\ |\psi| < 1$, and $\theta > 0$ is arbitrary.

P r o o f. Since X and Y are independent, from (2.15) we get

$$(2.16)\qquad \frac{f_\alpha(t)g(0)}{\sum_{i=0}^t f_\alpha(i)g(t-i)} = \frac{(p+\alpha\phi)^t}{1+\sum_{i=1}^t t!\bigl((t-i)!\bigr)^{-1}\phi^i}$$

and

$$\frac{f_\alpha(t-1)g(1)}{\sum_{i=0}^t f_\alpha(i)g(t-i)} = \frac{t(p+\alpha\phi)^{t-1}(1-p)}{1+\sum_{i=1}^t t!\bigl((t-i)!\bigr)^{-1}\phi^i},$$

which leads us to

$$f_\alpha(t) = \frac{g(1)}{g(0)}\frac{1}{t}\frac{(p+\alpha\phi)^t}{(p+\alpha\phi)^{t-1}}\frac{1}{1-p}f_\alpha(t-1).$$

It then follows, by recursion, that for $t \geqslant 1$

$$f_\alpha(t) = \frac{\theta^t}{t!}\left(\frac{p+\alpha\phi}{1-p}\right)^t f_\alpha(0), \tag{2.17}$$

where $\theta = g(1)/g(0)$. Referring now to (2.13) et seq., we infer that here also

$$f_\alpha(0) = \exp\left(-\frac{\theta p}{1-p}\right)\left(1-\frac{\theta\phi}{1-p}\right)$$

and

$$f_\alpha(t) = \frac{1}{t!}(\lambda+\alpha\psi)^t(1-\psi)e^{-\lambda},$$

where $\lambda = \theta p/(1-p),\ \psi = \theta\phi/(1-p),\ |\psi| < 1$. Using (2.16), we get

$$\frac{f_\alpha(t)\left(1+\sum_{i=1}^t t!\big((t-i)!\big)^{-1}\phi^i\right)g(0)}{(p+\alpha\phi)^t} = \sum_{i=0}^t f_\alpha(i)g(t-i). \tag{2.18}$$

Putting (2.17) in (2.18) we obtain

$$\begin{aligned} &\frac{\theta^t}{t!}\frac{g(0)}{(1-p)^t}\left(1+\sum_{i=1}^t \frac{t!}{(t-i)!}\phi^i\right) \\ &\qquad = \sum_{i=0}^t \frac{\theta^i}{i!}\frac{g(t-i)}{(1-p)^i}\left(p^i+\sum_{j=1}^i \frac{i!}{(i-j)!}\phi^j p^{i-j}\right). \end{aligned} \tag{2.19}$$

Note that $g(t)$ is the solution of (2.19) given by

$$g(m) = g(0)\frac{\theta^m}{m!}.$$

Since $\sum_{m=0}^\infty g(m) = 1$, we have $g(0) = e^{-\theta}$, which completes the proof. ■

3. A CHARACTERIZATION OF α-MODIFIED DISTRIBUTION OF TYPE j

Chakraborty investigated in [2] an α-modified binomial and Poisson distributions of type j. Recall that (cf. Chakraborty [2]) a random variable X is said to have an *α-modified binomial distribution of type j* ($X \sim MB_j(N,p,q,\phi)$) if

$$\begin{aligned} P[X=x] &= \binom{N}{x}\frac{\big(p+\phi\alpha(j)\big)^x q^{N-x}}{\big(q+p+\phi\alpha(j)\big)^N} \\ &= \binom{N}{x}\frac{\big(p+\phi\alpha(j)\big)^x q^{N-x}}{(q+p)^N+\sum_{i=1}^N\binom{N}{i}\phi^i\alpha^i(j)(q+p)^{N-i}}, \quad x = 0,1,\ldots,N, \end{aligned}$$

where $q+p\neq 1,\ q\geqslant 0,\ \phi\geqslant 0,\ p+\phi\geqslant 0$, and for $i=0,1,\ldots$

$$\alpha^i(j)=\begin{cases}\binom{i+j-1}{i}i! & \text{for } j\geqslant 1,\\ 0 & \text{for } j=0.\end{cases}$$

We are interested here in a special case of α-modified binomial distribution of type j $(X\sim MB_j(N,p,\phi))$ defined by

$$\begin{aligned}P[X=x]&=\binom{N}{x}\frac{\big(p+\phi\alpha(j)\big)^x q^{N-x}}{\big(1+\phi\alpha(j)\big)^N}\\ &=\binom{N}{x}\frac{\big(p+\phi\alpha(j)\big)^x q^{N-x}}{1+\sum_{i=1}^{N}\binom{N}{i}\phi^i\alpha^i(j)},\quad x=0,1,\ldots,N,\end{aligned}$$

where $q+p=1,\ q\geqslant 0,\ \phi\geqslant 0,\ p+\phi\geqslant 0$.

A random variable X is said to have an *α-modified Poisson distribution of type j* $(X\sim MP_j(\lambda,\psi))$ if

$$P[X=x]=\frac{1}{x!}\big(\lambda+\psi\alpha(j)\big)^x(1-\psi)^j e^{-\lambda},\quad x=0,1,2\ldots,$$

where λ and ψ are parameters such that $\lambda+\psi\geqslant 0,\ 0<\psi<1$.

A random variable X is said to have a *negative binomial distribution* $(X\sim NB(j,p))$ if

$$P[X=x]=\binom{j+x-1}{x}p^x(1-p)^j,\quad x=0,1,2,\ldots,$$

where $j>0$ and $0<p<1$.

It is known that if $X\sim P(\lambda)$ and $Y\sim NB(j,\psi)$ are independent, then we have $X+Y\sim MP_j(\lambda,\psi)$. This distribution is known as the *Delaporte distribution* (see the references in the Introduction). Here we give a generalization of the results in Section 2.

If a random variable X has the α-modified Poisson distribution of type j, i.e., $X\sim MP_j(\lambda_x,\psi)$, and a random variable Y has the Poisson distribution $Y\sim P(\lambda_y)$, and if additionally X and Y are independent, then

$$\begin{aligned}P[X=k|X+Y=t]&=\binom{t}{k}\frac{\big(\lambda_x+\alpha(j)\psi\big)^k\lambda_y^{t-k}}{\big(\lambda_x+\lambda_y+\alpha(j)\psi\big)^t}\\ &=\binom{t}{k}\frac{\big(p+\alpha(j)\phi\big)^k(1-p)^{t-k}}{\big(1+\alpha(j)\phi\big)^t}\\ &=\binom{t}{k}\frac{\big(p+\alpha(j)\phi\big)^k(1-p)^{t-k}}{1+\sum_{i=1}^{t}\binom{t}{i}\phi^i\alpha^i(j)}\end{aligned}$$

with $p = \lambda_x/(\lambda_x + \lambda_y)$ and $\phi = \psi/(\lambda_x + \lambda_y)$, i.e. the conditional distribution of X given $X + Y$ is an α-modified binomial distribution of type j $(MB_j(t, p, \phi))$ (cf. Section 1 for $j = 1$).

Now we present some characterization of the α-modified Poisson distribution of type j.

We say that a random variable X has an $\alpha(j)$-*modified discrete distribution with a support in* $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, \ldots\}$ if its probability function depends on $\alpha(j)$ symbol.

Let $C_{\alpha(j)}$ denote the class of random variables X such that $0 < P[X = k] = f_{\alpha(j)}(k)$, $k = 0, 1, 2, \ldots$, and $\sum_{k=0}^{\infty} f_{\alpha(j)}(k) = 1$.

THEOREM 3.1. *Let X and Y be random variables from $C_{\alpha(j)}$ and C, respectively. Suppose that X and Y are independent and*

$$P[X = k|X + Y = t] = \begin{cases} \binom{t}{k} \dfrac{\big(p_t + \alpha(j)\phi\big)^k (1 - p_t)^{t-k}}{1 + \sum_{i=1}^{t} \binom{t}{i} \phi^i \alpha^i(j)}, & 0 \leqslant k \leqslant t, \\ 0, & k > t. \end{cases}$$

Then $p_t \equiv p,\ 0 < p < 1,\ t = 0, 1, 2\ \ldots$, and

$$f_{\alpha(j)}(k) = \frac{1}{k!}\big(\lambda + \alpha(j)\psi\big)^k (1 - \psi)^j e^{-\lambda}, \quad g(k) = \frac{\theta^k}{k!} e^{-\theta},$$

where $\lambda = \theta p/(1 - p),\ \psi = \theta\phi/(1 - p),\ 0 < \psi < 1$, and $\theta > 0$ is arbitrary.

P r o o f. The proof is analogous to that of Theorem 2.1. ■

The following theorem contains a characterization of the Delaporte distribution.

THEOREM 3.2. *Let X and Y be independent random variables from $C_{\alpha(j)}$ and C, respectively. If*

$$P[X = t|X + Y = t] = \frac{\big(p + \alpha(j)\phi\big)^t}{1 + \sum_{i=1}^{t} \binom{t}{i} \phi^i \alpha^i(j)}, \qquad t \geqslant 1,\ 0 < p < 1,$$

$$P[X = t - 1|X + Y = t] = \frac{t\big(p + \alpha(j)\phi\big)^{t-1}(1 - p)}{1 + \sum_{i=1}^{t} \binom{t}{i} \phi^i \alpha^i(j)},$$

then

$$f_{\alpha(j)}(k) = \frac{1}{k!}\big(\lambda + \alpha(j)\psi\big)^k (1 - \psi)^j e^{-\lambda}, \quad g(k) = \frac{\theta^k}{k!} e^{-\theta},$$

where $\lambda = \theta p/(1 - p),\ \psi = \theta\phi/(1 - p),\ 0 < \psi < 1$, and $\theta > 0$ is arbitrary.

P r o o f. The proof is similar to that of Theorem 2.2. ■

Theorems 2.1 and 2.2 are special cases of Theorems 3.1 and 3.2 for $j = 1$, respectively.

Acknowledgments. We thank very much our friend Kerwin Morris from Adelaide University for his mathematical and English suggestions.

REFERENCES

[1] S. Berg and J. Jaworski, *Modified binomial and Poisson distributions with application in random mapping theory*, J. Statist. Plann. Inference 18 (1988), pp. 313–322.
[2] S. Chakraborty, *On some new α-modified binomial and Poisson distributions and their applications*, Comm. Statist. Theory Methods 37 (2008), pp. 1755–1769.
[3] S. D. Chatterji, *Some elementary characterizations of the Poisson distribution*, Amer. Math. Monthly 70 (1963), pp. 958–964.
[4] P. Delaporte, *Quelques problèmes de statistique mathématique posés par l'assurance automobile et le bonus pour non sinistre*, Bull. Trimest. Inst. Actuaires Fr. 227 (1959), pp. 87–102.
[5] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, second edition, Wiley, New York 1957.
[6] N. Johnson, S. Kotz, and A. Kemp, *Univariate Discrete Distributions*, second edition, Wiley, New York 1992.
[7] S. A. Klugman, H. H. Panjer, and G. E. Willmot, *Loss Models: From Data to Decisions*, Wiley, New York 1998.
[8] M. Murat and D. Szynal, *Moments of counting distributions satisfying recurrence relations*, Mat. Stosow. 1 (42) (2000), pp. 92–129.
[9] H. H. Panjer and G. E. Willmot, *Insurance Risk Models*, Society of Actuaries, 1992.
[10] J. Riordan, *Combinatorial Identities*, Wiley, New York 1968.
[11] K. Steliga and D. Szynal, *Note on a characterization of α-modified and truncated Poisson distribution*, Int. J. Pure Appl. Math. 67, No. 2 (2011), pp. 141–147.
[12] B. Sundt and R. Vernic, *Recursions for Convolutions and Compound Distributions with Insurance Applications*, Springer, Berlin–Heidelberg 2009.
[13] G. E. Willmot and B. Sundt, *On evaluation of the Delaporte distribution and related distributions*, Scand. Actuar. J. (1989), pp. 101–113.

Department of Mathematics
Faculty of Economics
Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 5
20-031 Lublin, Poland
E-mail: katarzyna.steliga@gmail.com

Wydział Zamiejscowy KUL
(Department of Economics)
ul. Ofiar Katynia 6
37-450 Stalowa Wola, Poland
E-mail: szynal@poczta.umcs.lublin.pl

Received on 20.12.2011;
revised version on 25.9.2012